A Gravitized Standard Model Is Found as the Solution to an Optimization Problem on Information and Geometric Measurements

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May 15, 2023

Abstract

A method to construct a provably optimal quantum theory is presented. The method comprises solving an optimization problem related to information and measurements. In the case of scalar measurements the solution reduces to the Gibbs measure of statistical mechanics. In the case of phase-invariant measurements, the solution is an equivalent formulation of non-relativistic quantum mechanics. Finally, in the case of geometric measurements, the solution extends the basis of quantum physics to support quantum gravity and the standard model; notably, it disallows dimensions other than four as well as gauges other than those of the standard model.

1 Introduction

An optimization problem relating to information and measurements is presented. Solving the problem for geometric measurements yields the solution of interest. Specifically, below 4D, the solution is vacuous; above 4D, it admits no observables; and finally in 4D, the solution contains gravity for fermions and bosons from the quotient bundle $FX/\mathrm{Spin}^c(3,1)$, electromagnetism from the U(1)-bundle, and the standard model from the gauge group $SU(3)\times SU(2)\times U(1)$. No other structures are possible within the theory, making it hyper-specific to what we observe in the universe.

Let us begin by reviewing how statistical mechanics uses an optimization problem on entropy and measurements to derive the Gibbs measure.

Measurements and expectation values are used as constraints in statistical mechanics to derive the Gibbs measure using Lagrange multipliers[1] by maximizing the entropy.

For instance, an energy constraint on the entropy is expressed as

$$\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q), \tag{1}$$

which is associated with an energy meter that measures the system's energy and produces a series of energy measurements $E_1, E_2, \ldots,$ convergent to an expectation value $\overline{E}$.

Another common constraint is related to the volume as

$$\overline{V} = \sum_{q \in \mathbb{Q}} \rho(q) V(q), \tag{2}$$

which is associated with a volume meter acting on a system and produces a sequence of measured volumes $V_1, V_2, \ldots,$ converging to an expectation value $\overline{V}$.

Moreover, the sum over the statistical ensemble must equal one, as follows:

$$1 = \sum_{q \in \mathbb{Q}} \rho(q). \tag{3}$$

Using Equations (1) and (3), a typical statistical mechanical system is obtained by maximizing the entropy using the corresponding Lagrange equation. The Lagrange multiplier method is expressed as

$$\mathcal{L}(\rho, \lambda, \beta) = -k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right), \tag{4}$$

where λ and β are the Lagrange multipliers.

By solving $\frac{\partial \mathcal{L}(\rho, \lambda, \beta)}{\partial \rho} = 0$ for ρ, the Gibbs measure is obtained as

$$\rho(q, \beta) = \frac{1}{Z(\beta)} \exp\bigl(-\beta E(q)\bigr), \tag{5}$$

where

$$Z(\beta) = \sum_{q \in \mathbb{Q}} \exp\bigl(-\beta E(q)\bigr). \tag{6}$$

Let us now return to the present optimization problem. The goal is to support all possible measurements of nature.

What measurements are missing from statistical mechanics? As the constraints used in statistical mechanics are scalar (e.g., energy and volume meters), they cannot support those that are geometric. In general, geometric measurements include some scalar measurements, such as those produced by a dilation meter, and geometric measurements, such as those produced by protractors and phase, boost, spin, and shear meters.

A constraint will be introduced that extends the scope of statistical mechanics to geometric measurements.

Consistent with the identified missing types of measurements, the construction of the extended constraint requires a mathematical object coherent with both scalars and geometry. As such, multivectors are used. A link between geometry and probability via the trace is also utilized. The trace of a matrix can be understood as the expected eigenvalue multiplied by the vector space dimension, and the eigenvalues as the ratios of the distortion of the linear transformation associated with the matrix[2].

Axiom 1 (The Geometric Constraint)**.**

$$\frac{1}{d}\operatorname{tr}\overline{\mathbf{u}} = \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{d}\operatorname{tr}\mathbf{u}(q), \tag{7}$$

where $\operatorname{tr}\mathbf{u}(q)$ *is an observable,* $\operatorname{tr}\overline{\mathbf{u}}$ *is its average,* $\mathbf{u}$ *corresponds to a multivector of the geometric algebra* $\mathcal{G}(\mathbb{R}^{m,n})$ *such that* $d = m+n$, ρ *is a probability measure, and* $\mathbb{Q}$ *is a statistical ensemble. It is also noted that the trace of a multivector can be obtained by mapping the multivector to its matrix representation (Section 2) and taking the trace of the matrix.*

As the multivectors of $\mathcal{G}(\mathbb{R}^2)$ and $\mathcal{G}(\mathbb{R}^{3,1})$ are group isomorphic to $\mathbb{M}(2,\mathbb{R})$ and $\mathbb{M}(4,\mathbb{R})$, respectively, the domain of the geometric constraint can be reckoned to be that of general linear measurements. The use of multivectors instead of matrices merely presents a preferred geometric representation of said general linear measurements.

To formulate the proposed optimization problem, Equation 1, which corresponds to a scalar measurement constraint, will be replaced with Axiom 1, which is the geometric constraint. Instead of energy or volume meters, protractors and phase, boost, dilation, spin, and shear meters will be supported.

Let us now rigorously state the optimization problem, then we will further discuss the rationale.

Theorem 1 (A Provably Optimal Formulation of Physics)**.**

$$\underbrace{\mathcal{L}(\rho,\lambda,\boldsymbol{\tau})}_{\substack{\text{a}\\ \text{maximization}\\ \text{problem}}} = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{on the quantity of information}\\ \text{associated with the}\\ \text{production of a measurement}\\ \text{event}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\text{over all predictive theories}} \underbrace{+\tau\left(\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}}-\sum_{q\in\mathbb{Q}}\rho(q)\frac{1}{n}\operatorname{tr}\mathbf{u}(q)\right)}_{\text{of nature}}, \tag{8}$$

where λ *and* τ *are Lagrange multipliers. The theorem yields the optimal formulation of physics as its solution. The solution and proof of this theorem are given in Section 3.*

Now, its rationale is discussed.

As the techniques of statistical mechanics are used abundantly, it is relevant to identify and discuss the correspondence between ordinary statistical mechanics and the present proposal.

Table 1: Correspondence between statistical mechanics and the present proposal.

Constraint	Energy (Scalar) Constraint	Geometric Constraint
Ontology	Ergodic system	Production of a measurement event
Entropy	Boltzmann	Shannon
Probability measure	Gibbs	Section 3 (Generalized Born Rule)
Micro-state	Energy levels	Collapsed state
Lagrange multiplier	Temperature	Section 3.1 (Entropic flow)

In the present proposal, the information is quantified by the relative Shannon entropy (in base e), and not by Boltzmann entropy. Consequently, the ontology is not that of ergodic systems but that of the production of a message (in the sense of the Shannon communication theory [3]) in which the elements are measurement events. Measurements are inline with the purpose of a predictive theory; it is therefore normal that they constitute the core of this optimization problem. It is clarified that the notion of a message here is not to be interpreted as a signal (i.e., the *physical transmission* of a message); rather the present message is a mathematical device to quantify the information associated to measurement events. Specifically, a message is a n-tuple whose elements are in $\mathbb{Q}$ and selected according to the probability measure ρ. It is also clarified that the message is not intended to be interpreted as an exchange of information between two observers; rather, it is a message defined by the observer following the production of a measurement event in nature. It is used to exactly specify the sequence of measurement events that have so far transpired in the system. For instance, it may be associated with the registration of a "click"[4] on a screen or an incidence counter. It is analogous to the micro-state of an ergodic system which exactly specify the position and momentum of all particles in a gas.

The remainder of the manuscript is organized as follows.

The Methods section introduces tools using geometric algebra, based on the study by Lundholm et al. [5, 6]. Specifically, the notion of a determinant for multivectors and the Clifford conjugate for generalizing the complex conjugate are used. These tools enable the geometric expression of the results.

The Results section presents two solutions for the general Lagrange equation. The first is applicable to an ensemble $\mathbb{Q}$ that is at most countably infinite, whereas the second is applicable to the continuum ($\sum \to \int$) where $\mathbb{Q}$ is uncountable.

The Analysis section details the solution in $\mathbb{R}$, $\mathbb{C}$, 2D, and 4D, and the defects in (2n+1)D and 2nD>4. Specifically, in 4D, the solution automatically contains gravity for fermions and bosons from the quotient bundle FX/Spin^c(3,1), electromagnetism from the U(1)-bundle, and the standard model from the gauge group SU(3)×SU(2)×U(1), and admits no freedom for alternatives. These structures do not need to be inserted manually; they are automatically included in the solution.

2 Methods

2.1 Notation

- Typography:

  Sets are written using the blackboard bold typography (e.g., $\mathbb{L}$, $\mathbb{W}$, and $\mathbb{Q}$) unless a prior convention assigns it another symbol.

  Matrices are in bold uppercase (e.g., $\mathbf{P}$ and $\mathbf{M}$); tuples, vectors, and multivectors are in bold lowercase (e.g., $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{g}$); and most other constructions (e.g., scalars and functions) have plain typography (e.g., a, and A).

  The unit pseudo-scalar (of geometric algebra), imaginary number, and identity matrix are $\mathbf{i}$, i, and $\mathbf{I}$, respectively.

- Sets:

  The projection of a tuple $\mathbf{p}$ is $\text{proj}_i(\mathbf{p})$.

  As an example, the elements of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$ are denoted as $\mathbf{p} = (x, y)$.

  The projection operators are $\text{proj}_1(\mathbf{p}) = x$ and $\text{proj}_2(\mathbf{p}) = y$; if projected over a set, the corresponding results are $\text{proj}_1(\mathbb{R}^2) = \mathbb{R}_1$ and $\text{proj}_2(\mathbb{R}^2) = \mathbb{R}_2$, respectively.

  The size of a set $\mathbb{X}$ is $|\mathbb{X}|$.

  The symbol $\cong$ indicates an isomorphism and $\rightarrow$ denotes a homomorphism.

- Analysis:

  The dagger $z^\dagger$ denotes the complex conjugate of z.

- Matrix:

  The Dirac gamma matrices are γ_0, γ_1, γ_2, and γ_3.

  The Pauli matrices are σ_x, σ_y, and σ_z.

  The dagger $\mathbf{M}^\dagger$ denotes the conjugate transpose of $\mathbf{M}$.

  The commutator is defined as $[\mathbf{M}, \mathbf{P}] : \mathbf{MP} - \mathbf{PM}$, and the anti-commutator is defined as $\{\mathbf{M}, \mathbf{P}\} : \mathbf{MP} + \mathbf{PM}$.

- <u>Geometric algebra:</u>

  The elements of an arbitrary curvilinear geometric basis are denoted as $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ (such that $\mathbf{e}_\nu \cdot \mathbf{e}_\mu = g_{\mu\nu}$), and $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n$ (such that $\hat{\mathbf{x}}_\mu \cdot \hat{\mathbf{x}}_\nu = \eta_{\mu\nu}$) if they are orthonormal.

  A geometric algebra of $m+n$D over field $\mathbb{F}$ is denoted as $\mathcal{G}(\mathbb{F}^{m,n})$.

  The grades of a multivector are denoted as $\langle \mathbf{v} \rangle_k$.

  Specifically, $\langle \mathbf{v} \rangle_0$ is a scalar, $\langle \mathbf{v} \rangle_1$ is a vector, $\langle \mathbf{v} \rangle_2$ is a bivector, $\langle \mathbf{v} \rangle_{n-1}$ is a pseudo-vector, and $\langle \mathbf{v} \rangle_n$ is a pseudo-scalar.

  A scalar and vector such as $\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_1$ form a para-vector; a combination of even grades ($\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_2 + \langle \mathbf{v} \rangle_4 + \ldots$) or odd grades ($\langle \mathbf{v} \rangle_1 + \langle \mathbf{v} \rangle_3 + \ldots$) form even or odd multivectors, respectively.

  Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set. A general multivector of $\mathcal{G}(\mathbb{R}^2)$ can be formulated as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

  Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set. A general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be formulated as $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

2.2 Geometric representation in 2D

Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^2)$ is expressed as

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}, \tag{9}$$

where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

Each multivector has a structure-preserving (addition/multiplication) matrix representation.

Definition 1 (2D geometric representation)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{10}$$

Thus, the trace of $\mathbf{u}$ is a. The converse is also true: each 2×2 real matrix is represented as a multivector of $\mathcal{G}(\mathbb{R}^2)$.

In geometric algebra, the determinant[6] of a multivector $\mathbf{u}$ can be defined as:

Definition 2 (Geometric representation of the determinant 2D)**.**

$$\begin{aligned} \det \quad : \quad & \mathcal{G}(\mathbb{R}^2) \longrightarrow \mathbb{R} \\ & \mathbf{u} \longmapsto \mathbf{u}^\ddagger \mathbf{u}, \end{aligned} \tag{11}$$

where $\mathbf{u}^{\ddagger}$ is

Definition 3 (Clifford conjugate 2D)**.**

$$\mathbf{u}^{\ddagger} := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2. \tag{12}$$

For example,

$$\det \mathbf{u} = (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) \tag{13}$$
$$= a^2 - x^2 - y^2 + b^2 \tag{14}$$
$$= \det \begin{bmatrix} a + x & -b + y \\ b + y & a - x \end{bmatrix}. \tag{15}$$

Finally, the Clifford transpose is defined.

Definition 4 (2D Clifford transpose)**.** *The Clifford transpose is the geometric analog to the conjugate transpose and is interpreted as a transpose followed by an element-by-element application of the complex conjugate. Likewise, the Clifford transpose is a transpose followed by an element-by-element application of the Clifford conjugate.*

$$\begin{bmatrix} \mathbf{u}_{00} & \dots & \mathbf{u}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{mn} \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{u}_{00}^{\ddagger} & \dots & \mathbf{u}_{m0}^{\ddagger} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{nm}^{\ddagger} \end{bmatrix} \tag{16}$$

If applied to a vector, then

$$\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{v}_1^{\ddagger} & \dots \mathbf{v}_m^{\ddagger} \end{bmatrix}. \tag{17}$$

2.3 Geometric representation in 3+1D

Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set. A general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be written as

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}, \tag{18}$$

where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

Similarly, each multivector has a structure-preserving (addition/multiplication) matrix representation.

The multivectors of $\mathcal{G}(\mathbb{R}^{3,1})$ are represented as follows:

Definition 5 (4D geometric representation).

$$
\begin{aligned}
&a + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\
&\quad + f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 + f_{13}\gamma_1 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 \\
&\quad + v_t\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + v_x\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v_y\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + v_z\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\
&\quad + b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3
\end{aligned}
$$

$$
\cong \begin{bmatrix}
a + x_0 - if_{12} - iv_3 & f_{13} - if_{23} + v_2 - iv_1 & -ib + x_3 + f_{03} - iv_0 & x_1 - ix_2 + f_{01} - if_{02} \\
-f_{13} - if_{23} - v_2 - iv_1 & a + x_0 + if_{12} + iv_3 & x_1 + ix_2 + f_{01} + if_{02} & -ib - x_3 - f_{03} - iv_0 \\
-ib - x_3 + f_{03} + iv_0 & -x_1 + ix_2 + f_{01} - if_{02} & a - x_0 - if_{12} + iv_3 & f_{13} - if_{23} - v_2 + iv_1 \\
-x_1 - ix_2 + f_{01} + if_{02} & -ib + x_3 - f_{03} + iv_0 & -f_{13} - if_{23} + v_2 + iv_1 & a - x_0 + if_{12} - iv_3
\end{bmatrix} \tag{19}
$$

Thus, the trace of $\mathbf{u}$ is a.

In 3+1D, the determinant is defined solely using the constructs of geometric algebra[6].

The determinant of $\mathbf{u}$ is as follows:

Definition 6 (3+1D geometric representation of determinant).

$$
\det \quad : \quad \mathcal{G}(\mathbb{R}^{3,1}) \longrightarrow \mathbb{R} \tag{20}
$$

$$
\mathbf{u} \longmapsto \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}, \tag{21}
$$

where $\mathbf{u}^\ddagger$ is

Definition 7 (3+1D Clifford conjugate).

$$
\mathbf{u}^\ddagger := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 + \langle \mathbf{u} \rangle_3 + \langle \mathbf{u} \rangle_4, \tag{22}
$$

and where $\lfloor \mathbf{u} \rfloor_{\{3,4\}}$ is the blade-conjugate of degrees three and four (the plus sign is reversed to a minus sign for blades three and four).

$$
\lfloor \mathbf{u} \rfloor_{\{3,4\}} := \langle \mathbf{u} \rangle_0 + \langle \mathbf{u} \rangle_1 + \langle \mathbf{u} \rangle_2 - \langle \mathbf{u} \rangle_3 - \langle \mathbf{u} \rangle_4 \tag{23}
$$

3 Results

3.1 Phase-invariant measurements in 0+1D

In this first result, which also serves as an introductory example, non-relativistic quantum mechanics is recovered using the Lagrange multiplier method and a linear constraint on the relative Shannon entropy.

As previously mentioned, the relative Shannon entropy (in base e) is applied instead of Boltzmann entropy to achieve the aforementioned goal.

$$
S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} \tag{24}
$$

In statistical mechanics, scalar measurement constraints are used on the entropy, such as energy and volume meters, which are sufficient for recovering the Gibbs ensemble. However, applying such scalar measurement constraints is insufficient to recover quantum mechanics.

A *complex measurement pattern*, a subset of the geometric constraint invariant for a complex phase, is used to overcome this limitation. It is defined[1] as

$$\operatorname{tr}\begin{bmatrix}0 & -\overline{E}\\ \overline{E} & 0\end{bmatrix} = \sum_{q\in\mathbb{Q}} \rho(q)\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0,\end{bmatrix}. \tag{25}$$

It may be recalled that $\begin{bmatrix}a(q) & -b(q)\\ b(q) & a(q)\end{bmatrix} \cong a(q) + ib(q)$ is the matrix representation of the complex numbers. In terms of multivectors, this constraint corresponds to the matrix representation of the pseudoscalar of $\mathcal{G}(\mathbb{R}^{0,1})$.

Similar to energy or volume meters, linear instruments produce a sequence of measurements that converge to an expected value but with phase invariance. In the solution, this phase invariance originates from the trace.

The Lagrangian equation that describes this optimization problem is

$$\mathcal{L}(\rho,\lambda,\tau) = -\sum_{q\in\mathbb{Q}} \rho(q)\ln\frac{\rho(q)}{p(q)} + \lambda\left(1 - \sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\operatorname{tr}\begin{bmatrix}0 & -\overline{E}\\ \overline{E} & 0\end{bmatrix} - \sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right). \tag{26}$$

This equation is maximized for ρ by imposing the condition $\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} = 0$. The following results are obtained:

$$\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} = -\ln\frac{\rho(q)}{p(q)} - 1 - \lambda - \tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix} \tag{27}$$

$$0 = \ln\frac{\rho(q)}{p(q)} + 1 + \lambda + \tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix} \tag{28}$$

$$\implies \ln\frac{\rho(q)}{p(q)} = -1 - \lambda - \tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix} \tag{29}$$

$$\implies \rho(q) = p(q)\exp(-1-\lambda)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right) \tag{30}$$

$$= \frac{1}{Z(\tau)}p(q)\det\exp\left(-\tau\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right), \tag{31}$$

[1]The consideration that $d = 1$ (in Axiom 1) if the matrix is 2×2 may be of concern. Here, only the imaginary part of the complex numbers $a + ib\,|_{a\to 0} = ib$ is used, rendering the constraint one-dimensional.

where $Z(\tau)$ is obtained as

$$1 = \sum_{q\in\mathbb{Q}} p(q)\exp(-1-\lambda)\exp\left(-\tau\,\mathrm{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right) \tag{32}$$

$$\implies \big(\exp(-1-\lambda)\big)^{-1} = \sum_{q\in\mathbb{Q}} p(q)\exp\left(-\tau\,\mathrm{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right) \tag{33}$$

$$Z(\tau) \coloneqq \sum_{q\in\mathbb{Q}} p(q)\det\exp\left(-\tau\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right). \tag{34}$$

The exponential of the trace is equal to the determinant of the exponential according to the relation $\det\exp\mathbf{A} \equiv \exp\mathrm{tr}\,\mathbf{A}$.

Finally,

$$\rho(q,\tau) = \frac{1}{Z(\tau)}p(q)\det\exp\left(-\tau\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right) \tag{35}$$

$$\cong p(q)|\exp -i\tau E(q)|^2. \tag{36}$$

With the equality $\tau = t/\hbar$ (analogous to $\beta = 1/(k_B T)$), the familiar form of

$$\rho(q,t) = \frac{1}{Z(t)}p(q)\Big|\exp\big(-itE(q)/\hbar\big)\Big|^2 \tag{37}$$

can be recovered, or, in general,

$$\rho(q,t) = \frac{1}{Z}\big|\psi(q,t)\big|^2,\ \text{where } \psi(q,t) = \exp\big(-itE(q)/\hbar\big)\psi(q), \tag{38}$$

where $|\psi(q)|^2 = p(q)$ is the initial preparation.

Here, the time t emerges as a Lagrange multiplier, which is the same manner in which T, the temperature, emerges in ordinary statistical mechanics. t may be qualified as a “thermal time” or as an “entropic flow.”

It can be shown that the Dirac–von Neumann axioms and the Born rule are satisfied.

To this end, the wavefunction is identified as a vector of a complex Hilbert space and the partition function as its inner product; this is expressed as

$$Z = \langle\psi|\psi\rangle\,. \tag{39}$$

As the solution is automatically normalized by the entropy-maximization procedure, the physical states are associated with the unit vectors and the probability of any particular state is expressed as

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger \psi(q,t). \tag{40}$$

As the solution is invariant under unitary transformations, it can be transformed out of its eigenbasis. Further, the energy $E(q)$ is generally represented by a Hamiltonian operator as follows:

$$\left|\psi(t)\right\rangle = \exp\!\left(-it\mathbf{H}/\hbar\right)\left|\psi(0)\right\rangle. \tag{41}$$

Any self-adjoint operator, defined as $\langle \mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$, corresponds to a real-valued statistical mechanics observable if measured in its eigenbasis, thereby completing the equivalence.

The dynamics are governed by the Schrödinger equation, obtained by taking the derivative with respect to the Lagrange multiplier:

$$\begin{aligned} \frac{\partial}{\partial t}\left|\psi(t)\right\rangle &= \frac{\partial}{\partial t}(\exp\!\left(-it\mathbf{H}/\hbar\right)\left|\psi(0)\right\rangle) && (42)\\ &= -i\mathbf{H}/\hbar \exp\!\left(-it\mathbf{H}/\hbar\right)\left|\psi(0)\right\rangle && (43)\\ &= -i\mathbf{H}/\hbar\left|\psi(t)\right\rangle && (44)\\ \implies \mathbf{H}\left|\psi(t)\right\rangle &= i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle, && (45) \end{aligned}$$

which is the Schrödinger equation.

Finally, the measurement postulate is imported as a direct consequence of $\rho(q,\tau)$ being a probability measure of statistical mechanics like any other; as it is parametrized over $\mathbb{Q}$, it describes the probability of finding the state at parameter q upon measurement (in the continuum case, this is a Dirac delta that associates with the state of the wavefunction immediately after measurement). The Shannon entropy quantities the information associated to such a measurement event.

Consequently, all axioms of non-relativistic quantum mechanics (including the Born rule and measurement postulate) have been reduced to a specific solution to the optimization problem, which only depends on a single axiom regarding the measurement pattern of nature. This demonstrates, so far in the case of non-relativistic quantum mechanics, that the axioms pertaining to the laws of physics (but not those relating to the measurement pattern) are redundant.

3.2 General case

As stated in Theorem 1, the Lagrange equation that defines the optimization problem is

$$
\mathcal{L}(\rho,\lambda,\tau) = -\sum_{q\in\mathbb{Q}} \rho(q) \ln\frac{\rho(q)}{p(q)} + \lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\frac{1}{d}\operatorname{tr}\overline{\mathbf{u}} - \sum_{q\in\mathbb{Q}}\rho(q)\frac{1}{d}\operatorname{tr}\mathbf{u}(q)\right), \tag{46}
$$

where λ and τ are the Lagrange multipliers and $\mathbf{u}(q)$ is an arbitrary multivector of $d = m + n$ dimensions.

To maximize this equation for ρ, the criterion $\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} = 0$ is used as follows:

$$
\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} = -\ln\frac{\rho(q)}{p(q)} - 1 - \lambda - \tau\frac{1}{d}\operatorname{tr}\mathbf{u}(q) \tag{47}
$$

$$
0 = \ln\frac{\rho(q)}{p(q)} + 1 + \lambda + \tau\frac{1}{d}\operatorname{tr}\mathbf{u}(q) \tag{48}
$$

$$
\implies \ln\frac{\rho(q)}{p(q)} = -1 - \lambda - \tau\frac{1}{d}\operatorname{tr}\mathbf{u}(q) \tag{49}
$$

$$
\implies \rho(q) = p(q)\exp(-1-\lambda)\exp\left(-\tau\frac{1}{d}\operatorname{tr}\mathbf{u}(q)\right) \tag{50}
$$

$$
= \frac{1}{Z(\boldsymbol{\tau})}p(q)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right), \tag{51}
$$

where $Z(\tau)$ is obtained as

$$
1 = \sum_{q\in\mathbb{Q}} p(q)\exp(-1-\lambda)\exp\left(-\tau\frac{1}{d}\operatorname{tr}\mathbf{u}(q)\right) \tag{52}
$$

$$
\implies \big(\exp(-1-\lambda)\big)^{-1} = \sum_{q\in\mathbb{Q}} p(q)\exp\left(-\tau\frac{1}{d}\operatorname{tr}\mathbf{u}(q)\right) \tag{53}
$$

$$
Z(\boldsymbol{\tau}) \coloneqq \sum_{q\in\mathbb{Q}} p(q)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right). \tag{54}
$$

The resulting probability measure is

$$
\rho(q,\tau) = \frac{1}{Z(\tau)}p(q)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right), \tag{55}
$$

where

$$
Z(\tau) = \sum_{q\in\mathbb{Q}} p(q)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right). \tag{56}
$$

Finally, it can be rewritten as:

$$\rho(q,\tau) = \frac{1}{Z(\tau)} \det\psi(q,\tau), \text{ where } \psi(q,\tau) = \exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right)\psi(q), \tag{57}$$

where $p(q) = \det\psi(q)$.

3.3 Continuum case

In his original paper, Shannon did not derive differential entropy as a theorem; instead, he posited that discrete entropy should be extended by replacing the sum with the integral:

$$-\sum_{q\in\mathbb{Q}} \rho(q)\ln\rho(q) \to -\int_{\mathbb{R}} \rho(x)\ln\rho(x)\,\mathrm{d}x \tag{58}$$

However, it was later discovered that differential entropy is not always positive, and neither is it invariant under a change of parameters. Specifically, it transforms as follows:

$$-\int_{\mathbb{R}} \rho(x)\ln\rho(x)\,\mathrm{d}x \to -\int_{\mathbb{R}} \tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\ln\left(\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x \tag{59}$$

$$= -\int_{\mathbb{R}} \tilde{\rho}(y)\ln\left(\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}y. \tag{60}$$

Furthermore, owing to an argument by Jaynes[7, 8], this is known to not be the correct limiting case of the Shannon entropy. Rather, the limiting case is relative entropy:

$$S = -\int_{\mathbb{R}} \rho(x)\ln\frac{\rho(x)}{p(x)}\,\mathrm{d}x, \tag{61}$$

where $p(x)$ is the initial preparation.

Relative entropy, in contrast to differential entropy, is invariant with respect to a change of parameter:

$$-\int_{\mathbb{R}} \rho(x)\ln\frac{\rho(x)}{p(x)}\,\mathrm{d}x \to -\int_{\mathbb{R}} \tilde{\rho}(y(x))\frac{dy}{dx}\ln\frac{\tilde{\rho}(y(x))\frac{dy}{dx}}{\tilde{p}(y(x))\frac{dy}{dx}}\,\mathrm{d}x \tag{62}$$

$$= -\int_{\mathbb{R}} \tilde{\rho}(y)\ln\frac{\tilde{\rho}(y)}{\tilde{p}(y)}\,\mathrm{d}y. \tag{63}$$

Let us also show that the normalization constraint is invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \tilde{\rho}(y(x))\frac{dy}{dx}\,\mathrm{d}x \tag{64}$$

$$= \int_{\mathbb{R}} \tilde{\rho}(y)\,\mathrm{d}y. \tag{65}$$

Let us now investigate the differential observable. A differential observable is typically formulated as

$$\overline{O} = \int_{\mathbb{R}} O(x)\rho(x)\,\mathrm{d}x. \tag{66}$$

However, this expression is not invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} O(x)\rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \tilde{O}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \tag{67}$$

$$= \int_{\mathbb{R}} \tilde{O}(y)\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}y. \tag{68}$$

To correct this, the relative (with respect to a reference) observable is introduced. For instance, if space is stretched by a factor of 2 ($x \to 2x$), the reference must also be stretched by the same amount for the observable to remain invariant. The consequence is that the following ratio is observed:

$$\overline{M/R} = \int_{\mathbb{R}} \frac{M(x)}{R(x)}\rho(x)\,\mathrm{d}x, \tag{69}$$

where R is the reference and the ratio $\overline{O} = \overline{U/R}$ is observable.

It is now shown to be invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} \frac{M(x)}{R(x)}\rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \frac{\tilde{M}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{R}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\rho(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \tag{70}$$

$$= \int_{\mathbb{R}} \frac{\tilde{M}(y)}{\tilde{R}(y)}\rho(y)\,\mathrm{d}y. \tag{71}$$

With these definitions, the Lagrange equation becomes:

$$\mathcal{L}(\rho,\lambda,\tau) = -\int_{\mathbb{R}} \rho(x)\ln\frac{\rho(x)}{p(x)}\,\mathrm{d}x + \lambda\left(1 - \int_{\mathbb{R}} \rho(x)\,\mathrm{d}x\right) + \tau\left(\frac{1}{d}\operatorname{tr}\frac{\overline{\mathbf{m}}}{\overline{\mathbf{r}}} - \int_{\mathbb{R}} \frac{1}{d}\operatorname{tr}\frac{\mathbf{m}(x)}{\mathbf{r}(x)}\rho(x)\,\mathrm{d}x\right). \tag{72}$$

Maximizing this equation with respect to ρ yields

$$\rho(x,\tau)\mid_a^b = \frac{1}{Z(\tau)}\int_a^b p(x)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(x)\right)\mathrm{d}x, \tag{73}$$

where

$$Z(\tau) = \int_{\mathbb{R}} p(q)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(x)\right)\mathrm{d}x, \tag{74}$$

where $\mathbf{u}(x) = \frac{\mathbf{m}(x)}{\mathbf{r}(x)}$.

The probability measure is now invariant with respect to a change of parameter:

$$\frac{\int_a^b p(x)\det\exp\left(-\tau\frac{1}{d}\frac{\mathbf{m}(x)}{\mathbf{r}(x)}\right)\mathrm{d}x}{\int_{\mathbb{R}} p(x)\det\exp\left(-\tau\frac{1}{d}\frac{\mathbf{m}(x)}{\mathbf{r}(x)}\right)\mathrm{d}x} \to \frac{\int_a^b \tilde{p}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\det\exp\left(-\tau\frac{1}{d}\frac{\tilde{\mathbf{m}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{\mathbf{r}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\right)\mathrm{d}x}{\int_{\mathbb{R}} \tilde{p}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\det\exp\left(-\tau\frac{1}{d}\frac{\tilde{\mathbf{m}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{\mathbf{r}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\right)\mathrm{d}x} \tag{75}$$

$$= \frac{\int_a^b \tilde{p}(y)\det\exp\left(-\tau\frac{1}{d}\frac{\tilde{\mathbf{m}}(y)}{\tilde{\mathbf{r}}(y)}\right)\mathrm{d}x}{\int_{\mathbb{R}} \tilde{p}(y)\det\exp\left(-\tau\frac{1}{d}\frac{\tilde{\mathbf{m}}(y)}{\tilde{\mathbf{r}}(y)}\right)\mathrm{d}y}. \tag{76}$$

3.4 The entropic flow of time

The role of τ is now elucidated in the 2D and 4D cases.

In 0+1D, τ associated with the time t. Further, the Schrödinger equation was recovered by taking the derivative of the wavefunction with respect to t:

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \mathbf{H}\left|\psi(t)\right\rangle. \tag{77}$$

In both 2D and 4D, a Schrödinger-like equation can also be recovered by deriving the wavefunction (with respect to τ).

First, the 2D case is considered.

A naive way to treat the dynamics would be to consider that τ constitutes a third dimension (2+1D). In this case, the 2D Schrödinger equation is

$$\frac{\partial}{\partial\tau}\psi(x,y,\tau) = -\frac{1}{2}\mathbf{u}(x,y)\psi(x,y,\tau). \tag{78}$$

The question now is how the dynamics can be understood.

Consider that in 0+1D, the non-relativistic Schrödinger equation generates in-time rotations in the complex plane (i.e., $\exp it$ generates the U(1) group with $\exp E(q)$ as the magnitude) for the probability amplitude. Similarly, in 2D, τ generates a one-parameter group that causes the probability amplitude to cycle over the possible geometric configuration of the system in a manner that preserves the probabilities. The U(1) group is replaced with a one-parameter realization of the $\mathrm{GL}^+(2,\mathbb{R})$ group. This is completely analogous to how time cycles the probability amplitude over the U(1) group in non-relativistic quantum mechanics, except that the geometry the system cycles over is richer than U(1).

The only problem with this naive method is that one has to introduce a third dimension to what should be 2D only. Although we may come to accept this for the 2D case (if we consider it to be embedded in a larger 2+1D spacetime), in 4D this is unacceptable.

Before considering the 4D case, it may be recalled that the Hamiltonian in the non-relativistic Schrödinger (0+1D) equation can be rendered as time dependent. In this case, the equation is

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \mathbf{H}(t)\left|\psi(t)\right\rangle. \tag{79}$$

Further, its solution[9] is

$$\left|\psi(t)\right\rangle = \sum_n c_n \exp\left(-\frac{i}{\hbar}\int_0^t E_n(t')\,\mathrm{d}t'\right)|n,t\rangle. \tag{80}$$

Let us now consider the 4D case. The corresponding Schrödinger equation would be

$$\frac{\partial}{\partial\tau}\psi(x,y,z,t,\tau) = -\frac{1}{4}\mathbf{u}(x,y,z,t)\psi(x,y,z,t,\tau). \tag{81}$$

However, this would add an extra fifth dimension to spacetime, which is unwanted. To resolve this, the coordinates are first changed from $\psi(x,y,z,t)$ to $\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau)$, where τ is the proper time experienced by the observer. The equation would then be

$$\frac{\partial}{\partial\tau}\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau) = -\frac{1}{4}\mathbf{u}(\tilde{x},\tilde{y},\tilde{z},\tau)\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau). \tag{82}$$

This form is very similar to the non-relativistic Schrödinger equation with a time-dependent Hamiltonian, as shown above. Now, the solution involves an integral over τ'.

$$\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau) = \sum_n c_n \exp\left(-\frac{1}{4}\int_0^\tau \mathbf{u}(\tilde{x},\tilde{y},\tilde{z},\tau')\,\mathrm{d}\tau'\right)\tilde{\psi}_n(\tilde{x},\tilde{y},\tilde{z},\tau) \tag{83}$$

This expression admits an arbitrary general linear geometry at every event of spacetime via $\mathbf{u}(\tilde{x}, \tilde{y}, \tilde{z}, \tau)$. The time evolution causes the probability amplitude of the wavefunction to cycle over the possible geometric configurations of the system provided they preserve the probabilities. As shown in the analysis, this construction also allows for the definition of a time evolution, defined from the perspective of the observer (proper time) and valid for both general relativity and quantum mechanics[10].

4 Analysis

A *general linear* Hilbert space in 2D and a *double-copy* general linear Hilbert space in 4D are produced. It is further shown that the last two structures include gravity, while the last one additionally includes the standard model.

The time-independent geometry of the wavefunction will now be analysed. The dynamical case, having been discussed earlier, will not be necessary here. As such, a time-independent formulation of $\rho(q)$ will be considered:

$$\rho(q) = \frac{1}{Z}\det\psi(q), \text{ where } \psi(q) = \exp\left(-\frac{1}{d}\mathbf{u}(q)\right)\psi_0(q), \tag{84}$$

where $p_0(q) = \det\psi_0(q)$.

4.1 General linear Hilbert space in 2D

The complex Hilbert space is insufficient to support all possible geometric measurements in nature. The general arena for physics is discovered to be the *general linear* Hilbert space, a generalization of the complex Hilbert space that can support all such measurements. This space allows the quantum theoretical support of arbitrary geometry, including pseudo-Riemannian geometry.

Let us observe this in detail.

It may be recalled that the general time-independent solution to the optimization problem is

$$\rho(q) = \frac{1}{Z(\tau)}\det\psi(q,\tau), \text{ where } \psi(q,\tau) = \exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right)\psi(q), \tag{85}$$

where $p(q) = \det\psi(q)$.

In 2D, the multivector $\mathbf{u}$ is in $\mathcal{G}(\mathbb{R}^2)$. It contains a scalar a, vector $\mathbf{x}$, and pseudoscalar $\mathbf{b}$, and can be written as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$. Further, the determinant in 2D can be expressed as $\det\mathbf{u} = \mathbf{u}^\ddagger\mathbf{u}$, where $\mathbf{u}^\ddagger$ is the Clifford conjugate of $\mathbf{u}$.

Consequently, the solution can be written as

$$\rho(q) = \frac{1}{Z}\psi(q)^\ddagger\psi(q), \text{ where } \psi(q) = \exp\left(-\frac{1}{2}\mathbf{u}(q)\right)\psi_0(q), \tag{86}$$

where $p_0(q) = \psi_0(q)^\ddagger \psi_0(q)$.

Rewriting the determinant as the 2D multivector norm allows us to use a notation similar to the bra-ket notation used in physics. It also allows us to represent an inner product over the general linear group, analogous to how the complex norm corresponds to the inner product of the complex Hilbert space.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^2)$. A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

A) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the sesquilinear map

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \quad & \mathbb{V} \times \mathbb{V} \longrightarrow \mathcal{G}(\mathbb{R}^2) \\ & \langle \mathbf{u}, \mathbf{v} \rangle \longmapsto \mathbf{u}^\ddagger \mathbf{v} \end{aligned} \tag{87}$$

is positive-definite such that for $\boldsymbol{\psi} \neq 0$, $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

B) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$. Then, for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q)) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \psi(q)^\ddagger \psi(q) \tag{88}$$

is either positive or equal to zero.

The following comments and definitions may be noted:

- From A) and B), it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum to unity:

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q)) = 1. \tag{89}$$

- $\boldsymbol{\psi}$ is referred to as a *physical state*.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is referred to as the *partition function* of $\boldsymbol{\psi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is referred to as a unit vector.
- $\rho(q)$ is referred to as the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$, such that the sum of probabilities remains normalized.

$$\langle \mathbf{T}\boldsymbol{\psi}, \mathbf{T}\boldsymbol{\psi} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{90}$$

are the *physical transformations* of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \in \mathbb{V}$ and $\forall \mathbf{v} \in \mathbb{V}$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{v} \rangle \tag{91}$$

  is referred to as an observable.

- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle. \tag{92}$$

4.2 General linear self-adjoint operator in 2D

The general case of an observable in 2D is shown in this section. A matrix $\mathbf{O}$ is observable if it is a self-adjoint operator defined as

$$\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle, \tag{93}$$

$\forall \boldsymbol{\phi} \in \mathbb{V}$ and $\forall \boldsymbol{\psi} \in \mathbb{V}$.

Setup: Let $\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix}$ be an observable.

Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be two two-state multivectors $\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix}$ and $\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}$. Here, the components $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, and $\mathbf{o}_{11}$ are multivectors of $\mathcal{G}(\mathbb{R}^2)$.

Derivation: 1. Calculate $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle &= (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2) \\ &\quad + (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2) \qquad (94) \\ &= \boldsymbol{\phi_1}^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\phi}_2 \\ &\quad + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\phi}_2. \qquad (95) \end{aligned}$$

2. Next, calculate $\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle &= \boldsymbol{\phi}_1^\ddagger (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2) + (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2) + (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 \qquad (96) \\ &= \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\phi}_1. \qquad (97) \end{aligned}$$

To realize $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi}\rangle$, the following relations must hold:

$$\mathbf{o}_{00}^{\ddagger} = \mathbf{o}_{00}, \tag{98}$$
$$\mathbf{o}_{01}^{\ddagger} = \mathbf{o}_{10}, \tag{99}$$
$$\mathbf{o}_{10}^{\ddagger} = \mathbf{o}_{01}, \tag{100}$$
$$\mathbf{o}_{11}^{\ddagger} = \mathbf{o}_{11}. \tag{101}$$

Therefore, $\mathbf{O}$ must be equal to its own Clifford transpose, indicating that $\mathbf{O}$ is observable if

$$\mathbf{O}^{\ddagger} = \mathbf{O}, \tag{102}$$

which is the geometric generalization of the self-adjoint operator $\mathbf{O}^{\dagger} = \mathbf{O}$ of complex Hilbert spaces.

4.3 General linear spectral theorem in 2D

The application of the spectral theorem to $\mathbf{O}^{\ddagger} = \mathbf{O}$ such that its eigenvalues are real is shown below:

Consider

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}. \tag{103}$$

Then, $\mathbf{O}^{\ddagger}$ is expressed as

$$\mathbf{O}^{\ddagger} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}. \tag{104}$$

It follows that $\mathbf{O}^{\ddagger} = \mathbf{O}$. This example is the most general 2×2 matrix $\mathbf{O}$ such that $\mathbf{O}^{\ddagger} = \mathbf{O}$.

The eigenvalues are obtained as

$$0 = \det(\mathbf{O} - \lambda \mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} - \lambda \end{bmatrix}, \tag{105}$$

which implies that

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12})(a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} + a_{11}) \tag{106}$$
$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2). \tag{107}$$

Finally,

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{108}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\}. \tag{109}$$

The roots are complex if $a^2 - x^2 - y^2 + b^2 < 0$. As $a^2 - x^2 - y^2 + b^2$ is the determinant of the multivector, the complex case is ruled out for orientation-preserving multivectors. Consequently, it follows that $\mathbf{O}^\ddagger = \mathbf{O}$ constitutes an observable with real-valued eigenvalues for orientation-preserving multivectors.

4.4 Invariant transformations in 2D

A left action on the wavefunction $\mathbf{T}\left|\psi\right\rangle$ connects to the bilinear form as $\left\langle\psi\right|\mathbf{T}^\ddagger\mathbf{T}\left|\psi\right\rangle$.

The invariance requirement on $\mathbf{T}$ is

$$\left\langle\psi\right|\mathbf{T}^\ddagger\mathbf{T}\left|\psi\right\rangle = \left\langle\psi|\psi\right\rangle. \tag{110}$$

Therefore, the group of matrices obeying

$$\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I} \tag{111}$$

are of interest.

Let a two-state system be considered, with a general transformation represented by

$$\mathbf{T} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}, \tag{112}$$

where u, v, w, x are the 2D multivectors.

The expression $\mathbf{T}^\ddagger\mathbf{T}$ represents

$$\mathbf{T}^\ddagger\mathbf{T} = \begin{bmatrix} v^\ddagger & u^\ddagger \\ w^\ddagger & x^\ddagger \end{bmatrix}\begin{bmatrix} v & w \\ u & x \end{bmatrix} = \begin{bmatrix} v^\ddagger v + u^\ddagger u & v^\ddagger w + u^\ddagger x \\ w^\ddagger v + x^\ddagger u & w^\ddagger w + x^\ddagger x \end{bmatrix}. \tag{113}$$

For $\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}$, the following relations must hold:

$$v^\ddagger v + u^\ddagger u = 1, \tag{114}$$

$$v^\ddagger w + u^\ddagger x = 0, \tag{115}$$

$$w^\ddagger v + x^\ddagger u = 0, \tag{116}$$

$$w^\ddagger w + x^\ddagger x = 1. \tag{117}$$

This is the case if

$$\mathbf{T} = \frac{1}{\sqrt{v^\ddagger v + u^\ddagger u}} \begin{bmatrix} v & u \\ -e^{\varphi} u^\ddagger & e^{\varphi} v^\ddagger \end{bmatrix}, \tag{118}$$

where u, v are the 2D multivectors and e^{φ} is a unit multivector.

Here, $\mathbf{T}$ is the geometric generalization (in 2D) of unitary transformations.

Comparatively, the unitary case is obtained when the vector part of the multivector vanishes, that is, $\mathbf{x} \to 0$. The following results:

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta} b^\dagger & e^{i\theta} a^\dagger \end{bmatrix}. \tag{119}$$

4.5 Gravity in FX/SO(2)

The quotient bundle associated with the structure reduction from $\mathrm{GL}^+(2,\mathbb{R})$ to SO(2) is now investigated.

Let X^2 be a smooth orientable real-valued manifold in 2D, and let TX be its tangent bundle and FX be its associated frame bundle. As X^2 is orientable, its structure group is $\mathrm{GL}^+(2,\mathbb{R})$. The action by the proposed wavefunction, valued in $\exp \mathcal{G}(\mathbb{R}^2) \cong \exp \mathbb{M}(2,\mathbb{R})$, generates $\mathrm{GL}^+(2,\mathbb{R})$, and thus acts on FX. A reduction of the structure group of FX to SO(2) can now be considered.

Let us begin by investigating the cosets of SO(2) in $\mathrm{GL}^+(2,\mathbb{R})$. Let $g_1 \in \mathrm{GL}^+(2,\mathbb{R})$, $g_2 \in \mathrm{GL}^+(2,\mathbb{R})$, and $s \in \mathrm{SO}(2)$. The relation $g_2 = g_1 s$ is now identified. Further, $g_2^T = s^T g_1^T$. Finally, the product $g_2 g_2^T = g_1 s s^T g_1^T \implies g_2 g_2^T = g_1 g_1^T$. As $g_1 g_1^T$ and $g_2 g_2^T$ are symmetric positive-definite 2×2 matrices, a diffeomorphism between $\mathrm{GL}^+(2,\mathbb{R})/\mathrm{SO}(2)$ and the inner products is verified.

The global section of the quotient bundle FX/SO(2) is a tetrad field $h^a_\mu(x)$ and is associated to a Riemannian metric on X^2 via the identity $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$. The connections that preserve the structure SO(2) across the manifold are the metric connections[11], and with the additional requirement of no torsion, the connections reduce to the Levi-Civita connection. It has been shown recently[12] that the Goldstone fields associated with the quotient bundle have sufficient degrees of freedom to create a metric and a covariant derivative. Finally, the frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^2[13]. This is the geometric setting for gravity.

Without introducing any other assumptions, the optimization problem affords a general linear quantum theory that holds in the $\mathrm{GL}^+(2,\mathbb{R})$ group, whose symmetry breaks into a theory of gravity (FX/SO(2)) and a quantum theory of the special orthogonal group (valued in SO(2)).

4.6 Wavefunction in SO(2)

With its structure reduced to SO(2), a quantum theory of the special orthogonal group is obtained, wherein the wavefunction defines the action on a vector of the tangent space of the manifold as follows:

$$\psi(x,y)^\ddagger \hat{\mathbf{x}}_0 \psi(x,y) = \exp\left(\frac{1}{2}\mathbf{i}B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\mathbf{i}B(x,y)\right) \tag{120}$$

$$= \exp\left(\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right). \tag{121}$$

The expression $\exp\left(\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)$ maps $\hat{\mathbf{x}}_0$ to a curvilinear basis $\mathbf{e}_0$ via the application of the rotor and its reverse:

$$\exp\left(\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right) = \mathbf{e}_0. \tag{122}$$

Consequently, a 2D relativistic wavefunction (with a Euclidean signature in this case) is obtained. This is the 2D version of the geometric algebra formulation of the relativistic wavefunction by Hestenes[14].

4.7 Metric interference in 2D

A transformation $\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}$ and wavefunction $|\psi\rangle = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$ are now considered, such that a multivector $\mathbf{u}$ is mapped to a linear combination of two multivectors. The following transformation may be considered:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{u}+\mathbf{v}\\ \mathbf{u}-\mathbf{v}\end{bmatrix}. \tag{123}$$

The following probability can be investigated:

$$\rho(\mathbf{u}+\mathbf{v}) = \frac{1}{Z}\det(\mathbf{u}+\mathbf{v}),\ \text{where } Z = \det(\mathbf{u}+\mathbf{v}) + \det(\mathbf{u}-\mathbf{v}). \tag{124}$$

The investigation proceeds as follows:

$$\det(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) \tag{125}$$

$$= (\mathbf{u}^\ddagger + \mathbf{v}^\ddagger)(\mathbf{u}+\mathbf{v}) \tag{126}$$

$$= (\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}) \tag{127}$$

$$= \det\mathbf{u} + \det\mathbf{v} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} \tag{128}$$

$$= \det\mathbf{u} + \det\mathbf{v} + \mathbf{u}\cdot\mathbf{v}, \tag{129}$$

where the dot product between multivectors is defined as follows:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\ddagger \mathbf{v} + \mathbf{v}^\ddagger \mathbf{u}. \tag{130}$$

As $\det \mathbf{u} > 0$ and $\det \mathbf{v} > 0$, $\mathbf{u} \cdot \mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det \mathbf{u}$ or $\det \mathbf{v}$ (whichever is greater). Therefore, it also satisfies the conditions of an interference term capable of destructive and constructive interference.

In the case $\mathbf{x} \to 0$, the interference pattern reduces to a form identical to the unitary case:

$$\det\Big(\psi_1 e^{-\frac{1}{2}\mathbf{b}_1} + \psi_2 e^{-\frac{1}{2}\mathbf{b}_2}\Big) = \det \psi_1 + \det \psi_2 + 2\psi_1\psi_2 e^{-\frac{1}{2}\mathbf{b}_1 - \frac{1}{2}\mathbf{b}_2} \tag{131}$$

$$= |\psi_1|^2 + |\psi_2|^2 + 2\psi_1\psi_2 e^{-\frac{1}{2}\mathbf{b}_1 - \frac{1}{2}\mathbf{b}_2}, \tag{132}$$

whereas in the general linear case,

$$\det\left(\psi_1 e^{-\frac{1}{2}(a_1+\mathbf{x}_1+\mathbf{b}_1)} + \psi_2 e^{-\frac{1}{2}(a_2+\mathbf{x}_2+\mathbf{b}_2)}\right) \tag{133}$$

$$= \det \psi_1 + \det \psi_2 + 2\psi_1\psi_2 \left(e^{-\frac{1}{2}(a_1+\mathbf{x}_1+\mathbf{b}_1)} + e^{-\frac{1}{2}(a_2+\mathbf{x}_2+\mathbf{b}_2)}\right), \tag{134}$$

which includes non-commutative effects in the interference pattern.

4.8 A *double-copy* general linear Hilbert space in 4D

In 2D, the determinant can be expressed using only the product $\psi^\ddagger\psi$, which can be interpreted as the inner product of two multivectors. This form allowed us to extend the complex Hilbert space to a *general linear* Hilbert space. It was then found that the familiar properties of the complex Hilbert spaces were transferable to the general linear Hilbert space, eventually yielding a gravitized quantum theory.

Although a similar correspondence exists in 4D, it is less recognizable because a *double-copy* inner product (i.e., $\rho = \lfloor \phi^\ddagger \phi \rfloor_{3,4} \phi^\ddagger \phi$) is needed to produce a real-valued probability in 4D. Thus, in 4D, an inner product cannot be produced as in the 2D case. The absence of a satisfactory inner product indicates no Hilbert space in the usual sense of a complete *inner product* vector space. This section aims to find a construction that supports the general linear wavefunction in 4D.

To build the right construction, a double-copy inner product of four terms is devised, superseding the inner product in the Hilbert space, mapping any four vectors to an element of $\mathcal{G}(\mathbb{R}^{3,1})$, and yielding a complete *double-copy* inner product vector space or, simply, a *double-copy* Hilbert space.

The construction is more familiar than it may first appear. The familiar quantum mechanical features (e.g., linear transformations, unit vectors, and linear superposition in the probability measure) are supported in the construction. Similarly to how it breaks in 2D, the construction would also break into a familiar inner-product Hilbert space whose Dirac current is invariant for SU(3)×SU(2)×U(1) and a theory of gravity and electromagnetism for charged fermions and bosons in the quotient FX/Spin$^c(3,1)$.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$. A subset of vectors in $\mathbb{V}$ forms a double-copy algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, the double-copy inner product form

$$\begin{aligned} \langle \cdot,\cdot,\cdot,\cdot\rangle \quad : \qquad & \mathbb{V}\times\mathbb{V}\times\mathbb{V}\times\mathbb{V} \longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ & \langle \mathbf{u},\mathbf{w},\mathbf{y},\mathbf{z}\rangle \longmapsto \sum_{i=1}^{m} \lfloor u_i^\ddagger w_i \rfloor_{3,4} y_i^\ddagger z_i \end{aligned} \tag{135}$$

   is positive-definite when $\boldsymbol{\phi} \neq 0$; that is $\langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle > 0$.

2. $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, then for each element $\phi(q) \in \boldsymbol{\phi}$, the function

$$\rho(\phi(q)) = \frac{1}{\langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle} \det \phi(q) \tag{136}$$

   is either positive or equal to zero.

The following properties, features, and comments may be noted:

- From A) and B), it follows that $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, and the probabilities sum to unity.

$$\sum_{\phi(q)\in\boldsymbol{\phi}} \rho(\phi(q)) = 1 \tag{137}$$

- $\boldsymbol{\phi}$ is referred to as a *physical state*.
- $\langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle$ is referred to as the *partition function* of $\boldsymbol{\phi}$.
- If $\langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle = 1$, then $\boldsymbol{\phi}$ is referred to as a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\phi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\phi}$ such as $\mathbf{T}\boldsymbol{\phi} \to \boldsymbol{\phi}'$ renders the sum of probabilities normalized (invariant):

$$\langle \mathbf{T}\boldsymbol{\phi},\mathbf{T}\boldsymbol{\phi},\mathbf{T}\boldsymbol{\phi},\mathbf{T}\boldsymbol{\phi}\rangle = \langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle \tag{138}$$

  are the *physical transformations* of $\boldsymbol{\phi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{y} \forall \mathbf{z} \in \mathbb{V}$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{w}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w}, \mathbf{O}\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{O}\mathbf{z} \rangle \tag{139}$$

  is referred to as an observable.

- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle}{\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle}. \tag{140}$$

4.9 Wavefunction in 3+1D

In the notation by Hestenes[14], the 3+1D wavefunction is expressed as

$$\psi = \sqrt{\rho e^{-ib}} R, \tag{141}$$

where ρ represents a scalar probability density, e^{ib} is a complex phase, and R is a rotor.

Comparatively, the proposed wavefunction in $\mathcal{G}(\mathbb{R}^{3,1})$ is

$$\phi = e^{-\frac{1}{4}(a+\mathbf{x}+\mathbf{f}+\mathbf{v}+\mathbf{b})} \phi_0. \tag{142}$$

To recover the formulation of the wavefunction by Hestenes, it is sufficient to eliminate the terms $a \to 0$, $\mathbf{x} \to 0$, and $\mathbf{v} \to 0$, and to perform a substitution of the entries of the double-copy inner product (Equation 150), as follows:

$$\mathbf{w} \to \mathbf{u}^{\ddagger}, \tag{143}$$

$$\mathbf{y} \to \mathbf{z}^{\ddagger}. \tag{144}$$

As one of the copies is destroyed by the substitution, the double-copy inner product reduces to an inner product. Furthermore, with the elimination, the blade-3,4 conjugate is also reduced to the blade-4 conjugate, yielding

$$\langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle \to \langle \mathbf{u}, \mathbf{u}^{\ddagger}, \mathbf{z}^{\ddagger}, \mathbf{z} \rangle \cong \langle \mathbf{u}, \mathbf{z} \rangle = \sum_{i=1}^{m} \lfloor u_i^2 \rfloor_{2,4} (z_i^2). \tag{145}$$

Consequently, the proposed wavefunction ϕ reduces to

$$\phi^2 = e^{-\frac{1}{2}(\mathbf{f}+\mathbf{b})} \phi_0^2. \tag{146}$$

This shows that the 3+1D wavefunction (comprising a rotor $R = e^{-\frac{1}{2}\mathbf{f}}$, a pseudo-scalar $e^{-\frac{1}{2}\mathbf{b}}$, and a prior probability $\phi_0^2 = \sqrt{\rho}$) is a sub-structure of the general $\mathcal{G}(\mathbb{R}^{3,1})$ wavefunction. The primary difference is that the present formulation exists in a grade 2-4 geometric Hilbert space.

In this sub-structure, the observables are satisfied when

$$\lfloor \mathbf{O} \rfloor_{2,4} = \mathbf{O}. \tag{147}$$

Let us now analyze the symmetry group of this wavefunction.

First, the term $\mathbf{b}$ commutes with $\mathbf{f}$. They can be factored out as

$$e^{-\frac{1}{2}(\mathbf{f}+\mathbf{b})}\phi_0^2 = e^{-\frac{1}{2}\mathbf{b}}e^{-\frac{1}{2}\mathbf{f}}\phi_0^2. \tag{148}$$

Second, the term $\exp \mathbf{f}$ can be understood as the exponential map from the bivectors to the $\mathrm{Spin}_+(3,1)$ group and the term $\exp \mathbf{b}$ to $\mathrm{U}(1)$.

Finally, as $\mathrm{Spin}_+(3,1) \cap \exp \mathbf{b} = \{\pm 1\}$, it must be removed from the group product[15].

Thus, the geometric components of the wavefunction correspond to the following group

$$\mathrm{U}(1) \times (\mathrm{Spin}_+(3,1)/\{\pm 1\}) \cong \mathrm{Spin}^c(3,1). \tag{149}$$

4.10 Geometric Hilbert space in 3+1D (broken symmetry)

The substitution given by Equation 145 yields the following algebra of geometric observables.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$. A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the inner product form

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \qquad & \mathbb{V} \times \mathbb{V} \longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ & \langle \mathbf{u}, \mathbf{w} \rangle \longmapsto \sum_{i=1}^{m} \lfloor u_i^2 \rfloor_{2,4} w_i^2 \end{aligned} \tag{150}$$

   is positive-definite when $\boldsymbol{\psi} \neq 0$; that is $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$.

2. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q)) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \det \psi(q) \tag{151}$$

   is either positive or equal to zero.

The following properties, features, and comments may be noted:

- From A) and B), it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum to unity.

$$\sum_{\psi(q)\in\boldsymbol{\psi}} \rho(\psi(q)) = 1 \tag{152}$$

- $\boldsymbol{\psi}$ is referred to as a *physical state*.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is referred to as the *partition function* of $\boldsymbol{\phi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is referred to as a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ such that $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$ renders the sum of probabilities normalized (invariant),

$$\langle \mathbf{T}\boldsymbol{\psi}, \mathbf{T}\boldsymbol{\psi} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{153}$$

  are the *physical transformations* of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{w} \in \mathbb{V}$,

$$\langle \mathbf{O}\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{w} \rangle \tag{154}$$

  is referred to as an observable.

- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle}. \tag{155}$$

4.11 Gravity and electromagnetism in 3+1D

In 2D, a coincidence of low dimensions, wherein the matrix representation of $\mathcal{G}(\mathbb{R}^2)$ is in $\mathbb{M}(2,\mathbb{R})$, was utilized. As such, the wavefunction generated is $\mathrm{GL}^+(2,\mathbb{R})$, which acts as the structure group of the frame bundle FX. Following a structure reduction from $\mathrm{GL}^+(2,\mathbb{R})$ to SO(2), a tetrad field was associated with the global section of the quotient bundle FX/SO(2).

In 4D, in contrast to 2D where SO(2) = Spin(2), the geometry of the wavefunction is not in SO but rather in Spin^c (because 4D also contains a pseudoscalar in addition to bivectors). Moreover, as Spin^c is not, in general, in GL^+, the same coincidence as in 2D does not occur.

Typically, to reach $\mathrm{Spin}(p,q)$ from the structure group $\mathrm{GL}(p+q)$, one would reduce $\mathrm{GL}(p+q)$ to $\mathrm{O}(p,q)$ and then lift it to $\mathrm{Spin}(p,q)$. Here, a different approach is used to obtain the spin connection.

Remarkably, 4D admits a coincidence that facilitates the embedding of the $\mathrm{Spin}^c(3,1)$ group into the $\mathrm{GL}^+(4,\mathbb{R})$ group. Then, the quotient $\mathrm{FX}/\mathrm{Spin}^c(3,1)$ can be reached without having to lift it to a larger geometric structure; the solution already contains the necessary aspects to take this quotient.

The coincidence originates from the standard classification of real Clifford algebra[16] and the fact that $\exp(\mathbf{f}+\mathbf{b}) \cong \mathrm{Spin}^c(3,1) \subset \exp\mathcal{G}(\mathbb{R}^{3,1})$. The following diagram commutes by group homomorphisms.

$$\begin{array}{ccc} \mathcal{G}(\mathbb{R}^{3,1}) & \xrightarrow{f} & \mathbb{M}(4,\mathbb{R}) \\ \big\downarrow{\scriptstyle exp} & & \big\downarrow{\scriptstyle exp} \\ \exp\mathcal{G}(\mathbb{R}^{3,1}) & \xrightarrow{f} & \mathrm{GL}^+(4,\mathbb{R}) \end{array} \tag{156}$$

As $\exp(\mathbf{f}+\mathbf{b}) \cong \mathrm{Spin}^c(3,1) \subset \exp\mathcal{G}(\mathbb{R}^{3,1})$, the map f embeds $\mathrm{Spin}^c(3,1)$ into $\mathrm{GL}^+(4,\mathbb{R})$. The inclusion of $\mathrm{Spin}^c(3,1)$ in $\exp\mathcal{G}(\mathbb{R}^{3,1})$ is required to break the symmetry into exactly a theory of gravity and of electromagnetism for charged fermions and into a $\mathrm{Spin}^c(3,1)$-valued quantum theory.

Let X^4 be a world manifold. The tangent bundle TX is first considered along with its associated frame bundle FX. The proposed wavefunction acts on the frame bundle using the exponential map of multivectors $\exp\mathcal{G}(\mathbb{R}^{3,1}) \cong \exp\mathbb{M}(4,\mathbb{R})$, which generates $\mathrm{GL}^+(4,\mathbb{R})$.

The desired reduction is from $\exp\mathcal{G}(\mathbb{R}^{3,1})$ to the $\mathrm{Spin}^c(3,1)$ group. With its symmetry reduced, the wavefunction assigns an element of $\mathrm{Spin}^c(3,1)$ to each event $x \in X^4$. The connection that preserves the structure is a $\mathrm{Spin}^c(3,1)$ preserving connection relating to a theory of gravity and electromagnetism for charged fermions. Notably, $\mathrm{SO}(3,1)\times\mathrm{U}(1)$ is a quotient of $\mathrm{Spin}^c(3,1)$. The cosets of $\mathrm{SO}(3,1)$ are further associable with the inner products. Thus, the global section of the quotient bundle $\mathrm{FX}/\mathrm{SO}(3,1)$ associates with a tetrad field that uniquely determines a pseudo-Riemannian metric. As for the U(1)-bundle, it is simply the geometric setting for electromagnetism. Finally, the frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^4

4.12 Dirac current

Hestenes[14] defined the Dirac current in the language of geometric algebra as

$$\mathbf{j} = \psi^{\ddagger}\gamma_0\psi = \rho R^{\ddagger}\gamma_0 R = \rho e_0 = \rho v, \tag{157}$$

where v is the proper velocity.

In the formulation herein, this relation also holds: the Dirac current represents the action of the wavefunction on the unit time-like vector in the tangent space on X^4. Specifically, the Dirac current is a statistically weighted Lorentz action on γ_0:

$$\mathbf{j} = \psi^\ddagger \gamma_0 \psi \tag{158}$$
$$= e^{-\frac{1}{2}\mathbf{f}+\frac{1}{2}\mathbf{b}}\phi_0\gamma_0 e^{\frac{1}{2}\mathbf{f}+\frac{1}{2}\mathbf{b}}\phi_0 \tag{159}$$
$$= \phi_0^2 e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} \tag{160}$$
$$= \rho e_0 \tag{161}$$
$$= \rho v. \tag{162}$$

4.13 SU(2) × U(1) group

The proposed wavefunction transforms as a group under multiplication. The following question can now be posed: what is the most general multivector $e^{\mathbf{u}}$ that renders the Dirac current invariant?

$$\psi^\ddagger (e^{\mathbf{u}})^\ddagger \gamma_0 e^{\mathbf{u}} \psi = \psi^\ddagger \gamma_0 \psi \iff (e^{\mathbf{u}})^\ddagger \gamma_0 e^{\mathbf{u}} = \gamma_0 \tag{163}$$

When is this satisfied?

The bases of the bivector part $\mathbf{f}$ of $\mathbf{u}$ are $\gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$. Among these, only $\gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$ commute with γ_0, and the rest anti-commute; therefore, the rest must be equated to zero. Finally, the base $\gamma_0\gamma_1\gamma_2\gamma_3$ anti-commutes with γ_0 and cancels out.

Consequently, the most general exponential multivector of the form $e^{\mathbf{u}}$, where $\mathbf{u} = \mathbf{f} + \mathbf{b}$, which preserves the Dirac current as

$$e^{\mathbf{u}} = \exp\left(\frac{1}{2}F_{12}\gamma_1\gamma_2 + \frac{1}{2}F_{13}\gamma_1\gamma_3 + \frac{1}{2}F_{23}\gamma_2\gamma_3 + \frac{1}{2}\mathbf{b}\right). \tag{164}$$

The bivector basis can be rewritten using the Pauli matrices

$$\gamma_2\gamma_3 = \mathbf{i}\sigma_x, \tag{165}$$
$$\gamma_1\gamma_3 = \mathbf{i}\sigma_y, \tag{166}$$
$$\gamma_1\gamma_2 = \mathbf{i}\sigma_z, \tag{167}$$
$$\mathbf{b} = \mathbf{i}b. \tag{168}$$

After replacement, the following is obtained:

$$e^{\mathbf{u}} = \exp\frac{1}{2}\mathbf{i}(F_{12}\sigma_z + F_{13}\sigma_y + F_{23}\sigma_x + b). \tag{169}$$

The terms $F_{23}\sigma_x + F_{13}\sigma_y + F_{12}\sigma_z$ and b are responsible for SU(2) and U(1) symmetries, respectively[17, 18].

4.14 SU(3) group

The invariance transformations identified by the 3+1D algebra of geometric observables (Equation 153) are $\mathbf{T}^\ddagger \mathbf{T} = \mathbf{I}$, $\mathbf{T}^\dagger \mathbf{T} = \mathbf{I}$, and $\lfloor \mathbf{T} \rfloor_{2,4} \mathbf{T} = \mathbf{I}$. In the first case, the identified evolution is bivectorial rather than unitary.

Similar to the SU(2) × U(1) case, the following question can be posed: in this case, what is the most general bivectorial evolution that renders the Dirac current invariant?

$$\mathbf{f}^\ddagger \gamma_0 \mathbf{f} = \gamma_0, \tag{170}$$

where $\mathbf{f}$ is a bivector:

$$\mathbf{f} = F_{01}\gamma_0\gamma_1 + F_{02}\gamma_0\gamma_2 + F_{03}\gamma_0\gamma_3 + F_{23}\gamma_2\gamma_3 + F_{13}\gamma_1\gamma_3 + F_{12}\gamma_1\gamma_2 \tag{171}$$

Explicitly, the expression $\mathbf{f}^\ddagger \gamma_0 \mathbf{f}$ is

$$\begin{aligned} \mathbf{f}^\ddagger \gamma_0 \mathbf{f} = -\mathbf{f}\gamma_0\mathbf{f} = (F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2)\gamma_0 & \quad (172) \\ + (-2F_{02}F_{12} + 2F_{03}F_{13})\gamma_1 & \quad (173) \\ + (-2F_{01}F_{12} + 2F_{03}F_{23})\gamma_2 & \quad (174) \\ + (-2F_{01}F_{13} + 2F_{02}F_{23})\gamma_3. & \quad (175) \end{aligned}$$

For the Dirac current to remain invariant, the cross-product must vanish:

$$\begin{aligned} -2F_{02}F_{12} + 2F_{03}F_{13} = 0, & \quad (176) \\ -2F_{01}F_{12} + 2F_{03}F_{23} = 0, & \quad (177) \\ -2F_{01}F_{13} + 2F_{02}F_{23} = 0, & \quad (178) \end{aligned}$$

leaving only

$$\mathbf{f}^\ddagger \gamma_0 \mathbf{f} = (F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2)\gamma_0. \tag{179}$$

Finally, $F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2$ must equal one.

Notably, $\mathbf{f}$ can be rewritten as a 3-vector with complex components:

$$\mathbf{f} = (F_{01} + \mathbf{i}F_{23})\gamma_0\gamma_1 + (F_{02} + \mathbf{i}F_{13})\gamma_0\gamma_2 + (F_{03} + \mathbf{i}F_{12})\gamma_0\gamma_3. \tag{180}$$

Consequently, when invariant for the Dirac current, the bivectorial evolution can be understood to be a realization of the SU(3) group[18].

4.15 Satisfiability of geometric observables in 4D

In 4D, an observable must satisfy Equation 139. Let us now verify that geometric observables are satisfiable in 4D. For simplicity, let us set m in Equation 150 as one. Then,

$$\lfloor(\mathbf{O}u)^\ddagger w\rfloor_{3,4} y^\ddagger z = \lfloor u^\ddagger \mathbf{O} w\rfloor_{3,4} y^\ddagger z = \lfloor u^\ddagger w\rfloor_{3,4}(\mathbf{O}y)^\ddagger z = \lfloor u^\ddagger w\rfloor_{3,4} y^\ddagger \mathbf{O} z, \quad (181)$$

where u_1, w_1, y_1 and z_1 are multivectors.

Let us investigate.

If $\mathbf{O}$ contained a vector, bivector, pseudo-vector, or pseudo-scalar, the equality would not be satisfied as these terms do not commune with the multivectors and cannot be factored out. The equality is satisfied if $\mathbf{O} \in \mathbb{R}$. Indeed, as a real value, $\mathbf{O}$ commutes with all multivectors, and hence can be factored out to satisfy the equality.

Thus, the observables are satisfied in the general 4D case. Furthermore, in 3+1D, the observable reduces to $\lfloor \mathbf{O} \rfloor_{2,4} = \mathbf{O}$, which is also satisfiable.

4.16 Unsatisfiability of geometric observables in 6D and above

At six dimensions or above, the corresponding observable relation cannot be satisfied. To explain this observation, the results by Acus et al. [19] regarding the 6D multivector norm are examined. They performed an exhaustive computer-assisted search for the geometric algebra expression for the determinant in 6D; as conjectured, they found no norm defined via self-products. The norm is a linear combination of self-products.

The system of linear equations is too long to list in its entirety; the author provides this mockup:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials}\rangle = 0, \quad (182)$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials}\rangle = 0, \quad (183)$$

$$\langle 74 \text{ monomials}\rangle = 0, \quad (184)$$

$$\langle 74 \text{ monomials}\rangle = 0. \quad (185)$$

The author then produces the special case of this norm that holds only for a 6D multivector comprising a scalar and a grade 4 element:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))). \quad (186)$$

Even in this simplified special case, formulating a linear relationship for observables is doomed to fail. Indeed, the real portion of the observable cannot be extracted from the equation. For any observable, the following equality is frustrated unless $\mathbf{O} = 1$:

$$b_1\mathbf{O}Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2Bg_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{187}$$
$$= b_1Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2\mathbf{O}Bg_5(g_4(B)g_3(g_2(B)g_1(B))). \tag{188}$$

Consequently, the relation for observables in 6D is unsatisfiable even by real numbers. Furthermore, as the norms involve more sophisticated systems of linear equations in higher dimensions, this result is conjectured to generalize to all dimensions above six.

4.17 Defective probability measure in 3D and 5D

The 3D and 5D cases (and possibly all odd-dimensional cases of higher dimensions) contain several irregularities that render them defective for use in this framework. Let us investigate.

In $\mathcal{G}(\mathbb{R}^3)$, the matrix representation of a multivector is as follows:

$$\mathbf{u} = a + x\sigma_x + y\sigma_y + z\sigma_z + q\sigma_y\sigma_z + v\sigma_x\sigma_z + w\sigma_x\sigma_y + b\sigma_x\sigma_y\sigma_z \tag{189}$$

is

$$\mathbf{u} \cong \begin{bmatrix} a+ib+iw+z & iq-v+x-iy \\ iq+v+x+iy & a+ib-iw-z \end{bmatrix}, \tag{190}$$

and the determinant is

$$\det\mathbf{u} = a^2 - b^2 + q^2 + v^2 + w^2 - x^2 - y^2 - z^2 + 2i(ab - qx + vy - wz). \tag{191}$$

The result is a complex-valued probability. Because a probability must be real-valued, the 3D case is defective in the present solution and cannot be used. In theory, it can be fixed by defining a complex norm to apply to the determinant:

$$\langle\mathbf{u},\mathbf{u}\rangle = (\det\mathbf{u})^\dagger \det\mathbf{u}. \tag{192}$$

However, defining such a norm would entail a double-copy inner product of four multivectors, but the space is only 3D, not 4D (so why four?). It would also break the relationship between trace and probability that justified its usage in statistical mechanics.

Consequently, this case is defective.

Instead of $\mathcal{G}(\mathbb{R}^3)$ multivectors, should 3×3 matrices be used in 3D? Alas no: 3×3 matrices do not admit a geometric algebra representation because they are not isomorphic with $\mathcal{G}(\mathbb{R}^3)$. $\mathcal{G}(\mathbb{R}^3)$ has eight parameters and 3×3 matrices have nine. 3×3 matrices are not representable geometrically in the same sense that 2×2 matrices are with $\mathcal{G}(\mathbb{R}^2)$.

In $\mathcal{G}(\mathbb{R}^{4,1})$, the algebra is isomorphic to complex 4×4 matrices. In this case, the determinant and probability would be complex-valued, rendering the case defective. Furthermore, 5×5 matrices have 25 parameters; however, $\mathcal{G}(\mathbb{R}^{4,1})$ multivectors have 32 parameters.

4.18 Dimensions that admit observable geometry

The solution is non-defective in the following dimensions:

- $\mathbb{R}$: This case corresponds to familiar statistical mechanics. The constraints are scalar $\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q)$, and the probability measure is the Gibbs measure $\rho(q) = \frac{1}{Z(\beta)} \exp\big(-\beta E(q)\big)$.

- $\mathrm{Im}(\mathbb{C}) \cong \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$: This case corresponds to familiar non-relativistic quantum mechanics.

However, neither of these cases contain geometry. The only cases that contain observable geometry are:

- $\mathcal{G}(\mathbb{R}^2)$: This case corresponds to the geometric quantum theory in 2D. Its $\mathrm{GL}^+(2)$ symmetry breaks into a theory of gravity FX/SO(2) and a quantum theory valued in SO(2). However, it is vacuous.

- $\mathcal{G}(\mathbb{R}^{3,1})$: Similar to the 2D case, this case corresponds to a geometric quantum theory. As such, its symmetry breaks into a theory of gravity and a relativistic wavefunction. However, in contrast to the 2D case, the wavefunction further admits an invariance with respect to the $\mathrm{SU}(2)\times\mathrm{U}(1)$ and SU(3) gauge groups.

In contrast, the solution is defective in the following dimensions:

- $\mathcal{G}(\mathbb{R}^3)$: In this case, the probability measure is complex-valued.

- $\mathcal{G}(\mathbb{R}^{4,1})$: In this case, the probability measure is complex-valued.

- 6D and above: For $\mathcal{G}(\mathbb{R}^n)$, where $n \geq 6$, no observables satisfy the corresponding observable equation, in general.

It may thus be concluded that the 3D and 5D cases fail to normalize and the 6D and above cases fail to satisfy observables. Consequently, in the general case of the solution, normalizable geometric observables cannot be satisfied beyond 4D. This suggests an intrinsic limit to the dimensionality of observable geometry and, by extension, to spacetime.

4.19 Metric interference in 3+1D

A geometric wavefunction would allow a larger class of interference patterns than complex interference. The geometric interference pattern includes the ways in which the geometry of a probability measure can interfere constructively or destructively and includes interference from rotations, phases, boosts, shears, spins, and dilations.

In the case of 4D *metric interference* (shown below), the interference pattern is associated with a superposition of elements of the group $\mathrm{Spin}^c(3,1)$, whose subgroup $\mathrm{SO}(3,1)$ is associated with a superposition of inner products in the quotient.

It is possible that an Aharonov–Bohm effect experiment on gravity[20] could detect special cases of the geometric phase and interference patterns identified in this section.

An interference pattern follows from a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the application of the determinant:

$$\det(\mathbf{u}+\mathbf{v}) = \det\mathbf{u} + \det\mathbf{v} + \mathbf{u}\cdot\mathbf{v}. \tag{193}$$

The determinants $\det\mathbf{u}$ and $\det\mathbf{v}$ are a sum of probabilities, whereas the dot product term $\mathbf{u}\cdot\mathbf{v}$ represents the interference term.

Following a transformation of a wavefunction $|\psi\rangle = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$ can be obtained, such that the multivectors are mapped to a linear combination of two multivectors:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{u}+\mathbf{v}\\ \mathbf{u}-\mathbf{v}\end{bmatrix}. \tag{194}$$

The dot product defines a bilinear form.

$$\begin{aligned} \cdot \quad : \qquad & \mathcal{G}(\mathbb{R}^{m,n}) \times \mathcal{G}(\mathbb{R}^{m,n}) \longrightarrow \mathbb{R} & (195)\\ & \mathbf{u}\cdot\mathbf{v} \longmapsto \frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) & (196)\end{aligned}$$

If $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$, then $\mathbf{u}\cdot\mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$ (whichever is greater). Therefore, it also satisfies the conditions of an interference term.

In 2D, the dot product takes the following form:

$$\begin{aligned} &\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) & (197)\\ &= \frac{1}{2}\left((\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v}\right) & (198)\\ &= \mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v} - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v} & (199)\\ &= \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u}. & (200)\end{aligned}$$

In 3+1D, takes has the following form:

$$\frac{1}{2}(\det(\mathbf{u}+\mathbf{v})-\det\mathbf{u}-\det\mathbf{v}) \tag{201}$$
$$=\frac{1}{2}\left(\lfloor(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})\rfloor_{3,4}(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})-\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}-\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\right) \tag{202}$$
$$=\frac{1}{2}\left(\lfloor\mathbf{u}^\ddagger\mathbf{u}+\mathbf{u}^\ddagger\mathbf{v}+\mathbf{v}^\ddagger\mathbf{u}+\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}(\mathbf{u}^\ddagger\mathbf{u}+\mathbf{u}^\ddagger\mathbf{v}+\mathbf{v}^\ddagger\mathbf{u}+\mathbf{v}^\ddagger\mathbf{v})-\dots\right) \tag{203}$$

$$\begin{aligned}=&\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}-\dots\end{aligned} \tag{204}$$

$$\begin{aligned}=&\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}.\end{aligned} \tag{205}$$

Simpler interference patterns are now considered.

Interference in 3+1D:

As seen previously, the substituted double-copy inner product reduces to an inner product (Equation 145). The interference pattern[21] is expressed as follows:

$$\begin{aligned}\det(\mathbf{u}+\mathbf{v})&=\lfloor\mathbf{u}+\mathbf{v}\rfloor_{2,4}(\mathbf{u}+\mathbf{v}) &(206)\\
&=\lfloor\mathbf{u}\rfloor_{2,4}(\mathbf{u}+\mathbf{v})+\lfloor\mathbf{v}\rfloor_{2,4}(\mathbf{u}+\mathbf{v}) &(207)\\
&=\lfloor\mathbf{u}\rfloor_{2,4}\mathbf{u}+\lfloor\mathbf{u}\rfloor_{2,4}\mathbf{v}+\lfloor\mathbf{v}\rfloor_{2,4}\mathbf{u}+\lfloor\mathbf{v}\rfloor_{2,4}\mathbf{v} &(208)\\
&=\det\mathbf{u}+\det\mathbf{v}+\lfloor\mathbf{u}\rfloor_{2,4}\mathbf{v}+\lfloor\mathbf{v}\rfloor_{2,4}\mathbf{u}. &(209)\end{aligned}$$

Now, replacing $\mathbf{u}=\rho_u e^{-\frac{1}{2}\mathbf{b}_u}e^{-\frac{1}{2}\mathbf{f}_u}$ and $\mathbf{v}=\rho_v e^{-\frac{1}{2}\mathbf{b}_v}e^{-\frac{1}{2}\mathbf{f}_v}$,

$$=|\rho_u|^2+|\rho_v|^2+\rho_u\rho_v\left(e^{\frac{1}{2}\mathbf{b}_u}e^{\frac{1}{2}\mathbf{f}_u}e^{-\frac{1}{2}\mathbf{b}_v}e^{-\frac{1}{2}\mathbf{f}_v}+e^{\frac{1}{2}\mathbf{b}_v}e^{\frac{1}{2}\mathbf{f}_v}e^{-\frac{1}{2}\mathbf{b}_u}e^{-\frac{1}{2}\mathbf{f}_u}\right). \tag{210}$$

Owing to the presence of $\mathbf{f}$ and $\mathbf{b}$, the geometric richness of the interference pattern exceeds that of the unitary case. The term $\mathbf{f}$ associates with a non-commutative interference effect in the interference pattern, which distinguishes it from (the entirely commutative) complex interference and could presumably be identified experimentally in a properly constructed interference experiment.

5 Conclusion

The information associated with measurement events is maximized under the geometric constraint. The solution supports a geometry richer than that previously supported in either statistical physics or quantum mechanics alone. Accommodating all possible general linear measurements entails a general linear wavefunction, for which the Born rule is extended to the determinant. This substantially extends the opportunity to capture all fundamental physics within a single framework. In this proposal, the measurements acquire a foundational role and the wavefunction is derived. It is assumed that an observer receives or produces a message (according to the theory of communication/Shannon entropy) of measurement events, and the probability measure, maximizing the information of this message, is the general linear wavefunction accompanied by the general linear Born rule. The states of this wavefunction exist in a general linear Hilbert space, which generalizes the complex Hilbert space to arbitrary geometry. The framework produces solutions for 2D and 4D, wherein the general observables are normalizable. The 2D case contains gravity but is otherwise vacuous. In the 4D case, a gravitized standard model results from the frame bundle FX of a world manifold, whose structure group is generated by $\exp \mathcal{G}(\mathbb{R}^{3,1})$ (which is group isomorphic to $\exp \mathbb{M}(4, \mathbb{R})$ and as such generates to $\mathrm{GL}^+(4, \mathbb{R})$ up to group isomorphism), undergoing symmetry breaking to $\mathrm{Spin}^c(3, 1)$. The global sections of the quotient bundle $\mathrm{FX}/\mathrm{SO}(3, 1)$ identify a pseudo-Riemannian metric. The connection is a Spin^c-preserving connection. The group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ is recovered in the broken symmetry and associates with the invariant transformations under the action of the wavefunction on a unit time-like vector of the tangent space, thus preserving the Dirac current. Finally, it is stressed that only a single axiom (Axiom 1: The Geometric Constraint) and a single theorem (Theorem 1: A Provably Optimal Formulation of Physics) are required to obtain these results. In consideration of the extreme generalizability of the optimization problem, the solution obtained is remarkably specific for the present universe.

6 Statements and Declarations

The author declares no competing interests. The authors did not receive support from any organization for the submitted work.

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