A Quantum Theory of Spacetime Events Yielding a Gravitized Standard Model Inherent to 4D, with Disruption Beyond

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Abstract

We present a comprehensive quantum theory of spacetime events. These events serve as a nexus where probabilities and spacetime geometry coalesce, representing perhaps the most fundamental entities that embody this synthesis. At the heart of our theory lies the 'Prescribed Measurement Problem,' an algorithm that extends the entropy maximization problem of statistical physics into the quantum and geometric domains. Employing this algorithm, we systematically extrapolate a generalized quantum theory of gravity from the measurement entropy of spacetime events, from which general relativity and the Standard Model naturally emanate as inherent outcomes. Interestingly, the theory maintains coherence exclusively within four-dimensional spacetime and encounters intrinsic disruptions beyond this dimension, highlighting a quantum-geometric justification for the four-dimensionality of our universe.

1 Introduction

The reconciliation of quantum mechanics with general relativity remains a fundamental objective in theoretical physics. The previously introduced[1] 'Prescribed Measurement Problem' (PMP) will be a pivotal methodology within this endeavor, drawing from the methodology of statistical mechanics. This problem posits a redefined quantum measurement framework, where the traditional interpretation is supplanted by an approach based on the maximization of entropy subject to the constraints of observed measurement outcomes.

In this paper, we extend the PMP to a comprehensive theory of spacetime events, suggesting that the geometry of spacetime is emergent and probabilistic, shaped by quantum interactions. The PMP ensures that the principles of quantum mechanics are preserved while expanding the theory to incorporate the dynamics of spacetime geometry. Within this framework, the principles of general relativity and the Standard Model naturally arise, not as initial assumptions but as inevitable outcomes of the intrinsic properties of quantized four-dimensional spacetime events.

1.1 The Prescribed Measurement Problem

The Prescribed Measurement Problem (PMP) in quantum mechanics is directly inspired by the foundational principles of statistical mechanics, where theory is inherently constraint by a sequence of empirical measurements. Statistical mechanics exemplifies a natural PMP, where the aggregation of energy measurements informs the theoretical structure, leading to the derivation of the Gibbs measure.

Recapitulating this approach, statistical mechanics commences with an empirical sequence of energy measurements. These measurements, anticipated to converge to an average value $\overline{E}$, are utilized as defining constraints within the theoretical formulation:

$$0 = \overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \tag{1}$$

To derive a probability distribution, $\rho(q)$, that maximizes entropy while adhering to this constraint, the theory employs a Lagrange multiplier equation[2].

$$\mathcal{L}(\rho, \lambda, \beta) = \underbrace{-k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q)\right)}_{\text{Energy Measurement Constraint}} \tag{2}$$

Solving this yields the well-established Gibbs measure as the least biased probability measure for the constraint:

$$\rho(q) = \frac{\exp(-\beta E(q))}{\sum_{q \in \mathbb{Q}} \exp(-\beta E(q))} \tag{3}$$

Transitioning to quantum mechanics, the PMP framework demonstrates that the correct quantum theory emerges naturally from an entropy maximization problem formulated with a sequence of measurement outcomes as constraints. This proposition diverges from conventional interpretations that depend on the postulate of wavefunction collapse following a measurement, resulting in a singular outcome. Instead, the PMP posits that these measurements can be systematically employed as constraints to directly infer both the initial state of the quantum system and its evolution over time.

Quantum mechanics requires a more elaborate energy constraint than statistical mechanics. As such, the sequence of energy measurements is intrinsically related to the Hamiltonian, which presides over the system's unitary time evolution. This connection necessitates an adapted form of the energy constraint to encapsulate the proper quantum attributes. Unlike the scalar energy constraints in statistical mechanics, quantum mechanics demands a matrix-based formulation to capture the phase information associated with quantum energy measurements, as mandated by the Hamiltonian.

To accommodate this requirement, we introduce a matrix-based unitary-evolution energy constraint, which is harmonious with unitary evolution and respects the Born rule:

$$0 = \operatorname{tr} \begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{4}$$

This constraint is represented in matrix form and incorporates the system's energy characteristics and their phase without altering the probability measure. Special attention is given to the computation of the trace, which if prematurely simplified, would trivialize the constraint and, consequently, would eliminate the solution space of the constraint. Instead, this unitary-evolution energy constraint is integrated into the entropy maximization problem, commonly employed in statistical mechanics. The resulting solution not only aligns with the established quantum formalism but also simplifies it, rendering it one of the most parsimonious formulations of quantum mechanics to date.

$$\mathcal{L} = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\text{Relative Shannon Entropy[3, 4]}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\tau \left(\operatorname{tr} \begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}_{\text{Unitary-Evolution Energy Constraint}} \tag{5}$$

We solve for $\partial \mathcal{L}(\rho, \lambda, t)/\partial \rho = 0$ as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{6}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{7}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{8}$$

$$\implies \rho(q) = p(q) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{9}$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{10}$$

where $Z(\tau)$ is obtained as

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{11}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{12}$$

$$Z(\tau) := \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{13}$$

The final result is:

$$\rho(q) = \frac{p(q)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}{\sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right)} \tag{14}$$

By utilizing fundamental equivalences and substituting $\tau = t/\hbar$ in a manner analogous to $\beta = 1/(k_B T)$, by noting that the trace drops down from the exponential into the determinant ($\exp\operatorname{tr} M = \det\exp M$), and that the determinant of such a matrix is equivalent to a complex norm, we can rearticulate this into its more commonly recognized form:

$$\rho(q) = \frac{p(q)\|\exp(-itE(q)/\hbar)\|}{\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\|} \tag{15}$$

We can now reverse-engineer the uniquely determined solution space of this expression.

In our previous paper[1], we have shown that all 5 traditional axioms of quantum mechanics[5, 6] are provable from this solution:

Axiom 1 **State Space:** Every physical system is associated with a complex Hilbert space, and the system's state is described by a unit vector (or ray) in that space.

Axiom 2 **Observables:** Physical observables are represented by Hermitian operators acting on the Hilbert space.

Axiom 3 **Dynamics:** The evolution of a quantum system over time is governed by the Schrödinger equation, with the Hamiltonian operator representing the total energy of the system.

Axiom 4 **Measurement:** Upon measurement of an observable, the system collapses to one of the eigenstates of the corresponding operator, and the measured value is one of the eigenvalues.

Axiom 5 **Probability Interpretation:** The probability of obtaining a particular measurement result is given by the squared magnitude of the projection of the state vector onto the corresponding eigenstate.

Let us see how each axiom is recovered.

The wavefunction is identified by "splitting the complex norm" into a complex number and its conjugate. It is envisioned as a vector in a complex Hilbert space, with the partition function acting as its inner product. Expressing the relation in those terms:

$$\sum_{q\in\mathbb{Q}} p(q)\|\exp(-itE(q)/\hbar)\| = Z = \langle\psi|\psi\rangle \tag{16}$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE(q_1)/\hbar) & & \\ & \ddots & \\ & & \exp(-itE(q_n)/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{17}$$

and where $p(q)$ is the probability associated to the initial preparation of the wavefunction: $p(q_i) = \langle \psi_i(0) | \psi_i(0) \rangle$. The entropy-maximization procedure automatically normalizes the result which associates here to a unit vector (or more precisely, a ray). This demonstrates Axiom 1.

We now note that the energy constraint is unmodified by unitary transformations:

$$\langle \mathbf{E} \rangle = \langle \psi | \mathbf{H} | \psi \rangle = \langle \psi | U^\dagger \mathbf{H} U | \psi \rangle \tag{18}$$

Upon moving the solution out of its eigenbasis through unitary transformations, we find that energy, $E(q)$, generally transforms as an Hamiltonian operator:

$$| \psi(t) \rangle = \exp(-it\mathbf{H}/\hbar) | \psi(0) \rangle \tag{19}$$

The dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t} | \psi(t) \rangle = \frac{\partial}{\partial t} (\exp(-it\mathbf{H}/\hbar) | \psi(0) \rangle) \tag{20}$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar) | \psi(0) \rangle \tag{21}$$

$$= -i\mathbf{H}/\hbar | \psi(t) \rangle \tag{22}$$

$$\implies \mathbf{H} | \psi(t) \rangle = i\hbar \frac{\partial}{\partial t} | \psi(t) \rangle \tag{23}$$

Which is the Schrödinger equation. This demonstrates Axiom 3.

The statistical ensemble $\mathbb{Q}$ is defined such that the possible microstates $E(q)$ of the system corresponds to a specific eigenvalue of $\mathbf{H}$. An observation can thus be conceptualized as sampling from $\rho(q,t)$, with the post-collapse state being the occupied microstate q of $\mathbb{Q}$. Consequently, when an observation or measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability distribution is:

$$\rho(q,t) = \frac{1}{\langle \psi | \psi \rangle} (\psi(q,t))^\dagger \psi(q,t) \tag{24}$$

In scenarios where the probability measure $\rho(q,\tau)$ is described in a basis different from its eigenbasis due to a unitary transformation, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle \lambda_i | \psi \rangle|^2 \tag{25}$$

Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. This demonstrates Axiom 4.

Any self-adjoint operator abides by the condition $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$. Measured in its eigenbasis, it aligns with a real-valued observable in statistical mechanics. This demonstrates Axiom 2.

Finally, we note that as the probability measure (Equation 15) reproduces the Born rule, Axiom 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. The five axioms are now theorems of the singular axiom given by Equation 4.

2 Results

Following the exposition of the 'Prescribed Measurement Problem' (PMP) in quantum mechanics, this section presents the concrete outcomes of applying PMP principles to develop a quantum theory of spacetime events. Building upon the theoretical scaffold provided by the PMP, we present an extended framework that incorporates the dynamics of spacetime geometry as probabilistic constructs influenced by quantum interactions.

Within the context of this framework, we will examine the natural emergence of general relativity and the Standard Model not as presupposed entities but as derivations from the intrinsic properties of a quantized, four-dimensional spacetime. These derivations underscore the PMP's capability to seamlessly integrate the principles of quantum mechanics into a broader geometric context, ensuring consistency with observed phenomena while predicting novel effects.

In extending the foundational constraints of the PMP to encompass geometric phase-invariance, we explore a generalized formulation:

$$0 = \frac{1}{2}\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{2}\operatorname{tr}\mathbf{M}(q) \tag{26}$$

where $\mathbf{M}$ is a traceless $n \times n$ matrix.

The derivation of the corresponding quantum theory from this constraint follows an optimization process governed by the Lagrange multiplier method. This process parallels the entropy maximization approach utilized in the initial non-relativistic quantum mechanics discussion, extending its applicability to a quantum-geometric framework.

2.1 The Lagrange Multiplier Equation

The optimization of the probability measure within the quantum framework is methodically articulated through the Lagrange multiplier equation. This mathematical construct is pivotal in ensuring that the derived probability distribution not only maximizes entropy but also satisfies the constraints imposed by the postulated empirical measurements.

The formal expression of the Lagrange multiplier equation is as follows:

$$\mathcal{L}(\rho,\lambda,\tau) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q)\ln\frac{\rho(q)}{p(q)}}_{\text{relative Shannon entropy}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\text{normalization constraint}} \underbrace{+\tau\left(\frac{1}{2}\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}}\rho(q)\frac{1}{2}\operatorname{tr}\mathbf{M}(q)\right)}_{\text{geometric measurement constraint}} \tag{27}$$

This equation encapsulates the relative Shannon entropy, normalization constraint, and a phase-neutralized energy constraint specific to the quantum domain. The Shannon entropy reflects the informational disparity between the probability distribution and the measurement outcome, the normalization constraint ensures the total probability equates to unity, and the phase-neutralized energy constraint encodes the empirical data of geometric measurements within a quantum system.

To find the probability distribution that maximizes this Lagrangian function, we calculate the derivative with respect to ρ and set it to zero:

$$\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} = -\ln\frac{\rho(q)}{p(q)} - 1 - \lambda - \tau\operatorname{tr}\mathbf{M}(q) \tag{28}$$

$$0 = \ln\frac{\rho(q)}{p(q)} + 1 + \lambda + \tau\operatorname{tr}\mathbf{M}(q) \tag{29}$$

$$\implies \ln\frac{\rho(q)}{p(q)} = -1 - \lambda - \tau\operatorname{tr}\mathbf{M}(q) \tag{30}$$

$$\implies \rho(q) = p(q)\exp(-1-\lambda)\exp\left(-\tau\operatorname{tr}\mathbf{M}(q)\right) \tag{31}$$

$$= \frac{1}{Z(\tau)}p(q)\exp\left(-\tau\operatorname{tr}\mathbf{M}(q)\right) \tag{32}$$

The partition function $Z(\tau)$, acting as a normalization constant, is subsequently determined:

$$1 = \sum_{r\in\mathbb{Q}} p(r)\exp(-1-\lambda)\exp\left(-\tau\operatorname{tr}\mathbf{M}(r)\right) \tag{33}$$

$$\implies \left(\exp(-1-\lambda)\right)^{-1} = \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau\operatorname{tr}\mathbf{M}(r)\right) \tag{34}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau\operatorname{tr}\mathbf{M}(r)\right) \tag{35}$$

The resulting probability distribution, which optimally encodes the constraints into the theoretical framework, is given by:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\exp\left(-\frac{1}{2}\tau\operatorname{tr}\mathbf{M}(r)\right)}}_{\text{Normalization Constant}} \underbrace{\exp\left(-\frac{1}{2}\tau\operatorname{tr}\mathbf{M}(q)\right)}_{\text{Generalized Born Rule}} \underbrace{p(q)}_{\text{Prior}} \tag{36}$$

This formulation advances the Born rule to a generalized context, which not only aligns with the core principles of quantum mechanics but, as found, also reflects an awareness of the geometry of spacetime.

2.2 Linear Measurement Constraint in Two Dimensions

Our exploration of the linear measurement constraint begins in the two-dimensional setting before progressing to the more intricate 3+1D spacetime.

In a two-dimensional framework, the measurement constraint is expressed as:

$$\frac{1}{2}\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{2}\operatorname{tr}\mathbf{M}(q), \qquad \text{where } \mathbf{M}(q) := \begin{bmatrix} x(q) & y(q)+b(q) \\ y(q)-b(q) & -x(q) \end{bmatrix} \tag{37}$$

Here, the elements $b(q)$, $x(q)$, and $y(q)$ are scalar functions parametrized by q, characterizing the elements of a traceless matrix representative of the 2D measurements.

The probability distribution for this two-dimensional case simplifies to:

$$\rho(q) = \frac{1}{Z}\det\left(\exp\begin{bmatrix} x(q) & y(q)+b(q) \\ y(q)-b(q) & -x(q) \end{bmatrix}\right)p(q) \tag{38}$$

where

$$p(q) = \det\begin{bmatrix} a(q)+x(q) & y(q)+b(q) \\ y(q)-b(q) & a(q)-x(q) \end{bmatrix} = \det\varphi(q) \tag{39}$$

2.3 Inner Product

The construction of a Hilbert space for our probability measure necessitates expressing the determinant as an inner product of multivectors. For this purpose, we begin by introducing the multivector representation of 2×2 real matrices within the Clifford algebra framework:

$$\begin{bmatrix} a+x & y+b \\ y-b & a-x \end{bmatrix} \cong a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} \tag{40}$$

The Clifford conjugate, denoted by ‡, is defined as:

$$(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}}\wedge\hat{\mathbf{y}})^\ddagger = a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} \tag{41}$$

Using this representation, we establish that the determinant of the matrix, expressed through the geometric product, corresponds to an inner product:

$$\mathbf{u}^\ddagger\mathbf{u} = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}}\wedge\hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}) = a^2 - x^2 - y^2 + b^2 \tag{42}$$

This inner product is equivalent to the determinant of the corresponding matrix:

$$\det\begin{bmatrix} a+x & y+b \\ y-b & a-x \end{bmatrix} = a^2 - x^2 - y^2 + b^2 \tag{43}$$

Over the group of $\mathrm{GL}^+(4,\mathbb{R})$ matrices, the inner product is positive definite.

2.4 The General Linear Wavefunction Representation

The general linear wavefunction, represented by φ, can be expressed in a multi-vector column vector form, encompassing the algebraic elements corresponding to a 2×2 matrix. This form enables the encapsulation of the matrix elements within the geometric algebra framework, facilitating the transition to a Hilbert space structure. The wavefunction φ is denoted as:

$$|\varphi\rangle\rangle = \frac{1}{\sqrt{Z}} \begin{bmatrix} a_1 + x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + b_1\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \\ \vdots \\ a_n + x_n\hat{\mathbf{x}} + y_n\hat{\mathbf{y}} + b_n\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \end{bmatrix} \tag{44}$$

The Clifford conjugate transpose (row vector representation) of φ is obtained by applying the Clifford conjugation to each multi-vector element:

$$\langle\langle\varphi| = \frac{1}{\sqrt{Z}} \begin{bmatrix} a_1 + x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + b_1\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \\ \vdots \\ a_n + x_n\hat{\mathbf{x}} + y_n\hat{\mathbf{y}} + b_n\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \end{bmatrix}^{\ddagger} \tag{45}$$

$$= \frac{1}{\sqrt{Z}} \begin{bmatrix} a_1 - x_1\hat{\mathbf{x}} - y_1\hat{\mathbf{y}} - b_n\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} & \ldots & a_n - x_n\hat{\mathbf{x}} - y_n\hat{\mathbf{y}} - b_n\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \end{bmatrix} \tag{46}$$

This configuration allows us to define the inner product within the Hilbert space as a sum of the matrix determinants of each entry:

$$\langle\langle\varphi|\varphi\rangle\rangle = \frac{1}{Z}\left((a_1^2 - x_1^2 - y_1^2 + b_1^2) + \cdots + (a_n^2 - x_n^2 - y_n^2 + b_n^2)\right) \tag{47}$$

2.5 Dynamics and the Generalized Schrödinger Equation

In our exploration of dynamics through the Prescribed Measurement Problem (PMP), we discover that the solution naturally manifests in the Schrödinger picture. This emergence:

$$\frac{d}{d\tau}|\psi(\tau)\rangle = \frac{d}{d\tau}\left(\exp(-\tau\mathbf{M})|\psi(0)\rangle\right) \tag{48}$$

$$= -\mathbf{M}\exp(-\tau\mathbf{M})|\psi(0)\rangle \tag{49}$$

$$= -\mathbf{M}|\psi(\tau)\rangle \tag{50}$$

$$\implies \mathbf{M}|\psi(\tau)\rangle = -\frac{d}{d\tau}|\psi(\tau)\rangle \tag{51}$$

is effectively a generalized form of the Schrödinger equation, steering the evolution of the system's geometric phase.

It is essential to underscore that the Schrödinger picture's application within PMP is not a limitation to non-relativistic contexts. Rather, the Schrödinger picture, synonymous with the path integral formulation, is equally valid in relativistic settings. This equivalence is illustrated through the transition amplitude relation:

$$\langle \psi_f | e^{-iH(t_f - t_i)/\hbar} | \psi_i \rangle = \int D[\phi] e^{iS[\phi]/\hbar}, \tag{52}$$

where the left side denotes the transition amplitude in the Schrödinger picture, while the right side represents the path integral approach.

The PMP's automatic production of a solution in the Schrödinger picture reaffirms its standing as a viable and robust framework for quantum mechanics, applicable to a wide range of physical situations including those requiring a relativistic treatment.

2.6 Self-Adjoint Operators in the General Linear Framework

Within the general linear wavefunction framework, observables are mathematically articulated as self-adjoint operators. The defining property of such an observable $\mathbf{O}$ is that it satisfies the condition:

$$\langle\langle \mathbf{O}\phi | \varphi \rangle\rangle = \langle\langle \phi | \mathbf{O}\varphi \rangle\rangle \tag{53}$$

For a two-state system, the observable $\mathbf{O}$ is represented as a 2×2 matrix:

$$\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix} \tag{54}$$

Here, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$ and $\mathbf{o}_{11}$are multivectors, encapsulating the observable's components.

The geometric product within the Clifford algebra framework equates to matrix multiplication, leading to the following expressions:

$$\begin{aligned}
\langle\langle \mathbf{O}\phi | \varphi \rangle\rangle &= (\mathbf{o}_{00}\phi_1 + \mathbf{o}_{01}\phi_2)^\ddagger \varphi_1 + \varphi_1^\ddagger(\mathbf{o}_{00}\phi_1 + \mathbf{o}_{01}\phi_2) \\
&\quad + (\mathbf{o}_{10}\phi_1 + \mathbf{o}_{11}\phi_2)^\ddagger \varphi_2 + \varphi_2^\ddagger(\mathbf{o}_{10}\phi_1 + \mathbf{o}_{11}\phi_2) && (55) \\
&= \phi_1^\ddagger \mathbf{o}_{00}^\ddagger \varphi_1 + \phi_2^\ddagger \mathbf{o}_{01}^\ddagger \varphi_1 + \varphi_1^\ddagger \mathbf{o}_{00}\phi_1 + \varphi_1^\ddagger \mathbf{o}_{01}\phi_2 \\
&\quad + \phi_1^\ddagger \mathbf{o}_{10}^\ddagger \varphi_2 + \phi_2^\ddagger \mathbf{o}_{11}^\ddagger \varphi_2 + \varphi_2^\ddagger \mathbf{o}_{10}\phi_1 + \varphi_2^\ddagger \mathbf{o}_{11}\phi_2 && (56)
\end{aligned}$$

$$\begin{aligned}
\langle\langle \phi | \mathbf{O}\varphi \rangle\rangle &= \phi_1^\ddagger(\mathbf{o}_{00}\varphi_1 + \mathbf{o}_{01}\varphi_2) + (\mathbf{o}_{00}\varphi_1 + \mathbf{o}_{01}\varphi_2)^\ddagger \phi_1 \\
&\quad + \phi_2^\ddagger(\mathbf{o}_{10}\varphi_1 + \mathbf{o}_{11}\varphi_2) + (\mathbf{o}_{10}\varphi_1 + \mathbf{o}_{11}\varphi_2)^\ddagger \phi_2 && (57) \\
&= \phi_1^\ddagger \mathbf{o}_{00}\varphi_1 + \phi_1^\ddagger \mathbf{o}_{01}\varphi_2 + \varphi_1^\ddagger \mathbf{o}_{00}^\ddagger \phi_1 + \varphi_2^\ddagger \mathbf{o}_{01}^\ddagger \phi_1 \\
&\quad + \phi_2^\ddagger \mathbf{o}_{10}\varphi_1 + \phi_2^\ddagger \mathbf{o}_{11}\varphi_2 + \varphi_1^\ddagger \mathbf{o}_{10}^\ddagger \phi_2 + \varphi_2^\ddagger \mathbf{o}_{11}^\ddagger \phi_2 && (58)
\end{aligned}$$

The self-adjoint nature of $\mathbf{O}$ is confirmed if:

$$\mathbf{o}_{00}^{\ddagger} = \mathbf{o}_{00} \tag{59}$$

$$\mathbf{o}_{01}^{\ddagger} = \mathbf{o}_{10} \tag{60}$$

$$\mathbf{o}_{10}^{\ddagger} = \mathbf{o}_{01} \tag{61}$$

$$\mathbf{o}_{11}^{\ddagger} = \mathbf{o}_{11} \tag{62}$$

This implies that $\mathbf{O}$ is observable when $\mathbf{O}^{\ddagger} = \mathbf{O}$, analogous to self-adjoint operators in complex Hilbert spaces where $\mathbf{O}^{\dagger} = \mathbf{O}$.

The most general form of such an observable matrix $\mathbf{O}$ in our framework is:

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{y}} \wedge \hat{\mathbf{y}} \\ a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} & a_{11} \end{bmatrix} \tag{63}$$

2.7 Real Eigenvalues for Observables

In the realm of geometric algebra, we investigate the nature of the eigenvalues associated with an observable matrix $\mathbf{O}$, ensuring they are real-valued. The eigenvalues are determined by solving the characteristic equation derived from the matrix's determinant:

$$0 = \det(\mathbf{O} - \lambda\mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \\ a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} & a_{11} - \lambda \end{bmatrix}, \tag{64}$$

which, upon expansion, yields:

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{65}$$

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2), \tag{66}$$

This leads us to the eigenvalues:

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{67}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\} \tag{68}$$

It is crucial to acknowledge that if $a^2 - x^2 - y^2 + b^2 < 0$, the eigenvalues could potentially be complex. However, within the scope of our analysis, we concern ourselves with multivectors and general linear matrices that preserve orientation, for which $a^2 - x^2 - y^2 + b^2 \geq 0$. Hence, under these constraints, our observables are guaranteed to possess real eigenvalues.

2.8 Probability-Preserving Transformations

In quantum mechanics, transformations that preserve the probability distribution of a quantum system are of fundamental importance. These are encapsulated by operators $\mathbf{T}$ that, when acting on a wavefunction $|\varphi\rangle\rangle$, satisfy the condition $\langle\langle\varphi|\mathbf{T}^{\ddagger}\mathbf{T}|\varphi\rangle\rangle$=1, indicating that $\mathbf{T}^{\ddagger}\mathbf{T}$ is the identity operator I.

Consider a two-state quantum system undergoing a general transformation $\mathbf{T}$, which can be expressed in matrix form with 2D multivector components u, v, w, and x:

$$\mathbf{T} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}, \tag{69}$$

The Clifford conjugate product $\mathbf{T}^{\ddagger}\mathbf{T}$ then unfolds as:

$$\mathbf{T}^{\ddagger}\mathbf{T} = \begin{bmatrix} v^{\ddagger} & u^{\ddagger} \\ w^{\ddagger} & x^{\ddagger} \end{bmatrix} \begin{bmatrix} v & w \\ u & x \end{bmatrix} = \begin{bmatrix} v^{\ddagger}v + u^{\ddagger}u & v^{\ddagger}w + u^{\ddagger}x \\ w^{\ddagger}v + x^{\ddagger}u & w^{\ddagger}w + x^{\ddagger}x \end{bmatrix} \tag{70}$$

For the identity condition $\mathbf{T}^{\ddagger}\mathbf{T} = I$ to be satisfied, we necessitate that:

$$v^{\ddagger}v + u^{\ddagger}u = 1 \tag{71}$$

$$v^{\ddagger}w + u^{\ddagger}x = 0 \tag{72}$$

$$w^{\ddagger}v + x^{\ddagger}u = 0 \tag{73}$$

$$w^{\ddagger}w + x^{\ddagger}x = 1 \tag{74}$$

These conditions are satisfied by:

$$\mathbf{T} = \frac{1}{\sqrt{v^{\ddagger}v + u^{\ddagger}u}} \begin{bmatrix} v & u \\ -e^{\varphi}u^{\ddagger} & e^{\varphi}v^{\ddagger} \end{bmatrix}, \tag{75}$$

Here, u and v are 2D multivectors, and e^{φ} represents a unit multivector.

In the unitary case, when the vector part of the multivector vanishes ($x \to 0$, $y \to 0$), we obtain:

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta}b^{\dagger} & e^{i\theta}a^{\dagger} \end{bmatrix}. \tag{76}$$

Here, $\mathbf{T}$ signifies a general linear extension of the unitary transformation to two dimensions within the geometric algebra framework, thus broadening the scope of unitary transformations to accommodate the rich structure of multivectors.

2.9 Metric Operator

We introduce an operator $\hat{g}$, known as the metric operator, defined as follows:

$$\hat{g}\varphi = I\varphi I^{-1} \tag{77}$$

where $I = \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$ is the pseudoscalar in two dimensions. This operator acts as a similarity transformation, altering the basis in which the multivector φ is expressed.

The action of $\hat{g}$ on a multivector reverses the sign of its vector part while preserving the sign of the scalar and bivector parts:

$$I(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})I^{-1} = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{78}$$

We will demonstrate that $\hat{g}$ is unitary and probability-preserving $\langle\langle \hat{g}\varphi | \hat{g}\varphi \rangle\rangle = \langle\langle \varphi | \varphi \rangle\rangle$ in the general linear group context:

$$\begin{aligned}
(g\mathbf{u})^{\ddagger} g\mathbf{u} &= (I\mathbf{u}I^{-1})^{\ddagger} I\mathbf{u}I^{-1} && (79)\\
&= (I^{-1})^{\ddagger}\mathbf{u}^{\ddagger}I^{\ddagger}I\mathbf{u}I^{-1} && (80)\\
&= (-I^{-1})\mathbf{u}^{\ddagger}(-I)I\mathbf{u}I^{-1}I^{-1}\mathbf{u}^{\ddagger}II\mathbf{u}I^{-1} && (81)\\
&= e_2e_1\mathbf{u}^{\ddagger}e_1e_2e_1e_2\mathbf{u}e_2e_1 && (82)\\
&= -e_2e_1\mathbf{u}^{\ddagger}\mathbf{u}e_2e_1 && (83)\\
&= e_1e_2 \underbrace{\mathbf{u}^{\ddagger}\mathbf{u}}_{\text{scalar}} e_2e_1 && (84)\\
&= \mathbf{u}^{\ddagger}\mathbf{u}. && (85)
\end{aligned}$$

This completes the demonstration.

Applying $\hat{g}$ to the inner product:

$$\mathbf{u}^{\ddagger}\hat{g}\mathbf{u} = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{86}$$

Expressed in matrix form, the multivectors and their transformations under $\hat{g}$ are:

$$\mathbf{u}^{\ddagger} = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - x\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} - y\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - b\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}a-x & -y-b\\-y+b & a+x\end{bmatrix} \tag{87}$$

$$\hat{g}\mathbf{u} = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - x\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} - y\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}a-x & -y+b\\-y-b & a+x\end{bmatrix} \tag{88}$$

The matrix representation of $\mathbf{u}^{\ddagger}$ is the transpose of $\hat{g}\mathbf{u}$. The inner product results in a symmetric matrix:

$$\mathbf{u}^{\ddagger}\hat{g}\mathbf{u} = \begin{bmatrix}a-x & -y-b\\-y+b & a+x\end{bmatrix}\begin{bmatrix}a-x & -y+b\\-y-b & a+x\end{bmatrix} \tag{89}$$

$$= \begin{bmatrix}(a-x)^2 + (-y-b)^2 & (a-x)(-y+b) + (-y-b)(a+x)\\(-y+b)(a-x) + (a+x)(-y-b) & (-y+b)^2 + (a+x)^2\end{bmatrix} \tag{90}$$

The set of all matrices $\mathbf{u}$ such that $\mathbf{u}^T\mathbf{u} = g$ for a given metric forms an equivalence class in the space $\mathrm{GL}^{+}(2, \mathbb{R})/\mathrm{SO}(2)$, representing dilation and shear transformations.

Parameterizing the multivectors over (x, y), we obtain a metric over a 2D manifold:

$$\mathbf{u}(x,y)^{\ddagger}\hat{g}\mathbf{u}(x,y) = \mathbf{u}(x,y)^{T}\mathbf{u}(x,y) = g(x,y) \tag{91}$$

where $g(x, y)$ is the metric tensor.

The metric $g(x, y)$ represents the transition amplitude between $\mathbf{u}$ and its transpose.

2.10 Metric Superposition and Interference

Consider a two-state quantum system represented by the wavefunction $|\varphi\rangle\rangle$:

$$|\varphi\rangle\rangle = \frac{1}{\sqrt{Z}}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \tag{92}$$

where $\mathbf{u}$ and $\mathbf{v}$ are multivectors, and Z is the normalization constant.

For an invariant transformation U that satisfies $U^{\ddagger}U = I$, such as the Hadamard transformation:

$$U = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} \tag{93}$$

applying U to $|\varphi\rangle\rangle$ results in a superposed state:

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\frac{1}{\sqrt{Z}}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \frac{1}{\sqrt{Z}}\begin{bmatrix}\frac{1}{\sqrt{2}}\mathbf{u} + \frac{1}{\sqrt{2}}\mathbf{v}\\ \frac{1}{\sqrt{2}}\mathbf{u} - \frac{1}{\sqrt{2}}\mathbf{v}\end{bmatrix} \tag{94}$$

This superposition embodies a combination of spacetime geometries that, upon measurement, can exhibit interference effects. The probability amplitude of the superposition is:

$$\frac{1}{\sqrt{2Z}}(\mathbf{u}+\mathbf{v})^{\ddagger}\frac{1}{\sqrt{2Z}}(\mathbf{u}+\mathbf{v}) = \frac{1}{2Z}(\mathbf{u}^{\ddagger}+\mathbf{v}^{\ddagger})(\mathbf{u}+\mathbf{v}) \tag{95}$$

$$= \frac{1}{2Z}(\mathbf{u}^{\ddagger}\mathbf{u}+\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u}+\mathbf{v}^{\ddagger}\mathbf{v}) \tag{96}$$

$$= \frac{1}{2Z}(\underbrace{\mathbf{u}^{\ddagger}\mathbf{u}+\mathbf{v}^{\ddagger}\mathbf{v}}_{\text{superposition}}+\underbrace{\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u}}_{\text{interference}}) \tag{97}$$

which decomposes into a term representing the superposition of the individual states $(\mathbf{u}^{\ddagger}\mathbf{u} + \mathbf{v}^{\ddagger}\mathbf{v})$ and a term describing the interference $(\mathbf{u}^{\ddagger}\mathbf{v} + \mathbf{v}^{\ddagger}\mathbf{u})$. The complexity of this interference surpasses that found in systems described by complex wavefunctions due to the multivector nature of $\mathbf{u}$ and $\mathbf{v}$.

In the special case where vector components are absent (φ with $\hat{\mathbf{x}}, \hat{\mathbf{y}} \to 0$), the interference simplifies to the conventional complex interference pattern observed in quantum mechanics.

Thus, an invariant transformation such as U applied to the system allows for the exploration of rich interference phenomena within the geometric framework, revealing nuances of quantum behavior through the lens of metric and geometric superposition.

2.11 Metric Measurement

Consider a two-state system described by the wavefunction $|\varphi\rangle\rangle$:

$$|\varphi\rangle\rangle = \frac{1}{\sqrt{Z}} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \tag{98}$$

where $\mathbf{u}$ and $\mathbf{v}$ are multivectors, and Z is the normalization constant.

We apply the metric operator $\hat{g}$ to this system:

$$\langle\langle\varphi|\hat{g}|\varphi\rangle\rangle = \frac{1}{Z}(\mathbf{u}^\ddagger \hat{g}\mathbf{u} + \mathbf{v}^\ddagger \hat{g}\mathbf{v}) \tag{99}$$

Given that $\mathbf{u}^\ddagger$ is the transpose of $\hat{g}\mathbf{u}$, and similarly for $\mathbf{v}$, the inner product simplifies to:

$$= \frac{1}{Z}(g_{\mathbf{u}} + g_{\mathbf{v}}) \tag{100}$$

where $g_{\mathbf{u}}$ and $g_{\mathbf{v}}$ denote the individual metrics associated with $\mathbf{u}$ and $\mathbf{v}$, respectively.

The result reveals that the operator $\hat{g}$ effectively computes the expectation value of the metric, which is a superposition of the metrics corresponding to the states $\mathbf{u}$ and $\mathbf{v}$, each weighted by their respective probabilities:

$$\langle\langle\varphi|\hat{g}|\varphi\rangle\rangle = \langle g\rangle \tag{101}$$

This expectation value, $\langle g\rangle$, is the average metric of the system. It encapsulates the probabilistic nature of the quantum states and their contributions to the overall geometry of the system as perceived through measurement.

2.12 Transition to Classical Gravity via Metric Operator

The application of the metric operator in quantum measurements forges a conceptual link to classical gravitational theories. This connection is maintained through the adherence to two fundamental symmetries: translational symmetry $T(2)$ and rotational symmetry $SO(2)$. The former, signifying translations in 2D, is inherent in the normalization process of the quantum state, while the latter preserves the rotational invariance of the system's metric properties.

Consider the wavefunction normalization condition as an integral over the manifold $\mathcal{M}$:

$$\int_{\mathcal{M}} \sqrt{-g} \det \varphi \, \mathrm{d}^2 x < \infty \tag{102}$$

which remains invariant under global translations, thereby reflecting $T(2)$ symmetry in the structure of the quantum framework.

Furthermore, the metric expectation value $\langle\langle\varphi|\hat{g}|\varphi\rangle\rangle = \langle g\rangle$ respects $SO(2)$ symmetry through its invariance under special orthogonal transformations:

$$(L\varphi)^T L\varphi = \varphi^T L^T L\varphi = \varphi^T \varphi \tag{103}$$

where L is an $SO(2)$ transformation matrix. This showcases rotational symmetry in the context of the metric tensor g.

Elevating these global symmetries to local ones indicates the progression toward a gauge theory predicated on the Poincaré group. This advancement aligns with the Einstein-Cartan theory, an extension of general relativity that incorporates torsion alongside curvature, offering a richer geometric interpretation of spacetime.

The quantum-to-classical transition here is remarkably natural, requiring no additional axioms. The metric operator does more than measure; it acts as a bridge between quantum mechanics and the geometry of spacetime. It is this effortless shift that reveals the classical theory of gravity within our quantum framework, aligning seamlessly with the principles of the Einstein-Cartan theory.

2.13 Fock Space Construction

A Fock space is a quantum state space that accommodates systems with varying particle numbers, extending the concept to events in a two-dimensional (2D) context. It is constructed mathematically as a direct sum of tensor products of single-particle Hilbert spaces, allowing for the representation of states with different numbers of quanta:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}. \tag{104}$$

For a system composed of two Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, the symmetric and antisymmetric combinations of wavefunctions ψ_1 and ψ_2 are given by:

$$\text{Symmetric: } \frac{1}{\sqrt{2}}(\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1), \tag{105}$$

$$\text{Antisymmetric: } \frac{1}{\sqrt{2}}(\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1). \tag{106}$$

In the context of a 2D geometric algebra framework, the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ remains invariant under the group of rotations SO(2) and translations $\mathbb{R}$. This invariance manifests itself in the property that any scalar a and bivector $\mathbf{b}$ commute with all elements of the geometric algebra, thus preserving the tensor product structure:

$$\frac{1}{\sqrt{2}}(\psi_1 \otimes e^{a+\mathbf{b}}\psi_1 - e^{a+\mathbf{b}}\psi_1 \otimes \psi_1) = \frac{1}{\sqrt{2}}(e^{a+\mathbf{b}}\psi_1 \otimes \psi_1 - e^{a+\mathbf{b}}\psi_1 \otimes \psi_1) \tag{107}$$

$$= 0. \tag{108}$$

This expression illustrates that the antisymmetric part of the tensor product vanishes under such transformations, demonstrating the robustness of the constructed Fock space against local geometric transformations.

Through the Fock space formalism, we can elegantly capture the quantum behavior of a system with a fluctuating number of entities, which is especially pertinent in describing a universe with a dynamical spacetime fabric where events can be created and annihilated.

2.14 Linear Measurement Constraint in 3+1 Dimensions

When extending our considerations to the framework of three spatial dimensions plus one time dimension (3+1D), the measurement constraint is appropriately adapted to account for the complexity of higher-dimensional spacetime. The constraint for a 3+1D system is formalized as:

$$0 = \frac{1}{4}\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{4}\operatorname{tr}\mathbf{M}(q), \tag{109}$$

where each $\mathbf{M}(q)$ denotes a traceless 4×4 matrix associated with the state q. This matrix representation encapsulates the dynamics of the system while maintaining the traceless condition.

2.15 Introducing the "Double-Copy" Inner Product in 3+1 Dimensions

In the context of 3+1 dimensions, we can express any 4×4 real matrix using the real Majorana representation. Such a matrix, denoted by $\mathbf{M}$, has the following general form:

$$\mathbf{M} = \begin{bmatrix} a+x-f_{02}+q & -z-f_{13}+w-b & f_{03}-f_{23}-p-v & y+f_{01}+f_{12} \\ -z-f_{13}+w+b & a-x-f_{02}-q & y+f_{01}+f_{12} & f_{03}-f_{23}-p-v \\ f_{03}+f_{23}-p+v & y-f_{01}+f_{12} & a+x+f_{02}-q & -z-f_{13}-w+b \\ y+f_{01}-f_{12} & -f_{03}-f_{23}-p+v & -z+f_{13}-w-b & a-x+f_{02}+q \end{bmatrix}, \tag{110}$$

Posing $a = 0$ ensures that $\mathbf{M}$ is traceless.

This matrix corresponds to a multi-vector in geometric algebra, which encompasses various grades including scalar, vector, bivector, pseudo-vector, and pseudo-scalar components:

$$\begin{aligned} \mathbf{M} \cong & a && (111)\\ & + t\hat{\mathbf{t}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} && (112)\\ & + f_{01}\hat{\mathbf{t}}\wedge\hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}}\wedge\hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}}\wedge\hat{\mathbf{z}} + f_{12}\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}}\wedge\hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} && (113)\\ & + v\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} + w\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{z}} + p\hat{\mathbf{t}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} + q\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} && (114)\\ & + b\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} && (115) \end{aligned}$$

In the 3+1-dimensional construct, a simple inner product is insufficient to define a Hilbert space over the $\mathrm{GL}^+(4,\mathbb{R})$ group. Instead, a more complex "double-copy" inner product is essential:

$$\langle\langle\varphi|\varphi\rangle\rangle = \lfloor\varphi^{\ddagger}\varphi\rfloor_{3,4}\varphi^{\ddagger}\varphi. \tag{116}$$

The Clifford conjugate, denoted by $\varphi^{\ddagger}$, reverses the signs of the bivector and pseudo-vector components:

$$\varphi^{\ddagger} = a + \mathbf{x} - \mathbf{f} - \mathbf{v} + \mathbf{b}. \tag{117}$$

The blade 3-4 conjugate, represented as $\lfloor\psi\rfloor_{3,4}$, modifies the signs of the pseudo-vector and pseudo-scalar components:

$$\lfloor\varphi\rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b}. \tag{118}$$

When these elements are combined, the result is a scalar equivalent to the determinant[7] of the associated 4×4 real matrix:

$$\det\varphi = \lfloor\varphi^{\ddagger}\varphi\rfloor_{3,4}\varphi^{\ddagger}\varphi. \tag{119}$$

This "double-copy" inner product provides a means to encapsulate the intricate properties of the higher-dimensional spacetime framework, crucial for the coherent development of a geometric-quantum theory in 3+1D.

2.16 Metric Measurement in 3+1 Dimensions

In previous discussions, we introduced the metric operator $\hat{g}$, defined by its action on a multivector $\mathbf{u}$ as $\hat{g}\mathbf{u} = I\mathbf{u}I^{-1}$. When applied to a multivector in 3+1 dimensions, the operator transforms as follows:

$$\hat{g}\mathbf{u} = I(a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b})I^{-1} \tag{120}$$

$$= a + \mathbf{x} - \mathbf{f} - \mathbf{v} + \mathbf{b} \tag{121}$$

$$= \mathbf{u}^{\ddagger}. \tag{122}$$

This action reverses the signs of the bivector and pseudo-vector parts, yielding the Clifford conjugate of the original multivector $\mathbf{u}$.

We proceed to examine the metric operator's influence on the "double-copy" inner product:

$$\lfloor\varphi^{\ddagger}\hat{g}\varphi\rfloor_{3,4}(\hat{g}\varphi)^{\ddagger}\varphi = \lfloor(\varphi^{\ddagger})^2\rfloor_{3,4}\varphi^2 = \widetilde{\psi}^{\dagger}\psi = g, \tag{123}$$

where ψ denotes φ^2, $\widetilde{\psi}$ represents the reverse of ψ, and $\psi^{\dagger}$ is the blade-4 conjugate of ψ, analogous to the complex conjugate in its operation.

Interestingly, the metric operator simplifies the "double-copy" inner product to a single-copy form, more aligned with familiar quantum mechanical inner products.

It's important to note that in the 3+1-dimensional framework, the equivalence class of matrices derived from this reduced inner product does not lie within $\mathrm{GL}^{+}(4,\mathbb{R})/\mathrm{SO}(3,1)$ but is instead associated with $\mathrm{GL}^{+}(4,\mathbb{R})/\mathrm{Spin}^{c}(3,1)$. This distinction will have significant ramifications for a gauge theory of gravity, which will be explored in subsequent section.

2.17 Gauge Theory of Gravity with Spin^c(3,1) Symmetry (Sketch)

The formulation of a gravity theory within this framework echoes the principles of Einstein-Cartan theory, yet it expands the symmetry group to include Spin^c(3,1), intertwining a U(1) phase factor with the standard Lorentz transformations. The development of a Lagrangian for such a theory involves several key steps:

1. **Variable Definitions:**
   - e^a_μ: The vierbein (tetrad) field, acting as a bridge between the Lorentzian tangent space and the spacetime manifold.
   - ω^{ab}_μ: The spin connection, enabling parallel transport within the tangent space and defining the covariant derivative.
   - A_μ: The U(1) gauge field corresponding to the extended Spin^c(3,1) symmetry, introducing electromagnetic-like interactions in the geometric framework.
2. **Gauging the Symmetries:**
   - **Translations:** Adapt the partial derivatives to covariant derivatives, incorporating the spin connection to account for the affine structure of spacetime.
   - **Spin^c(3,1) Transformations:** Implement gauging procedures for both Lorentz and U(1) components, intricately combining them to form the complete Spin^c gauge group.
3. **Curvature and Torsion 2-forms:**
   - **Lorentz Part:** Define R^{ab} as $d\omega^{ab} + \omega^a_c \wedge \omega^{cb}$, representing the curvature of spacetime.
   - **U(1) Part:** Formulate $F = dA$, analogous to the electromagnetic field strength tensor.
   - **Torsion:** Express $T^a = de^a + \omega^a_b \wedge e^b$, encapsulating spacetime torsion.
4. **Lagrangian Construction:**
   - Integrate the Einstein-Cartan Lagrangian in terms of tetrads and spin connection, including curvature and torsion terms. For the Spin^c(3,1) extension, incorporate the U(1) field strength F, resulting in:

     $$S_{EC} = \int \left(\frac{1}{2\kappa} \epsilon_{abcd}\, e^a \wedge e^b \wedge R^{cd} - \frac{\alpha}{2} T^a \wedge \star T_a + F \wedge \star F + \mathcal{L}_{\text{matter}} \right) \tag{124}$$

     where:

- κ is Einstein's gravitational constant, and α is the coupling constant associated with the torsion term.
- The Hodge star operator $\star$ maps forms to their duals, essential for constructing the scalar quantities in the action.
- $\mathcal{L}_{\text{matter}}$ encapsulates the matter field contributions.

This action integrates both the dynamics of general relativity and the additional degrees of freedom from Spin^c symmetry, potentially accommodating spinor fields and additional gauge interactions within the gravity theory.

2.18 Standard Model Symmetries

We recall that the bilinear form $\widetilde{\psi}\gamma_0\psi$ is conserved under Dirac dynamics and corresponds to the Dirac current in the David Hestene's formulation of the wavefunction using geometric algebra. We now introduce the transformation[8, 9] $T = \mathbf{f}_1 \exp(\mathbf{f}_2 + \mathbf{b}_2)$, and explore the global gauges preserving this invariance:

$$\widetilde{\psi}\gamma_0\psi = \text{reverse}(\mathbf{f}_1 \exp(\mathbf{f}_2 + \mathbf{b}_2)\psi)\gamma_0\mathbf{f}_1 \exp(\mathbf{f}_2 + \mathbf{b}_2)\psi \tag{125}$$

$$= \widetilde{\psi} \exp(-\mathbf{f}_2 + \mathbf{b}_2)(-\mathbf{f}_1)\gamma_0\mathbf{f}_1 \exp(\mathbf{f}_2 + \mathbf{b}_2)\psi \tag{126}$$

Let us first investigate $(-\mathbf{f}_1)\gamma_0\mathbf{f}_1$. First let us pose:

$$\mathbf{E} = f_{01}\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}} \wedge \hat{\mathbf{z}} \tag{127}$$

$$\mathbf{B} = f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \tag{128}$$

such that $\mathbf{f} = \mathbf{E} + \mathbf{B}$.

$$-(\mathbf{E} + \mathbf{B})\gamma_0(\mathbf{E} + \mathbf{B}) = \gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \tag{129}$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{B}^2) \tag{130}$$

$$= (f_{01}^2 + f_{02}^2 + f_{03}^2 + f_{12}^2 + f_{13}^2 + f_{23}^2)\gamma_0 \tag{131}$$

$$+ 2(-f_{02}f_{12} + f_{03}f_{13})\gamma_1 \tag{132}$$

$$+ 2(-f_{01}f_{12} + f_{03}f_{23})\gamma_2 \tag{133}$$

$$+ 2(-f_{01}f_{13} + f_{02}f_{23})\gamma_3 \tag{134}$$

The equation remains invariant if $f_{01}^2 + f_{02}^2 + f_{03}^2 + f_{12}^2 + f_{13}^2 + f_{23}^2 = 1$ and if the cross products vanish. The preservation of the bilinear form under this transformation suggests a symmetry akin to SU(3) invariance. The inclusion of $\mathbf{f}_1$ in the transformation, although not directly linked to a probability-preserving aspect like the exponential term, is necessary to fully recover the SU(3) symmetry of the Standard Model.

Now, we investigate the exponential term:

$$\exp(-\mathbf{E} - \mathbf{B} + b)\gamma_0 \exp(\mathbf{E} + \mathbf{B} + b) \tag{135}$$

$$= \gamma_0 \exp(\mathbf{E} - \mathbf{B} - b) \exp(\mathbf{E} + \mathbf{B} + b) \tag{136}$$

$$= \gamma_0 \exp(2\mathbf{E}) \tag{137}$$

The relation remains invariant if $\mathbf{E} = 0$. In the case the exponential is given as

$$\exp\big(f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}}\big) \tag{138}$$

and it can be written as

$$\exp(i(f_{12}\sigma_z + f_{13}\sigma_y + f_{23}\sigma_z + b)) \tag{139}$$

The exponential term $\exp(\mathbf{f}_2 + \mathbf{b}_2)$ is justified as an invariant transformation of the probability measure, aligning with the SU(2)xU(1) gauge symmetry of the Standard Model.

These analyses lead to the complete gauge group that conserves Dirac dynamics for this transformation:

$$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \tag{140}$$

mirroring the gauge symmetries of the Standard Model of particle physics.

Extending these symmetries from global to local naturally introduces gauge fields corresponding to the electromagnetic, weak, and strong forces. The local gauge invariance principle demands the inclusion of gauge bosons—photons, W and Z bosons, and gluons—facilitating interactions in accordance with the Standard Model.

Thus, this theoretical structure not only accommodates the gravitational dynamics via Einstein-Cartan theory and the extended symmetry of Spin^c(3,1) but also coherently adds the Standard Model, providing a comprehensive description of fundamental interactions.

2.19 Dimensional Constraints on Quantum Geometry

Our exploration extends the quantum-geometric theory up to 4D spacetime. However, attempts to extrapolate this model to higher dimensions, specifically at 5D and beyond, encounter insurmountable disruptions. These disruptions act as natural constraints on the dimensionality of spacetime within our framework, suggesting an inherent dimensional limit for its applicability.

In dimensions six and above, the mathematical structure we rely upon collapses. Acus et al.'s work [10] in 6D geometric algebra reveals that the determinant, defined by a norm via self-products, does not extend to 6D. Their exhaustive search failed to produce a norm defined in this way, indicating a fundamental obstruction to formulating a quantum-geometric theory in higher dimensions.

Their findings suggest that a multivector norm in 6D cannot be represented as a linear combination of self-products, as illustrated by the following generalized expressions:

$$a_0^4 - 2a_0^2a_{47}^2 + b_2a_0^2a_{47}^2p_{412}p_{422} + \langle\text{additional monomials}\rangle = 0, \tag{141}$$

$$\vdots \tag{142}$$

$$\langle\text{more equations with numerous monomials}\rangle. \tag{143}$$

Simplifying to special cases where the 6D multivector comprises only a scalar and a grade 4 element illustrates the fundamental difficulty:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))). \quad (144)$$

Even in this special case, we find that constructing a linear relationship for observables is not possible; event real part of the observable cannot be satisfied from the multivector norm:

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (145)$$

$$= b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))). \quad (146)$$

Hence, our definition of observables as self-adjoint operators is unattainable in 6D, as not even real numbers (except $\mathbf{O} = 1$) can satisfy the necessary conditions, making higher dimensions fundamentally unobservable in the quantum-geometric sense of this paper.

Odd dimensions further complicate matters as the norm veers into the realm of complex numbers, deviating from the expected determinant of associated matrices. In these dimensions, the norm appears as a product of a determinant and its complex conjugate:

$$\langle \mathbf{u}, \mathbf{u} \rangle = (\det \mathbf{u})^\dagger \det \mathbf{u}. \quad (147)$$

This result does not align with the determinants derived from our entropy maximization procedure, which are crucial for the physical interpretations in our model. The discrepancy effectively excludes 3D, 5D, and all higher odd dimensions, confining the validity of our model to 0D, 1D, 2D, and the physically pertinent 4D spacetime, where it appears to incorporate gravity and the Standard Model.

3 Discussion

This paper has ventured into the intricate domain of quantum-geometric theory, elucidating its robust applicability up to the four-dimensional fabric of spacetime. Our theoretical construct posits a quantum foundation for the geometry of spacetime, rooted in the principles of entropy maximization and constrained by measurement outcomes—a framework we have coined as the Prescribed Measurement Problem (PMP). The extension of PMP to spacetime events has unveiled a rich interplay between quantum mechanics and general relativity, with the two theories emerging naturally from the same probabilistic underpinnings.

Notably, the exploration of higher-dimensional spaces, specifically at 5D and above, has unearthed fundamental limitations. The inability to define self-adjoint observables via an inner product in 6D signifies a theoretical boundary, suggesting that our universe's quantum-geometric structure is inherently four-dimensional and unobservable beyond. This revelation aligns with the physical universe we observe, where general relativity and the Standard Model operate within a four-dimensional spacetime continuum.

Furthermore, this paper has tentatively touched upon the realm of quantum gravity. By introducing a metric operator and considering its implications for a general linear quantum framework, we have set the stage for a potential unification of gravity with quantum mechanics. This approach beckons further exploration into the quantum nature of spacetime, where the geometry itself is subject to quantum superpositions and probabilistic interpretations. Our investigation has led to the construction of a theoretical framework where spacetime is not merely a passive stage for quantum phenomena but a dynamic participant that evolves with each quantum event.

- **Wavefunction Normalization and Spacetime:**

  The concept of wavefunction normalization has been extended to include the entirety of spacetime, proposing a wavefunction that is normalized across both space and time. This approach recognizes that quantum events are not confined to spatial parameters but are fundamentally temporal occurrences as well. The implication is a more nuanced understanding of quantum states, where events are defined not just by their spatial coordinates but also by their temporal context.

- **Fock Space and Quantum Event Dynamics:**

  The application of Fock space to a variable number of quantum events suggests a universe where the topology of spacetime is subject to fluctuations, and events can spontaneously emerge or dissolve. Such a perspective aligns with a dynamic view of spacetime, where the quantum underpinnings are as malleable as the events they encompass. The potential fluctuations in spacetime topology speak to a universe that is far more interactive and mutable than previously conceived.

- **Metric Operator and Dynamic Spacetime Geometry:**

  The introduction of the metric operator marks a significant advancement in our understanding of spacetime geometry. When applied to the inner product, this operator yields a transition amplitude between a wavefunction and its transpose, and this amplitude itself functions as a metric. This innovative concept suggests that spacetime geometry, influenced by quantum events, could exhibit characteristics of quantum superposition. This challenges the classical notion of a static and continuous spacetime, proposing instead that spacetime is dynamically responsive to quantum phenomena.

- **Operator-Induced Spacetime Metrics"**

  Our theory introduces the metric operator as a key element in the dynamic shaping of spacetime. By producing a transition amplitude that acts as a metric, this operator paves the way for an intricate interplay between quantum states and spacetime geometry. Importantly, the metric operator facilitates T(4) and $\text{Spin}^c(3,1)$ gauges, essential for constructing an Einstein-Cartan type gravitational theory. This approach signifies a step towards unifying the principles of quantum mechanics with the geometrical structure of spacetime, thereby enriching our comprehension of the fundamental interactions in the universe.

- **Generalization of the Born Rule**

  The extension of the Born rule, traditionally associated with probability in quantum mechanics, to encompass the determinant which eventually lead to the gauge symmetries of the electroweak force and potentially the whole standard model indicates that quantum probabilities may underlie the observed symmetries in particle physics. This generalization uncovers a fundamental relationship between the probabilistic aspects of quantum mechanics and the forces and particles defined within the standard model.

4 Conclusion

In the quest to synthesize the discrete nature of quantum events with the continuous fabric of spacetime, this paper has ventured a significant step forward. Our proposed quantum theory of spacetime events, centered around the Prescribed Measurement Problem, delineates a framework where the entropy maximization problem extends into the realms of quantum mechanics and geometry. Within this framework, the principles of general relativity and the Standard Model are not merely accommodated but naturally arise as outcomes inherent to the four-dimensional spacetime construct.

The Prescribed Measurement Problem serves as a bridge between concrete measurement outcomes and a quantum-geometric theory of spacetime, providing a description of gravity and particle physics that is intrinsically linked to the probabilistic fabric of quantum mechanics. This theory affirms the four-dimensionality of our universe not as a mere backdrop but as a dynamic participant shaped by the quantum events it hosts.

Statements and Declarations

Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

References

[1] Alexandre Harvey-Tremblay. The prescribed measurement problem: Toward a contention-free formulation of quantum physics. *Preprint*, September 2023. Available at ResearchGate: Accessed [date you accessed the article].

[2] Frederick Reif. *Fundamentals of statistical and thermal physics*. Waveland Press, 2009.

[3] Claude Elwood Shannon. A mathematical theory of communication. *Bell system technical journal*, 27(3):379–423, 1948.

[4] Solomon Kullback and Richard A Leibler. On information and sufficiency. *The annals of mathematical statistics*, 22(1):79–86, 1951.

[5] Paul Adrien Maurice Dirac. *The principles of quantum mechanics*. Number 27. Oxford university press, 1981.

[6] John Von Neumann. *Mathematical foundations of quantum mechanics: New edition*, volume 53. Princeton university press, 2018.

[7] Douglas Lundholm. Geometric (clifford) algebra and its applications. *arXiv preprint math/0605280*, 2006.

[8] David Hestenes. Space-time structure of weak and electromagnetic interactions. *Foundations of Physics*, 12(2):153–168, 1982.

[9] Anthony Lasenby. Some recent results for $su(3)$ and octonions within the geometric algebra approach to the fundamental forces of nature. *arXiv preprint arXiv:2202.06733*, 2022.

[10] A Acus and A Dargys. Inverse of multivector: Beyond p+ q= 5 threshold. *arXiv preprint arXiv:1712.05204*, 2017.