Multivector Amplitudes: A Superset of Complex Amplitudes Yielding the Standard Model and Gravity?

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April 9, 2024

Abstract

A quantum theory utilizing multivector amplitudes instead of complex amplitudes has been developed within the framework of geometric algebra. This theory generalizes the Born rule to a multivector probability measure that is invariant under a wide range of geometric transformations. In this formalism, the gamma matrices become operators, enabling the construction of the metric tensor as an observable. By requiring invariance of the metric tensor under specific multivector transformations, the gauge symmetries $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and their associated conserved currents naturally emerge, without the need for additional assumptions. Remarkably, the multivector amplitude formalism is found to be consistent only with 3+1-dimensional spacetime, encountering obstructions in other dimensional configurations. This finding aligns with the observed dimensionality of the universe and suggests a possible explanation for the specific gauge symmetries of the Standard Model. Furthermore, the incorporation of the metric tensor as an observable within the quantum framework provides a natural integration of gravity with quantum mechanics.

1 Introduction

In this paper, we introduce a novel quantum theory that employs multivector amplitudes instead of complex amplitudes. The theory is entirely derived by solving an entropy maximization problem, which intrinsically yields a probability measure and an associated non-negative Hilbert space in which the multivector-valued wavefunction resides. The maximization problem also generates the complete set of requisite mathematical tools for a comprehensive quantum mechanical treatment, including a non-negative inner product related

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to probabilities, an evolution operator, transition amplitudes, superposition, interference, and observables, all generalized to the geometric domain via multivectors. By formulating the theory as a solution to an entropy optimization problem, its consistency and well-definedness are mathematically assured.

Within this framework, we find that the gamma matrices are elevated to the status of operators, enabling the construction of the metric tensor as an observable. Remarkably, the gauge symmetries of the standard model of particle physics, namely U(1), SU(2), and SU(3), naturally emerge as the charge currents that preserve the invariance of the metric tensor. Furthermore, multivector amplitudes are found to be free of obstructions exclusively in 3+1D spacetime, potentially offering insights into the dimensional specificity of the universe.

This innovative approach to quantum mechanics extends the 'Prescribed Observation Problem' (POP), a methodology we previously proposed [1], which applies entropy maximization techniques, well-established in statistical mechanics, to derive the axioms of quantum mechanics from first principles. The natural extension of this methodology to multivectors gives rise to the most geometrically rich quantum theory that can be formulated in terms of a wavefunction residing in a non-negative Hilbert space.

In the results section, we will delve into the properties and implications of this multivector-based quantum mechanical theory. We commence with a concise overview of entropy maximization techniques as employed in statistical mechanics, followed by a summary of our previous work applying these techniques to quantum mechanics, and finally, their generalization to multivectors.

The microcanonical ensemble of statistical mechanics (SM) can be derived from an entropy maximization problem:

Definition 1 (Lagrange equation of SM).

$$\mathcal{L}(\rho, \lambda, \beta) = \underbrace{-k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right)}_{\text{Average Energy Constraint}}$$

(1)

Solving this optimization problem[2] yields the celebrated Gibbs' measure:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \beta)}{\partial \rho} = 0 \implies \rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} \exp(-\beta E(r))}}_{\text{Microcanonical Ensemble}} \underbrace{\exp(-\beta E(q))}_{\text{Gibbs' Measure}}$$

(2)

Inspired by the result of Gibbs, in our previous work [1], we reformulated QM as a solution to an entropy maximization problem. The Lagrange equation defining the optimization problem is:

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Definition 2 (Lagrange equation of QM)**.**

$$\mathcal{L}(\rho, \lambda, t) = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative} \\ \text{Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{t/\hbar \left(\operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right)}_{\text{Phase Anti-Constraint}}$$

$$(3)$$

The phase anti-constraint serves as a formal device to expand the solution space, allowing for the incorporation of complex phases into the probability measure. As it expands rather the constrict the solution space, the expression is the *opposite* of a constraint — hence we named it an anti-constraint.

Theorem 1. *Solving this optimization problem yields the Born rule as the probability measure, $p(q)$ as the wavefunction initial state, and a partition function that is unitarily invariant:*

$$\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho} = 0 \implies \rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \| \exp(-itE(r)/\hbar) \|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\| \exp(-itE(q)/\hbar) \|}_{\text{Born Rule}} \underbrace{p(q)}_{\text{Initial State}}$$

$$(4)$$

The solution resolves[1] into the five canonical axioms of QM [3, 4].

Proof. The optimization problem is solved as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{5}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{6}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{7}$$

$$\implies \rho(q) = p(q) \exp(-1 - \lambda) \exp \left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right) \tag{8}$$

$$= \frac{1}{Z(\tau)} p(q) \exp \left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right) \tag{9}$$

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The partition function is obtained as follows:

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-1 - \lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right)$$

(10)

$$\implies (\exp(-1 - \lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right)$$

(11)

$$Z(\tau) := \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right)$$

(12)

The probability measure is given by:

$$\rho(q) = \frac{p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}{\sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right)}$$

(13)

Transforming the representation of complex numbers from $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ to $a + ib$ and associating the exponential trace with the complex norm using $\exp \operatorname{tr} \mathbf{M} \equiv \det \exp \mathbf{M}$, we obtain:

$$\exp \operatorname{tr} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \quad (14)$$

$$= r^2(\cos^2(b) + \sin^2(b)) \quad (15)$$

$$= \|r(\cos(b) + i\sin(b))\| \quad (16)$$

$$= \|r \exp(ib)\| \quad (17)$$

Substituting $\tau = t/\hbar$ and applying the complex-norm representation to both the numerator and denominator yields the following probability measure:

$$\rho(q) = \frac{1}{\sum_{r \in \mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\|} \|\exp(-itE(q)/\hbar)\| p(q) \quad (18)$$

$\square$

Let us recall the five principal axioms of the canonical formalism of QM [3, 4]:

Axiom 1 **State Space:** Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

Axiom 2 **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

Axiom 3 **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

Axiom 4 **Measurement:** The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

Axiom 5 **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

We now explore how these axioms are recovered from the expanded solution space engendered by the anti-constraint.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate, visualized as a vector within a complex n-dimensional Hilbert space, with the partition function acting as the inner product:

$$\sum_{r \in \mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\| = Z = \langle \psi | \psi \rangle \tag{19}$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE(q_1)/\hbar) & & \\ & \ddots & \\ & & \exp(-itE(q_n)/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{20}$$

Here, $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p(q_i) = \langle \psi_i(0) | \psi_i(0) \rangle$, and Z is invariant under unitary transformations.

The axioms of quantum mechanics are recovered as follows:

1. The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle \psi | \psi \rangle}$, linking $|\psi\rangle$ to a unit vector in Hilbert space. As the POP formulation of QM associates physical states with its probability measure, and the probability is defined up to a phase, physical states map to rays within Hilbert space, demonstrating Axiom 1.

2. In Z, an observable must satisfy:

$$\overline{O} = \sum_{r \in \mathbb{Q}} p(r) O(r) \|\exp(-itE(r)/\hbar)\| \tag{21}$$

Since $Z = \langle \psi | \psi \rangle$, any self-adjoint operator satisfying $\langle \mathbf{O}\psi | \phi \rangle = \langle \psi | \mathbf{O}\phi \rangle$ will equate the above equation, demonstrating Axiom 2.

3. Transforming Equation 20 out of its eigenbasis through unitary operations, the energy $E(q)$ typically transforms as a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle \tag{22}$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier:

$$\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)\left|\psi(0)\right\rangle) \tag{23}$$

$$= -i\mathbf{H}/\hbar\exp(-it\mathbf{H}/\hbar)\left|\psi(0)\right\rangle \tag{24}$$

$$= -i\mathbf{H}/\hbar\left|\psi(t)\right\rangle \tag{25}$$

$$\implies \mathbf{H}\left|\psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle \tag{26}$$

which is the Schrödinger equation, demonstrating Axiom 3.

4. From Equation 20, the possible microstates $E(q)$ of the system correspond to the eigenvalues of $\mathbf{H}$. An observation can be conceptualized as sampling from $\rho(q,t)$, with the post-measurement state being the occupied microstate q of $\mathbb{Q}$. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, corresponding to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability distribution is:

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger\psi(q,t). \tag{27}$$

In scenarios where the probability measure $\rho(q,\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on an eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{28}$$

Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument holds for any observable, it demonstrates Axiom 4.

5. Since the probability measure (Equation 4) replicates the Born rule, Axiom 5 is also demonstrated.

Revisiting quantum mechanics from this perspective offers a coherent and unified narrative. Specifically, the phase anti-constraint is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4, and 5) through the principle of entropy maximization. The phase anti-constraint becomes the formulation's sole axiom, and Axioms 1, 2, 3, 4, and 5 now emerge as theorems. For a more in-depth analysis of the POP in the context of QM, the reader is invited to consult our previous work [1].

In this paper, we present a natural generalization of the reformulation of quantum mechanics based on the POP methodology. We extend the "phase anti-constraint" from our previous work to a more general "geometric anti-constraint," which is the geometrically richest anti-constraint that still yields a wavefunction living in a non-negative Hilbert space. This generalization leads to a quantum theory based on multivector amplitudes. The Lagrange multiplier equation for this generalized formulation becomes:

Definition 3 (Lagrange equation of multivector-valued QM)**.**

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative} \\ \text{Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q) \right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{\tau \left(\frac{1}{d} \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q) \mathbf{M}(q) \right)}_{\text{Geometric Anti-Constraint}}$$

(29)

where d is the dimension of the space or spacetime, $\mathbf{M}$ is a traceless square matrix and τ is a Lagrange multiplier that will represent the proper time.

As we will see, the resolution of this Lagrange equation generates an extension of the five canonical axioms of QM that incorporates multivector amplitudes. This multivector-based quantum mechanical theory provides a unified framework that naturally includes the metric tensor of gravity as a quantum mechanical observable and the standard model gauge symmetries U(1), SU(2), and SU(3). Solving the optimization problem also generates all the necessary tools for a consistent quantum mechanical treatment, from non-negative inner products to observables, to self-adjointness, to superposition, to sum over geometries and interference extending them to the realm of multivector amplitudes.

2 Results

Theorem 2. *The solution to the Lagrange multiplier equation (Equation 29) resolves to the following probability measure:*

$$\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho} = 0 \implies \rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \exp\left(-\frac{1}{d}\tau \operatorname{tr} \mathbf{M}(r)\right)}}_{\text{Geometrically Invariant Ensemble}} \underbrace{\exp\left(-\frac{1}{d}\tau \operatorname{tr} \mathbf{M}(q)\right)}_{\text{Geometric Born Rule}} \underbrace{p(q)}_{\text{Initial State}}$$

(30)

Proof.

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \frac{1}{d} \operatorname{tr} \mathbf{M}(q) \tag{31}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr} \frac{1}{d}\mathbf{M}(q) \tag{32}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr} \frac{1}{d}\mathbf{M}(q) \tag{33}$$

$$\implies \rho(q) = p(q) \exp(-1 - \lambda) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(q)\right) \tag{34}$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(q)\right) \tag{35}$$

The partition function $Z(\tau)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-1 - \lambda) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(r)\right) \qquad (36)$$

$$\implies (\exp(-1 - \lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(r)\right) \qquad (37)$$

$$Z(\tau) := \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(r)\right) \qquad (38)$$

Consequently, the optimal probability distribution is given by:

$$\rho(q) = \frac{1}{\sum_{r \in \mathbb{Q}} p(r) \det \exp\left(-\frac{1}{d}\tau \mathbf{M}(r)\right)} \det \exp\left(-\frac{1}{d}\tau \mathbf{M}(q)\right) p(q) \qquad (39)$$

where $\det \exp M = \exp \operatorname{tr} M$. $\qquad \square$

This theorem generalizes the Born rule to a probability measure that is invariant under a wide range of geometric transformations. The geometrically invariant ensemble serves as a normalization factor, while the initial state $p(q)$ represents the probability associated with the initial preparation of the system.

Corollary 2.1. *QM is a special solution of Theorem 2.*

Proof.

$$\rho(q)\bigg|_{d \to 1, \mathbf{M}(q) \to \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}} = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\text{Born Rule}} \underbrace{p(q)}_{\text{Initial State}}$$

$$(40)$$

$\square$

This corollary demonstrates that quantum mechanics is a special case of the generalized probability measure derived in Theorem 2. By setting the dimension $d = 1$ and choosing the traceless matrix $\mathbf{M}(q)$ to represent a complex phase within the energy of the system, we recover the familiar Born rule and the unitarily invariant ensemble of quantum mechanics from which the five canonical axioms of QM (Theorem 1) are provable.

Corollary 2.2. *SM is a special solution of Theorem 2*

Proof.

$$\rho(q)\big|_{d \to 1, \mathbf{M}(q) \to [E(q)], p(q) \to 1} = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} \exp(-\beta E(r))}}_{\text{Microcanonical Ensemble}} \underbrace{\exp(-\beta E(q))}_{\text{Gibbs Measure}} \qquad (41)$$

$\square$

Similarly, this corollary shows that statistical mechanics is another special case of the generalized probability measure. By setting the dimension $d = 1$, choosing the traceless matrix $\mathbf{M}(q)$ to represent the energy of the system, and assuming a uniform initial state $p(q) = 1$, we recover the Gibbs measure and the microcanonical ensemble of statistical mechanics.

These corollaries illustrate the unifying power of the generalized probability measure derived in Theorem 2, as it encompasses both quantum mechanics and statistical mechanics as special cases. The theorem provides a common framework for understanding the foundations of these theories and highlights the central role of entropy maximization in their construction.

2.1 Obstructions to Multivector amplitudes in 2D

In this section, we apply Theorem 2 to a two-dimensional (2D) space, where the dimension $d = 2$ and the traceless matrix $\mathbf{M}$ is a 2×2 matrix. The probability measure in this case takes the form:

$$\rho(q) = \frac{1}{\sum_{r \in \mathbb{Q}} p(r) \det \exp\left(-\frac{1}{2}\tau \left[\begin{smallmatrix} x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q) \end{smallmatrix}\right]\right)} \det \exp\left(-\frac{1}{2}\tau \left[\begin{smallmatrix} x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q) \end{smallmatrix}\right]\right) p(q)$$

$$(42)$$

To represent this probability measure in terms of multivectors, we choose a matrix representation that is group isomorphic to the geometric algebra in 2D over the reals, denoted as $\mathrm{GA}(2) \cong \mathbb{M}(2, \mathbb{R})$:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \tag{43}$$

where the basis elements of this geometric algebra are defined as:

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{44}$$

A more compact notation for this multivector $\mathbf{u}$ is as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b} \tag{45}$$

where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

Using this notation, the evolution operator in the probability measure can be written as:

$$\exp\left(-\frac{1}{2}\tau \begin{pmatrix} x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q) \end{pmatrix}\right) = e^{-\frac{1}{2}\tau(\mathbf{x}(q)+\mathbf{b}(q))} \tag{46}$$

We now introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 4 (Multivector conjugate (a.k.a Clifford conjugate)). *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* GA(2). *The multivector conjugate is defined as:*

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{b} \tag{47}$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 3 (Determinant as a Multivector Self-Product).

$$\mathbf{u}^{\ddagger}\mathbf{u} = \det \mathbf{M_u} \tag{48}$$

Proof. Let $\mathbf{u} = a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$, and let $\mathbf{M_u}$ be its matrix representation $\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix}$. Then:

$$1: \quad \mathbf{u}^{\ddagger}\mathbf{u} \tag{49}$$

$$= (a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})^{\ddagger}(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{50}$$

$$= (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{51}$$

$$= a^2 - x^2 - y^2 + b^2 \tag{52}$$

$$2: \quad \det \mathbf{M_u} \tag{53}$$

$$= \det \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \tag{54}$$

$$= (a + x)(a - x) - (y - b)(y + b) \tag{55}$$

$$= a^2 - x^2 - y^2 + b^2 \tag{56}$$

$$\square$$

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 5 (Multivector Conjugate Transpose). *Let* $|V\rangle\!\rangle \in (\text{GA}(2))^n$:

$$|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{57}$$

The multivector conjugate transpose of $|V\rangle\!\rangle$ *is defined as first taking the transpose and then the element-wise multivector conjugate:*

$$\langle\!\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{58}$$

Definition 6 (Bilinear Form). *Let* $|V\rangle\!\rangle$ *and* $|W\rangle\!\rangle$ *be two vectors valued in* $\mathrm{GA}(2)$. *We introduce the following bilinear form:*

$$\langle\!\langle V|W\rangle\!\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \ldots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{59}$$

The partition function in Equation 42 can be expressed in terms of the general linear wavefunction multiplied against its conjugate, as follows:

Theorem 4 (Partition Function). $Z = \langle\!\langle V|V\rangle\!\rangle$

Proof.

$$\langle\!\langle V|V\rangle\!\rangle = \sum_{q\in\mathbb{Q}} V(q)^{\ddagger}V(q) = \sum_{q\in\mathbb{Q}} \det \mathbf{M}_{V(q)} = Z \tag{60}$$

$\square$

Theorem 5 (Inner Product). *In the even sub-algebra of* $\mathrm{GA}(2)$, *the bilinear form is an inner product.*

Proof.

$$\langle\!\langle V|W\rangle\!\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \ldots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{61}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$. $\square$

Since the even sub-algebra of $\mathrm{GA}(2)$ is closed with respect to addition and multiplication, and the bilinear form is an inner product, it follows can it can be used to construct an Hilbert space. As it leads to a well-defined quantum theory, we will therefore study the $\mathbf{x} \to 0$ case going forward in this section.

We now introduce the wavefunction, which is rotor-valued:

Definition 7 (Rotor-valued Wavefunction). *The rotor-valued wavefunction is defined as follows:*

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1}R_1 \\ \vdots \\ \sqrt{\rho_n}R_n \end{bmatrix} \tag{62}$$

The rotor wavefunction leads to the (2D) Dirac current:

Definition 8 (Dirac Current). *Let* $\psi(q) = \sqrt{\rho(q)}R(q)$. *Then,*

$$J \equiv \psi(q)^{\ddagger}\hat{\mathbf{x}}_{\mu}\psi(q) = \rho(q)\mathbf{e}_{\mu}(q) \tag{63}$$

The Lagrange multiplier τ leads to a proper-time valued Schrödinger equation:

Definition 9 (Rotor/proper-time Schrödinger equation).

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|\psi(\tau)\rangle\!\rangle = -\frac{1}{2}\mathbf{b}|\psi(\tau)\rangle\!\rangle \tag{64}$$

The resulting theory is very similar to David Hestenes' geometric algebra formulation of QM[5], but applied to the 2D case.

2.1.1 Obstructions

We note two obstructions:

1. The Lagrangian multiplier requires the proper time τ, but 2D contains 2 space dimensions and 0 time dimensions.

2. The 1+1D theory leads to a split-complex quantum theory because the bilinear form is $(a - b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})(a + b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})$, which resolves to negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states (versus $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ for the euclidean 2D case).

In the next section, we will investigate the 3+1D, then we will investigate obstructions in higher dimensional configurations. This will show that the 3+1D is the only multivector quantum theory which is obstruction-free.

2.2 Multivector Amplitudes in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. We begin by defining a general multivector in the geometric algebra GA(3, 1).

Definition 10 (Multivector). *Let $\mathbf{u}$ be a multivector of* GA(3, 1). *Its general form is:*

$$\mathbf{u} = a \tag{65}$$

$$+ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} + t\hat{\mathbf{t}} \tag{66}$$

$$+ f_{01}\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}} \wedge \hat{\mathbf{z}} + f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \tag{67}$$

$$+ v_0\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + v_1\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + v_2\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + v_3\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \tag{68}$$

$$+ b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \tag{69}$$

A more compact notation for $\mathbf{u}$ is

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{70}$$

where a is a scalar, $\mathbf{x}$ a vector, $\mathbf{f}$ a bivector, $\mathbf{v}$ is pseudo-vector and $\mathbf{b}$ a pseudo-scalar.

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation, which establishes a connection between the geometric algebra and matrix algebra.

Definition 11 (Matrix Representation $\mathbf{M_u}$ of $\mathbf{u}$). *In a 3+1-dimensional context, a 4×4 real matrix, $\mathbf{M}$, can be expressed using the real Majorana representation. Such a matrix has the general form:*

$$\mathbf{M} = \begin{bmatrix} a + x - f_{02} + q & -z - f_{13} + w - b & f_{03} - f_{23} - p - v & t + y + f_{01} + f_{12} \\ -z - f_{13} + w + b & a - x - f_{02} - q & -t + y + f_{01} + f_{12} & f_{03} - f_{23} - p - v \\ f_{03} + f_{23} - p + v & t + y - f_{01} + f_{12} & a + x + f_{02} - q & -z - f_{13} - w + b \\ -t + y + f_{01} - f_{12} & -f_{03} - f_{23} - p + v & -z + f_{13} - w - b & a - x + f_{02} + q \end{bmatrix}, \tag{71}$$

To manipulate and analyze multivectors in $\mathrm{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 12 (Multivector Conjugate (in 4D)).

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{72}$$

Definition 13 (3,4 Blade Conjugate). *The 3,4 blade conjugate of* $\mathbf{u}$ *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{73}$$

We can now express the determinant of the matrix representation of a multivector via a self-product[6]:

Theorem 6 (Determinant as a Multivector Self-Product).

$$\lfloor \mathbf{u}^{\ddagger} \mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger} \mathbf{u} = \det \mathbf{M_u} \tag{74}$$

Proof. Omitted due to space constraint. See [6] for a proof. $\qquad\square$

These constructions allow us to express the probability measure in terms of the multivector self-product.

Definition 14 ($\mathrm{GA}(3,1)$-valued Vector).

$$|V\rangle\!\rangle = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{f}_1 + \mathbf{v}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{f}_n + \mathbf{v}_n + \mathbf{b}_n \end{bmatrix} \tag{75}$$

Definition 15 (Multilinear Form).

$$\langle\!\langle V|V|V|V\rangle\!\rangle = \left\lfloor \begin{bmatrix} \mathbf{u}_1^{\ddagger} & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \right\rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^{\ddagger} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^{\ddagger} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{76}$$

Theorem 7 (Partition Function). $Z = \langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle$

Proof.

$$\langle\!\langle \psi|\psi|\psi|\psi \rangle\!\rangle \tag{77}$$

$$= \Big\lfloor \begin{bmatrix} \mathbf{u}_1^{\ddagger} & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \Big\rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^{\ddagger} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^{\ddagger} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{78}$$

$$= \Big\lfloor \begin{bmatrix} \mathbf{u}_1^{\ddagger}\mathbf{u}_1 & \dots & \mathbf{u}_n\mathbf{u}_n \end{bmatrix} \Big\rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^{\ddagger}\mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^{\ddagger}\mathbf{u}_n \end{bmatrix} \tag{79}$$

$$= \lfloor \mathbf{u}_1^{\ddagger}\mathbf{u}_1 \rfloor_{3,4}\mathbf{u}_1^{\ddagger}\mathbf{u}_1 + \dots + \lfloor \mathbf{u}_n^{\ddagger}\mathbf{u}_n \rfloor_{3,4}\mathbf{u}_n^{\ddagger}\mathbf{u}_n \tag{80}$$

$$= \sum_{i=1}^{n} \det \mathbf{M}_{\mathbf{u}_i} \tag{81}$$

$$= Z \tag{82}$$

$\square$

We can reduce the multilinear form to a sesquilinear form, as follows:

Definition 16 (Sesquilinear Form). *Let V and W be $GA(3,1)$-valued vectors. Then:*

$$\langle V|W \rangle \equiv \langle\!\langle V|V|W|W \rangle\!\rangle = \sum_{q\in\mathbb{Q}} \lfloor V(q)^{\ddagger}V(q) \rfloor_{3,4} W(q)^{\ddagger}W(q) \tag{83}$$

Theorem 8 (Non-negative inner product). *The sesquilinear form, applied to the even sub-algebra of $GA(3,1)$, reduces to a non-negative inner product. The resulting non-negative Hilbert space is valued in even sub-algebra of $GA(3,1)$.*

Proof. We consider an inner product between two even multivector of $GA(3,1)$.

$$|V\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix} \text{ and } |W\rangle = \begin{bmatrix} a_1' + \mathbf{f}_1' + \mathbf{b}_1' \\ \vdots \\ a_n' + \mathbf{f}_n' + \mathbf{b}_n' \end{bmatrix}$$

The, the sesquilinear form is:

$$\langle V|W\rangle \tag{84}$$

$$= \langle\!\langle V|V|W|W\rangle\!\rangle \tag{85}$$

$$= \lfloor[(a_1 + \mathbf{f}_1 + \mathbf{b}_1)^{\ddagger}(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \quad \ldots]\rfloor_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^{\ddagger}(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{86}$$

$$= \lfloor[(a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \quad \ldots]\rfloor_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{87}$$

$$= \lfloor[a_1^2 + a_1\mathbf{f}_1 + a_1\mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1\mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1\mathbf{f}_1 + \mathbf{b}_1^2 \quad \ldots]\rfloor_{3,4}\ldots \tag{88}$$

$$= \lfloor[a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 \quad \ldots]\rfloor_{3,4}\ldots \tag{89}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0 e_1 e_2 e_3 (E_1 B_1 + E_2 B_2 + E_3 B_3)$

$$= \lfloor[a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0 e_1 e_2 e_3 (E_1 B_1 + E_2 B_2 + E_3 B_3) \quad \ldots]\rfloor_{3,4}\ldots \tag{90}$$

We note that the terms are now complex numbers, which we rewrite as $\text{Re}(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\text{Im}(z) = -4(E_1 B_1 + E_2 B_2 + E_3 B_3)$

$$= \lfloor[z_1 \quad \ldots \quad z_2]\rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{91}$$

$$= [z_1^{\dagger} \quad \ldots \quad z_2^{\dagger}] \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{92}$$

$$= z_1^{\ddagger} z_1 + \cdots + z_n^{\ddagger} z_n \tag{93}$$

Which is non-negative for all even multivectors of $GA(3,1)$. The even sub-algebra of $GA(3,1)$ forms a vector space, which now armed with an inner product, can be used to construct a non-negative Hilbert space. $\qquad\square$

We now define the even-sub-algebra-valued wavefunction, or $\text{Spin}^c(3,1)$-valued wavefunction:

Definition 17 ($\text{Spin}^c(3,1)$-valued Wavefunction).

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{4}(a_1 + \mathbf{f}_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{4}(a_n + \mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt[4]{\rho_1} R_1 B_1 \\ \vdots \\ \sqrt[4]{\rho_n} R_n B_n \end{bmatrix} \tag{94}$$

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where R_i is a rotor and B_i is a phase.

The evolution operator of the partition function, valued in the even sub-algebra, becomes:

Definition 18 (Spin^c(3, 1) Flow)**.**

$$e^{-\frac{1}{4}\tau(\mathbf{f}(q)+\mathbf{b}(q))} \tag{95}$$

In turn, this leads to the following Schrödinger equation:

Definition 19 (Spin^c(3, 1) Generating Schrödinger equation)**.**

$$\frac{d}{dt}\psi(t) = -\frac{1}{2}(\mathbf{f} + \mathbf{b})\psi(\tau) \tag{96}$$

The consistency and compatibility of the geometric algebra formalism with the standard quantum formalism are further re-enforced by the unitary evolution of states and the self-adjointness of observables.

Theorem 9 (Unitary Evolution)**.**

Proof.

$$\langle S\psi|S\phi\rangle \tag{97}$$
$$= \langle\!\langle S\psi|S\psi|S\phi|S\phi\rangle\!\rangle \tag{98}$$
$$= \sum_{q\in\mathbb{Q}} \lfloor (S\psi(q))^{\ddagger}S\psi(q)\rfloor_{3,4}(S\phi(q))^{\ddagger}S\phi(q) \tag{99}$$
$$= \sum_{q\in\mathbb{Q}} \lfloor \psi(q)^{\ddagger}S^{\ddagger}S\psi(q)\rfloor_{3,4}\phi(q)^{\ddagger}S^{\ddagger}S\phi(q) \tag{100}$$

It follows that for S with elements in $a + \mathbf{f} + \mathbf{b}$, then $S^{\ddagger}S = U$, where U is in $a + \mathbf{b}$. Then U commutes with $\psi(q)$ and $\phi(q)$.

$$= \sum_{q\in\mathbb{Q}} \lfloor \psi(q)^{\ddagger}\psi(q)\rfloor_{3,4}\phi(q)^{\ddagger}\lfloor U\rfloor_{3,4}U\phi(q) \tag{101}$$

which in the case where the elements of U are in $a + \mathbf{b}$, implies

$$U^{\dagger}U = I \tag{102}$$

which is the condition for an unitary operator in a complex Hilbert Space. $\square$

Theorem 10 (Self-Adjoint Observables)**.**

Proof. Let $|\psi\rangle\!\rangle$ and $|\phi\rangle\!\rangle$ be Spin^c(3, 1)-valued wavefunctions:

$$\langle A\psi|\phi\rangle \tag{103}$$
$$= \langle\!\langle A\psi|A\psi|\phi|\phi\rangle\!\rangle \tag{104}$$
$$= \sum_{q\in\mathbb{Q}} \lfloor (A\psi(q))^{\ddagger}A\psi(q)\rfloor_{3,4}\phi(q)^{\ddagger}\phi(q) \tag{105}$$
$$= \sum_{q\in\mathbb{Q}} \lfloor \psi(q)^{\ddagger}A^{\ddagger}A\psi(q)\rfloor_{3,4}\phi(q)^{\ddagger}\phi(q) \tag{106}$$

Then, from the other side we have:

$$\langle\psi|A\psi\rangle \tag{107}$$

$$= \langle\!\langle\psi|\psi|A\psi|A\psi\rangle\!\rangle \tag{108}$$

$$= \sum_{q\in\mathbb{Q}} \lfloor\psi(q)^{\ddagger}\psi(q)\rfloor_{3,4}(A\phi(q))^{\ddagger}A\phi(q) \tag{109}$$

$$= \sum_{q\in\mathbb{Q}} \lfloor\psi(q)^{\ddagger}\psi(q)\rfloor_{3,4}\phi(q)^{\ddagger}A^{\ddagger}A\phi(q) \tag{110}$$

It follows that for A with elements in $a + \mathbf{f} + \mathbf{b}$, then the elements of $A^{\ddagger}A$ are in $a + \mathbf{b}$. This implies that $(A^{\ddagger}A)^{\dagger} = A^{\ddagger}A$. Finally, posing $O = A^{\ddagger}A$, we obtain

$$O^{\dagger} = O \tag{111}$$

$\square$

The metric, however, requires the self-adjointness of the multilinear form itself; the bilinear form is insufficient:

Theorem 11 (Metric). *Let $\psi(q) = \sqrt[4]{\rho(q)}R(q)B(q)$, where $R(q)$ is a rotor, and $B(q)$ is a phase. Then,*

$$\rho(q)g_{\mu\nu}(q) = \langle\psi(q)|\gamma_{\mu}\psi(q)|\psi(q)|\gamma_{\nu}\psi(q)\rangle \tag{112}$$

We also note that $\rho(q)g_{\mu\nu}(q) = \langle\gamma_{\mu}\psi(q)|\psi(q)|\gamma_{\nu}\psi(q)|\psi(q)\rangle$ is also true. Consequently, in 3+1D, the basis of space time γ_{μ} and γ_{ν} are observables since $\langle\gamma_{\mu}\psi(q)|\psi(q)|\gamma_{\nu}\psi(q)|\psi(q)\rangle = \langle\psi(q)|\gamma_{\mu}\psi(q)|\psi(q)|\gamma_{\nu}\psi(q)\rangle$, leading to the metric tensor $g_{\mu\nu}$, when measured.

Proof.

$$\langle\psi(q)|\gamma_{\mu}\psi(q)|\psi(q)|\gamma_{\nu}\psi(q)\rangle \tag{113}$$

$$= \lfloor\sqrt[4]{\rho}\tilde{R}B\gamma_{\mu}\sqrt[4]{\rho}RB\rfloor_{3,4}\sqrt[4]{\rho}\tilde{R}B\gamma_{\nu}\sqrt[4]{\rho}RB \tag{114}$$

(We have dropped the dependence on (q) to improve legibility). We note that $B\gamma_{\mu}B = \gamma_{\mu}$, because the pseudo-scalar anti-commutes with vectors. Therefore,

$$= \rho\mathbf{e}_{\mu}\mathbf{e}_{\nu} \tag{115}$$

$$= \rho g_{\mu\nu} \tag{116}$$

For completeness, we also investigate the self-adjoint:

$$\langle\gamma_{\mu}\psi(q)|\psi(q)|\gamma_{\nu}\psi(q)|\psi(q)\rangle \tag{117}$$

$$= \lfloor\sqrt[4]{\rho}\tilde{R}B(-\gamma_{\mu})\sqrt[4]{\rho}RB\rfloor_{3,4}\sqrt[4]{\rho}\tilde{R}B(-\gamma_{\nu})\sqrt[4]{\rho}RB \tag{118}$$

$$= \lfloor\sqrt[4]{\rho}\tilde{R}B\gamma_{\mu}\sqrt[4]{\rho}RB\rfloor_{3,4}\sqrt[4]{\rho}\tilde{R}B\gamma_{\nu}\sqrt[4]{\rho}RB \tag{119}$$

$$= \rho g_{\mu\nu} \tag{120}$$

$\square$

As we recall in 2D, the insertion of the γ_μ within the probability measure produced the Dirac current ($\rho(q)\mathbf{e}_\mu = \psi(q)^{\ddagger}\hat{\mathbf{x}}_\mu\psi(q)$). Whereas, in 3+1D the analogous operation replaces the Dirac current (a probability density involving a 4-vector) with a probability density involving the metric tensor.

We will now show that the theory contains the U(1), SU(2), and SU(3) gauge symmetries, which play a fundamental role in the standard model of particle physics. The multilinear form contains no other symmetries with respect to the γ_0 basis.

To show this invariance with respect to the metric, we will utilize the γ_0 basis:

Theorem 12 (U(1) invariance). *[7, 8]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \quad (121)$$

Proof.

$$\langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \quad (122)$$

$$= \lfloor\psi(q)^{\ddagger}e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^{\ddagger}e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q) \quad (123)$$

$$= \lfloor\psi(q)^{\ddagger}\gamma_0 e^{-\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^{\ddagger}\gamma_0 e^{-\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{b}}\psi(q) \quad (124)$$

$$= \lfloor\psi(q)^{\ddagger}\gamma_0\psi(q)\rfloor_{3,4}\psi(q)^{\ddagger}\gamma_0\psi(q) \quad (125)$$

$$= \langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle \quad (126)$$

$\square$

Theorem 13 (SU(2) invariance). *[7, 8]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \quad (127)$$

implies $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$, *which generates* SU(2).

Proof.

$$\langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \quad (128)$$

$$= \lfloor\psi(q)^{\ddagger}e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rfloor_{3,4}\psi(q)^{\ddagger}e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q) \quad (129)$$

We now identify the relation $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0$, which is true only if $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_1 + \theta_3\gamma_0\gamma_3$:

$$e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_1-\theta_3\gamma_0\gamma_3-B_1\gamma_2\gamma_3-B_2\gamma_1\gamma_3-B_3\gamma_1\gamma_2}\gamma_0 \quad (130)$$

$$= \gamma_0 e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_1-\theta_3\gamma_0\gamma_3+B_1\gamma_2\gamma_3+B_2\gamma_1\gamma_3+B_3\gamma_1\gamma_2} \quad (131)$$

therefore the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $B_1 = B_2 = B_3 = 0$:

Finally, we note that $e^{\theta_1\gamma_0\gamma_1+\theta_2\gamma_0\gamma_1+\theta_3\gamma_0\gamma_3}$ generates SU(2). $\square$

Theorem 14 (SU(3) invariance). *[7, 8]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)|\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)\rangle \quad (132)$$

Proof. The relation that must remain invariant is $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$. Let $\mathbf{f} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_1 + E_3\gamma_0\gamma_1 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_1 + E_3\gamma_0\gamma_1 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \qquad (133)$$

Then the parts in γ_0 anti-commute with γ_0, and the parts commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_1 + E_3\gamma_0\gamma_1 - B_1\gamma_2\gamma_3 - B_2\gamma_1\gamma_3 - B_3\gamma_1\gamma_2)\mathbf{f} \qquad (134)$$

To can be written as

$$\gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \qquad (135)$$
$$= \gamma_0(\mathbf{E}^2 + \mathbf{EB} - \mathbf{BE} - \mathbf{B}^2) \qquad (136)$$

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we need 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{EB} - \mathbf{BE}$. The second requirement simply means that $\mathbf{E}$ and $\mathbf{B}$ commute, and the first means

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) \qquad (137)$$

which is simply the SU(3) symmetry group. $\qquad\square$

The following theorem provides a general expression for the interference pattern arising from the superposition of two general multivectors or even multivectors, which generalizes the complex interference commonly found in standard QM. This interference can lead to a sum over geometries:

Theorem 15 (Multivector Superposition and Interference)**.**

Proof. The general form of geometric interference, which includes diffeomorphisms, Spin^c and flux interference, for a superposition $\mathbf{u}_1$ with $\mathbf{u}_2$, is given as follows:

$$\lfloor(\mathbf{u}_1 + \mathbf{u}_2)^\ddagger(\mathbf{u}_1 + \mathbf{u}_2)\rfloor_{3,4}(\mathbf{u}_1 + \mathbf{u}_2)^\ddagger(\mathbf{u}_1 + \mathbf{u}_2) \qquad (138)$$
$$= (\mathbf{u}_1^* + \mathbf{u}_2^*)(\mathbf{u}_1^\dagger + \mathbf{u}_2^\dagger)(\mathbf{u}_1^\ddagger + \mathbf{u}_2^\ddagger)(\mathbf{u}_1 + \mathbf{u}_2) \qquad (139)$$
$$= (\mathbf{u}_1^*\mathbf{u}_1^\dagger + \mathbf{u}_1^*\mathbf{u}_2^\dagger + \mathbf{u}_2^*\mathbf{u}_1^\dagger + \mathbf{u}_2^*\mathbf{u}_2^\dagger)(\mathbf{u}_1^\ddagger\mathbf{u}_1 + \mathbf{u}_1^\ddagger\mathbf{u}_2 + \mathbf{u}_2^\ddagger\mathbf{u}_1 + \mathbf{u}_2^\ddagger\mathbf{u}_2) \qquad (140)$$
$$= \underbrace{\mathbf{u}_1^*\mathbf{u}_1^\dagger\mathbf{u}_1^\ddagger\mathbf{u}_1}_{\rho_1} + \underbrace{\mathbf{u}_2^*\mathbf{u}_2^\dagger\mathbf{u}_2^\ddagger\mathbf{u}_2}_{\rho_2} + \underbrace{\mathbf{u}_1^*\mathbf{u}_1^\dagger\mathbf{u}_1^\ddagger\mathbf{u}_2 + \mathbf{u}_1^*\mathbf{u}_1^\dagger\mathbf{u}_2^\ddagger\mathbf{u}_1 + 12 \text{ terms}}_{\text{geometric interference pattern}} \qquad (141)$$

$\qquad\square$

Let us now discuss gravity.

In the multivector amplitude formalism, the fundamental objects for a formulation of the Einstein-Hilbert action are available from within the theory. The action can be formulated using the metric tensor observable and an SO(3,1)-valued spin connection ω, which would be a gauge that leaves the Spin^c(3,1)-valued wavefunction invariant. A crude example for constructing spacetime

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could involve measurements identically-prepared quantum systems, using the gamma matrices γ_μ and γ_ν as operators. Such measurements produces $g_{\mu\nu}$ as the observable. As measurements of the spacetime geometry are accumulated, a progressively more complete picture of spacetime geometry emerges. This gives a quantum character to the metric tensor used in the EFE.

In summary, this section provides a foundation for a unified, geometrically intuitive formulation of quantum mechanics, gravity, and particle physics in 3+1 dimensions using multivector amplitudes. The geometric algebra $GA(3,1)$ offers a powerful framework for describing the fundamental interactions of particles and spacetime geometry, naturally incorporating U(1), SU(2), SU(3), Spin(3,1), and SO(3,1) gauge symmetries, as well as the metric tensor as a quantum mechanical object.

2.3 Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to extend the multivector amplitude formalism to dimensions other than 3+1D. We begin by examining the self-products associated with low-dimensional geometric algebras.

Definition 20. *From the results of [6], the self-products associated with low-dimensional geometric algebras are:*

$$CL(0,1): \qquad \varphi^\dagger \varphi \tag{142}$$

$$CL(2,0): \qquad \varphi^\ddagger \varphi \tag{143}$$

$$CL(3,0): \qquad \lfloor \varphi^\ddagger \varphi \rfloor_3 \varphi^\ddagger \varphi \tag{144}$$

$$CL(3,1): \qquad \lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi \tag{145}$$

$$CL(4,1): \qquad (\lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi)^\dagger (\lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi) \tag{146}$$

From Theorem 6, and the results obtained in the previous sections, we have seen that in the $CL(3,1)$ case, the self-product corresponds to the determinant of the matrix representation of the corresponding geometric algebra and can be interpreted as a probability measure associated with many physical phenomena. However, when we investigate other dimensions, we encounter several obstructions that prevent the construction of a consistent and physically meaningful probability measure.

The first obstruction arises in the case of $CL(0,1)$, $CL(3,0)$, and higher odd-dimensional geometric algebras, where the determinant of the matrix representation is complex-valued and, consequently, cannot represent a probability.

Theorem 16. *For* $CL(0,1)$*,* $CL(3,0)$*, and higher odd-dimensional geometric algebras, the determinant of the matrix representation is complex-valued and, consequently, cannot represent a probability.*

Proof. The probabilities in the POP framework are defined by the determinant of a matrix. 3D geometric algebra is represented by 2x2 complex matrices, and

the determinant of such matrices is complex, not real. Hence, the probabilities are complex-valued, not real-valued, making the solution unphysical in 3D. In 0+1D, the GA is isomorphic to the complex numbers, and the determinant of a complex number is the complex number itself. Since odd-dimensional geometric algebras map to complex-valued matrices, this is also the case with 5D geometric algebra and higher odd-dimensional spaces. □

This theorem highlights the fundamental issue with odd-dimensional geometric algebras, where the complex-valued determinant of the matrix representation cannot be interpreted as a physically meaningful probability measure.

The second obstruction concerns the lack of a corresponding geometric algebra formulation for certain matrix dimensions, which limits the ability to define a wavefunction in terms of multivectors, necessary for defining an amplitude.

Theorem 17. *For 1×1, 3×3, or any higher odd-dimensional matrices, there is no corresponding geometric algebra formulation. It is, therefore, not possible to represent the determinant as a self-product of multivectors, which limits the ability to define a wavefunction.*

Proof. All geometric algebras, regardless of signature or dimension, map to even-dimensional square matrices. This means that odd-dimensional square matrices, such as 3x3 matrices, do not have a corresponding geometric algebra formulation and thus cannot define an amplitude. □

This theorem emphasizes the importance of having a geometric algebra formulation for the matrix representation, as it allows for the definition of a wavefunction in terms of multivectors and the construction of an amplitude based on the multivector self-product.

As we move to higher dimensions, we encounter further obstructions that prevent the construction of a consistent probability measure and the satisfaction of observables. In particular, the multivector representation of the norm in 6D fails to extend the self-product patterns found in lower dimensions.

Conjecture 1. *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 20 collapse. The research by Acus et al.[9] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (147)$$

This equation is not a multivector self-product but a linear sum of two multivector self-products.

The full expression [9] is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials}\rangle = 0 \tag{148}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials}\rangle = 0 \tag{149}$$

$$\langle 74 \text{ monomials}\rangle = 0 \tag{150}$$

$$\langle 74 \text{ monomials}\rangle = 0 \tag{151}$$

From Equation 147, it is possible to see that no observable O can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{152}$$

Any equality of the above type between $b_1 \mathbf{O}$ and $b_2 \mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it questionable that a satisfactory quantum theory (with observables) be constructible in 6D. $\square$

This theorem demonstrates that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a consistent quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2. *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

Finally, we consider the specific case of four dimensions and show that the POP method requires a 3+1D signature to maintain consistency with the previously established results.

Theorem 18. *The POP method in four dimensions specifically requires a 3+1D signature.*

Proof. Starting with 4x4 real matrices as our solution, we are restricted to choosing a geometric algebra isomorphic to it. In 4D, the options are:

1. GA(3, 1) is isomorphic to the algebra of 4×4 real matrices, denoted as $M(4, \mathbb{R})$.

2. GA(1, 3) is isomorphic to the algebra of 2×2 quaternionic matrices, denoted as $M(2, \mathbb{H})$ or $\mathbb{H}(2)$.

3. GA(4, 0) is isomorphic to the direct sum of two copies of the algebra of 2×2 real matrices, denoted as $M(2, \mathbb{R}) \oplus M(2, \mathbb{R})$.

4. GA(2, 2) is isomorphic to the algebra of 4×4 real matrices, denoted as $M(4, \mathbb{R})$.

5. $GA(0,4)$ is isomorphic to the algebra of 2×2 quaternionic matrices, denoted as $M(2, \mathbb{H})$ or $\mathbb{H}(2)$.

This leaves only the choice of either $GA(3,1)$ or $GA(2,2)$ as signatures of interest. $\square$

Conjecture 3 (Obstruction in $GA(2,2)$). *The maximization problem introduces a single Lagrange multiplier τ, governing the time evolution of systems, leading to possible obstructions when applied to a spacetime with multiple time dimensions, such as $GA(2,2)$.*

Conjecture 4 (Obstruction in $GA(4,0)$ and $GA(0,4)$ and $GA(2,0)$). *The maximization problem introduces a single Lagrange multiplier τ, governing the time evolution of systems, leading to obstructions when applied to a spacetime with no time dimensions, such as $GA(0,4)$, $GA(4,0)$ or $GA(2,0)$.*

Theorem 19 (Obstruction in 1+1D). *We repeat the obstruction found in 1+1D, leading to negative probabilities because the bilinear norm resolves to $a^2 - b^2$.*

These conjectures provide additional insights into the unique role of the 3+1D signature in the POP method. The conjecture regarding the obstruction in $GA(2,2)$ suggests that the presence of multiple time dimensions may lead to complications in the formalism due to the introduction of a single Lagrange multiplier governing the time evolution of systems. This highlights the importance of the 3+1D signature, which has a single time dimension, in maintaining consistency with the Lagrange equation.

The dimensional obstructions encountered in this section provide valuable insights into the limitations of the geometric algebra approach and the specific requirements for constructing a consistent and physically meaningful quantum formalism. It suggests a plausible mechanism for the specific dimensional arrangement of the universe deeply linked to the mathematical good behavior of geometric probabilities measures.

3 Discussion

The Geometric Anti-Constraint as the Sole Axiom

The geometric anti-constraint, given by $0 = \frac{1}{d} \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}(q)$, serves as the sole axiom of the theory in our formulation. This constraint shapes the optimization problem and determines the structure of the resulting quantum theory. Just as the average energy constraint $\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q)$ in statistical mechanics yields the Gibbs measure, and the phase anti-constraint $0 = \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}$, a special case of the geometric anti-constraint, in our previous work leads to the five canonical axioms of quantum mechanics, the geometric anti-constraint resolves into a quantum theory that naturally incorporates multivector amplitudes.

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The power of multivector amplitudes lies in their ability to encapsulate the essential features of both particle physics and gravitation within a single framework. The gauge symmetries of the standard model, namely U(1), SU(2), and SU(3), arise naturally from the invariance of the probability measure under the transformations generated by the bivectors of the geometric algebra. Similarly, the geometry of spacetime emerges from the geometric anti-constraint through the invariance of the probability measure under infinitesimal diffeomorphisms and Lorentz transformations. This remarkable property suggests that the geometric anti-constraint contains the necessary information to describe the fundamental interactions of particles and fields, as well as the geometry of spacetime, without the need for ad hoc assumptions or additional postulates.

Addressing the Relativistic Nature of the Schrödinger Equation

A common objection to the relativistic nature of our theory arises from the use of the Schrödinger equation or Schrödinger-like time evolution. However, it is crucial to distinguish between the general Schrödinger equation itself and the non-relativistic single-particle Schrödinger equation. The Schrödinger equation, given by $|\psi(t)\rangle = e^{-itH/\hbar}|\psi(0)\rangle$, is relativistic provided H is relativistic. This formulation is equivalent to the Feynman path integral representation, which is manifestly compatible with relativity.

To illustrate this point, let's consider the example of a free scalar field. In the Feynman path integral representation, the action for a free scalar field $\phi(x)$ is given by:

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right), \tag{153}$$

where m is the mass of the scalar field. The path integral is then defined as:

$$Z = \int \mathcal{D}\phi \, e^{iS[\phi]/\hbar}, \tag{154}$$

which sums over all possible field configurations weighted by the exponential of the action.

In the Hamiltonian formulation, the Schrödinger equation for the free scalar field is given by:

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \tag{155}$$

where the Hamiltonian $\hat{H}$ is obtained from the Legendre transform of the Lagrangian:

$$\hat{H} = \int d^3x \left(\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 \right), \tag{156}$$

with $\hat{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ being the canonical momentum operator.

The solution to the Schrödinger equation is given by:

$$|\Psi(t)\rangle = e^{-it\hat{H}/\hbar}|\Psi(0)\rangle, \tag{157}$$

which describes the time evolution of the quantum state $|\Psi(t)\rangle$.

The relativistic compatibility of both the Feynman path integral representation and the Hamiltonian formulation using the Schrödinger equation are dependant on the Lagrangian or Hamiltonian used, and not on the choice of representation.

The POP methodology resolves to the Schrödinger picture, yet this does not prevent it from being relativistic.

Probability Density

Let us now extend the entropy maximization problem from the discreet Σ to the continuum $\int$, using a Riemann sum:

$$\mathcal{L} = -\lim_{n\to\infty}\left(\sum_{i=1}^{n}\rho(x_i)\ln\frac{\rho(x_i)}{p(x_i)} + \lambda\left(1 - \sum_{i=1}^{n}\rho(x_i)\right) + \tau\left(\text{tr}\sum_{i=1}^{n}\rho(x_i)\frac{1}{\varepsilon(x_i)}\begin{bmatrix}0 & -E(x_i)\\ E(x_i) & 0\end{bmatrix}\right)\right)\Delta x$$

$$(158)$$

where

- n is the number of subintervals,

- $\Delta x = (b-a)/n$ is the width of each subinterval,

- x_i is a point within the i-th subinterval $[x_{i-1}, x_i]$, often chosen to be the midpoint $(x_{i-1} + x_i)/2$.

- $1/\varepsilon(x_i)$ is a factor required to transform the energy $E(x)$ into an energy density $\mathcal{E}(x) = E(x)/\varepsilon(x)$, required for integration.

which yields an integral:

$$\mathcal{L} = -\int_a^b \rho(x)\ln\frac{\rho(x)}{p(x)}\mathrm{d}x + \lambda\left(1 - \int_a^b \rho(x)\mathrm{d}x\right) + \tau\left(\text{tr}\int_a^b \rho(x)\frac{1}{\varepsilon(x)}\begin{bmatrix}0 & -E(x)\\ E(x) & 0\end{bmatrix}\mathrm{d}x\right)$$

$$(159)$$

Solving this optimization problem yields a probability measure parametrized over the continuum.

We can extend this formulation to multivector amplitudes by using the geometric anti-constraint and parametrized over a world manifold X^4:

$$\mathcal{L} = -\int_a^b \rho(x^\mu)\ln\frac{\rho(x^\mu)}{p(x^\mu)}\sqrt{-g}\mathrm{d}^4x + \lambda\left(1 - \int_a^b \rho(x^\mu)\sqrt{-g}\mathrm{d}^4x\right) + \tau\left(\text{tr}\int_a^b \frac{1}{4}\rho(x^\mu)\frac{1}{\varepsilon(x^\mu)}\mathbf{M}(x^\mu)\sqrt{-g}\mathrm{d}^4x\right)$$

$$(160)$$

The solution to this optimization problem is a probability density:

$$\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho} = 0 \implies \rho(x^\mu) = \underbrace{\frac{1}{\int_a^b p(x^\mu) \exp\left(-\frac{1}{4}\tau \frac{1}{\varepsilon(x^\mu)} \operatorname{tr} \mathbf{M}(x^\mu)\right) \sqrt{-g} \mathrm{d}^4 x}}_{\text{Geometrically Invariant Ensemble}} \underbrace{\exp\left(-\frac{1}{4}\tau \frac{1}{\varepsilon(x^\mu)} \operatorname{tr} \mathbf{M}(x^\mu)\right)}_{\text{Geometric Born Rule}} \underbrace{p(x^\mu)}_{\text{Initial State}}$$

(161)

This formulation extends the multivector amplitude framework to the continuum, allowing for the description of continuous systems while preserving the geometric structure and invariance properties of the theory.

Double copy gauge theory

The U(1), SU(2), and SU(3) invariances in the multivector amplitude formalism lead to a double copy structure of gauge theories, as each side of the multilinear form can evolve independently. For instance in the SU(3) case:

$$\langle \psi(q) | \gamma_0 \psi(q) | \psi(q) | \gamma_0 \psi(q) \rangle = \langle \mathbf{f}_1 \psi(q) | \gamma_0 \mathbf{f}_1 \psi(q) | \mathbf{f}_2 \psi(q) | \gamma_0 \mathbf{f}_2 \psi(q) \rangle \qquad (162)$$

This results in two separately conserved SU(3) gauge theories:

$$- \mathbf{f}_1 \gamma_0 \mathbf{f}_1 = \gamma_0 \implies \text{SU(3) as copy 1} \qquad (163)$$

$$- \mathbf{f}_2 \gamma_0 \mathbf{f}_2 = \gamma_0 \implies \text{another SU(3) as copy 2} \qquad (164)$$

This argument also holds for U(1) and SU(2).

A potential future research direction could be to investigate whether this double copy structure is connected to the double copy theory[10], which aims to express gravity as a double copy of a gauge theory. Exploring this relationship may provide further insights into the interaction picture of quantum gravity.

4   Conclusion

In conclusion, this paper advances the 'Prescribed Observation Problem' (POP) into a multivector quantum theory, seamlessly bridging the realms of quantum mechanics and spacetime geometry. Our findings reveal the POP's exceptional ability to generate a mathematically well-behaved theory that generalizes quantum probabilities through the introduction of the multivector probability measure, a generalization of the Born rule. This measure is invariant under a wide range of geometric transformations, including those generated the gauge groups of the standard model, and leading to the metric tensor as a quantum mechanical observables, without the need for additional assumptions beyond the geometric anti-constraint. Remarkably, multivector amplitudes are found to be consistent only with a 3+1D spacetime, encountering obstructions in other dimensional configurations. This finding aligns with the observed dimensionality and gauge symmetries of the universe and suggests a possible explanation for

its specificity. This research represents a significant step in reconciling quantum mechanics with general relativity, challenging and expanding conventional methodologies in theoretical physics, and potentially paving the way for new insights in the field.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

- Data Availability Statement: No datasets were generated or analyzed during the current study.

- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

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