Quantum Gravity as the Solution to a Maximization Problem on the Entropy of All Geometric Measurements

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Abstract

A quantum theory utilizing multivector amplitudes instead of complex amplitudes has been developed within the framework of geometric algebra. This theory generalizes the Born rule to a multivector probability measure that is invariant under a wide range of geometric transformations. In this formalism, the gamma matrices become self-adjoint operators, enabling the construction of the metric tensor as a quantum observable, and the Schrödinger equation becomes the active generator of arbitrary metric transformations. Furthermore, by requiring time invariance of the probability measure under all multi-vectorial amplitude transformations, specifically the gauge symmetries $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ retain conserved charge density, thus introducing them without the need for additional assumptions. Remarkably, the multivector amplitude formalism is found to be consistent only with 3+1-dimensional spacetime, encountering various obstructions in other dimensional configurations. This finding aligns with the observed dimensionality of the universe and suggests a possible explanation for the specific gauge symmetries of the Standard Model. Furthermore, the incorporation of the metric tensor as a quantum observable provides a natural pathway to integrate gravity with quantum mechanics.

1 Introduction

In this paper, we introduce a novel quantum theory that employs multivector amplitudes instead of complex amplitudes. The theory is entirely derived by solving an entropy maximization problem, yielding a probability measure and an associated vector space in which the multivector-valued wavefunction resides. The maximization problem also generates the complete set of requisite mathematical tools for a comprehensive quantum mechanical treatment, including a

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product form yielding non-negative probabilities, an evolution operator, transition amplitudes, superposition, interference, and observables, all generalized to the geometric domain via multivectors. By formulating the theory as a solution to an entropy optimization problem, its consistency and well-definedness are mathematically assured.

Within this framework, we find that the gamma matrices are elevated to the status of self-adjoint operators, enabling the construction of the metric tensor as a quantum observable. Remarkably, the gauge symmetries of the standard model of particle physics, namely U(1), SU(2), and SU(3), along with their associated conserved charge density, naturally emerge to preserve the time invariance of the probability measure under multi-vectorial amplitude transformations. Furthermore, multivector amplitudes are found to be free of obstructions exclusively in 3+1D spacetime, potentially offering insights into the dimensional specificity of the universe.

This innovative approach to quantum mechanics extends the 'Prescribed Observation Problem' (POP), a methodology we previously proposed [1], which applies entropy maximization techniques, well-established in statistical mechanics, to derive the axioms of quantum mechanics from first principles. The natural extension of this methodology to multivectors gives rise to the most geometrically rich quantum theory that can be formulated in terms of a wavefunction residing in a vector space and possessing a product form yielding non-negative probabilities.

In the results section, we will delve into the properties and implications of this multivector-based quantum mechanical theory. We commence with a concise overview of entropy maximization techniques as employed in statistical mechanics, followed by a summary of our previous work applying these techniques to quantum mechanics, and finally, their generalization to multivectors.

Statistical Mechanics

Let us now begin with entropy maximization in the field of statistical mechanics (SM). The microcanonical ensemble of SM can be derived from an entropy maximization problem:

Definition 1 (Lagrange multiplier equation of SM)**.**

$$\mathcal{L}(\rho,\lambda,\beta) = \underbrace{-k_B \sum_{q\in\mathbb{Q}} \rho(q)\ln\rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\beta\left(\overline{E}-\sum_{q\in\mathbb{Q}}\rho(q)E(q)\right)}_{\text{Average Energy Constraint}} \tag{1}$$

Solving this optimization problem[2] yields the celebrated Gibbs' measure:

$$\frac{\partial\mathcal{L}(\rho,\lambda,\beta)}{\partial\rho} = 0 \implies \rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}}\exp(-\beta E(r))}}_{\text{Microcanonical Ensemble}}\underbrace{\exp(-\beta E(q))}_{\text{Gibbs' Measure}} \tag{2}$$

Quantum Mechanics

Inspired by the result of Gibbs, in our previous work [1], we reformulated QM as a solution to an entropy maximization problem. The Lagrange equation defining the optimization problem is:

Definition 2 (Lagrange multiplier equation of QM)**.**

$$\mathcal{L}(\rho,\lambda,t) = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{Relative}\\ \text{Shannon}\\ \text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + \underbrace{t/\hbar\left(\operatorname{tr}\sum_{q\in\mathbb{Q}}\rho(q)\left[\begin{smallmatrix}0 & -E(q)\\ E(q) & 0\end{smallmatrix}\right]\right)}_{\text{Phase Anti-Constraint}} \tag{3}$$

The phase anti-constraint serves as a formal device to expand the solution space, allowing for the incorporation of complex phases into the probability measure. As it expands rather the constrict the solution space, the expression is the *opposite* of a constraint — hence we named it an anti-constraint.

Theorem 1. *Solving this optimization problem yields the Born rule as the probability measure, $p(q)$ as the wavefunction initial state, and a partition function that is unitarily invariant:*

$$\frac{\partial\mathcal{L}(\rho,\lambda,t)}{\partial\rho} = 0 \implies \rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\|}}_{\textit{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\textit{Born Rule}} \underbrace{p(q)}_{\textit{Initial State}} \tag{4}$$

The solution resolves[1] into the five canonical axioms of QM [3, 4].

Proof. The optimization problem is solved as follows:

$$\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} = -\ln\frac{\rho(q)}{p(q)} - 1 - \lambda - \tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix} \tag{5}$$

$$0 = \ln\frac{\rho(q)}{p(q)} + 1 + \lambda + \tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix} \tag{6}$$

$$\implies \ln\frac{\rho(q)}{p(q)} = -1 - \lambda - \tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix} \tag{7}$$

$$\implies \rho(q) = p(q)\exp(-1-\lambda)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right) \tag{8}$$

$$= \frac{1}{Z(\tau)}p(q)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right) \tag{9}$$

The partition function is obtained as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r)\exp(-1-\lambda)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{10}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{11}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{12}$$

The probability measure is given by:

$$\rho(q) = \frac{p(q)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}{\sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right)} \tag{13}$$

Transforming the representation of complex numbers from $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ to $a + ib$ and associating the exponential trace with the complex norm using $\exp \operatorname{tr} \mathbf{M} \equiv \det \exp \mathbf{M}$, we obtain:

$$\exp \operatorname{tr}\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right] = \det \exp \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right] = r^2 \det \left[\begin{smallmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{smallmatrix}\right], \text{ where } r = \exp a \tag{14}$$

$$= r^2(\cos^2(b) + \sin^2(b)) \tag{15}$$

$$= \|r(\cos(b) + i\sin(b))\| \tag{16}$$

$$= \|r\exp(ib)\| \tag{17}$$

Substituting $\tau = t/\hbar$ and applying the complex-norm representation to both the numerator and denominator yields the following probability measure:

$$\rho(q) = \frac{1}{\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\|}\|\exp(-itE(q)/\hbar)\|p(q) \tag{18}$$

□

Let us recall the five principal axioms of the canonical formalism of QM [3, 4]:

Axiom 1 **State Space:** Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

Axiom 2 **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

Axiom 3 **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

Axiom 4 **Measurement:** The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

Axiom 5 **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

We now explore how these axioms are recovered from the expanded solution space engendered by the anti-constraint.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate, visualized as a vector within a complex n-dimensional Hilbert space, with the partition function acting as the inner product:

$$\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\| = Z = \langle\psi|\psi\rangle \tag{19}$$

where

$$\begin{bmatrix}\psi_1(t)\\ \vdots \\ \psi_n(t)\end{bmatrix} = \begin{bmatrix}\exp(-itE(q_1)/\hbar) & & \\ & \ddots & \\ & & \exp(-itE(q_n)/\hbar)\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \vdots \\ \psi_n(0)\end{bmatrix} \tag{20}$$

Here, $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p(q_i) = \langle\psi_i(0)|\psi_i(0)\rangle$, and Z is invariant under unitary transformations.

The axioms of quantum mechanics are recovered as follows:

1. The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle\psi|\psi\rangle}$, linking $|\psi\rangle$ to a unit vector in Hilbert space. As the POP formulation of QM associates physical states with its probability measure, and the probability is defined up to a phase, physical states map to rays within Hilbert space, demonstrating Axiom 1.

2. In Z, an observable must satisfy:

$$\overline{O} = \sum_{r\in\mathbb{Q}} p(r)O(r)\|\exp(-itE(r)/\hbar)\| \tag{21}$$

Since $Z = \langle\psi|\psi\rangle$, any self-adjoint operator satisfying $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$ will equate the above equation, demonstrating Axiom 2.

3. Transforming Equation 20 out of its eigenbasis through unitary operations, the energy $E(q)$ typically transforms as a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle \tag{22}$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}(\exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle) && (23)\\
&= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle && (24)\\
&= -i\mathbf{H}/\hbar |\psi(t)\rangle && (25)\\
\implies \mathbf{H} |\psi(t)\rangle &= i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle && (26)
\end{aligned}$$

which is the Schrödinger equation, demonstrating Axiom 3.

4. From Equation 20, the possible microstates $E(q)$ of the system correspond to the eigenvalues of $\mathbf{H}$. An observation can be conceptualized as sampling from $\rho(q,t)$, with the post-measurement state being the occupied microstate q of $\mathbb{Q}$. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, corresponding to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability distribution is:

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger \psi(q,t). \tag{27}$$

In scenarios where the probability measure $\rho(q,\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on an eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{28}$$

Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument holds for any observable, it demonstrates Axiom 4.

5. Since the probability measure (Equation 4) replicates the Born rule, Axiom 5 is also demonstrated.

Revisiting quantum mechanics from this perspective offers a coherent and unified narrative. Specifically, the phase anti-constraint is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4, and 5) through the principle of entropy maximization. The phase anti-constraint becomes the formulation's sole axiom, and Axioms 1, 2, 3, 4, and 5 now emerge as theorems. For a more in-depth analysis of the POP in the context of QM, the reader is invited to consult our previous work [1].

Multivector Amplitudes

In this paper, we present a natural generalization of the reformulation of quantum mechanics based on the POP methodology. We extend the "phase anti-constraint" from our previous work to a more general "geometric anti-constraint," which is the geometrically richest anti-constraint that resolves into a wavefunction living in a vector space and into a non-negative product form associated with probabilities. This generalization leads to a quantum theory based on multivector amplitudes. The Lagrange multiplier equation for this generalized formulation becomes:

Definition 3 (Lagrange multiplier equation of multivector-valued QM)**.**

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative} \\ \text{Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{\tau \left(\frac{1}{d} \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q)\mathbf{M}(q)\right)}_{\text{Geometric Anti-Constraint}} \tag{29}$$

where $\rho(q)$ is the probability distribution, d is the dimension of the space or spacetime, $\mathbf{M}$ is a traceless square matrix and τ is a Lagrange multiplier that will represent the proper time.

As we will see, the resolution of this Lagrange equation generates an extension of QM that incorporates multivector amplitudes. Solving the optimization problem also generates all the necessary tools for a consistent quantum mechanical treatment of the multivector-valued quantum theory.

2 Results

Theorem 2. *The solution to the Lagrange multiplier equation (Equation 29) resolves to the following distribution:*

$$\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho} = 0 \implies \rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \exp\left(-\frac{1}{d}\tau \operatorname{tr} \mathbf{M}(r)\right)}}_{\text{Geometrically Invariant Ensemble}} \underbrace{\exp\left(-\frac{1}{d}\tau \operatorname{tr} \mathbf{M}(q)\right)}_{\text{Geometric Born Rule}} \underbrace{p(q)}_{\text{Initial State}} \tag{30}$$

Proof.

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \frac{1}{d} \operatorname{tr} \mathbf{M}(q) \tag{31}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q) \tag{32}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q) \tag{33}$$

$$\implies \rho(q) = p(q) \exp(-1 - \lambda) \exp\left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q)\right) \tag{34}$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q)\right) \tag{35}$$

The partition function $Z(\tau)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(r)\right) \tag{36}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(r)\right) \tag{37}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \frac{1}{d}\mathbf{M}(r)\right) \tag{38}$$

Consequently, the optimal solution is given by:

$$\rho(q) = \frac{1}{\sum_{r\in\mathbb{Q}} p(r) \det\exp\left(-\frac{1}{d}\tau\mathbf{M}(r)\right)} \det\exp\left(-\frac{1}{d}\tau\mathbf{M}(q)\right) p(q) \tag{39}$$

where $\det\exp M = \exp\operatorname{tr} M$. □

The resulting distribution is invariant under a wide range of geometric transformations. The partition function serves as a normalization factor, $p(q)$ is the initial value of the distribution and the determinant generalizes the Born rule.

Corollary 2.1. *QM is a special solution of Theorem 2.*

Proof.

$$\rho(q)\Bigg|_{d\to 1, \mathbf{M}(q)\to\begin{bmatrix} 0 & -E(q) \\ E(q) & 0\end{bmatrix}} = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\text{Born Rule}} \underbrace{p(q)}_{\text{Initial State}} \tag{40}$$

□

This corollary demonstrates that quantum mechanics is a special case of the distribution $\rho(q)$. By setting the dimension d to 1 and choosing the traceless matrix $\mathbf{M}(q)$ to represent a complex phase within the energy of the system, we recover the familiar Born rule and the unitarily invariant ensemble of quantum mechanics from which the five canonical axioms of QM (Theorem 1) are provable.

Corollary 2.2. *SM is a special solution of Theorem 2*

Proof.

$$\rho(q)\big|_{d\to 1, \mathbf{M}(q)\to[E(q)], p(q)\to 1} = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} \exp(-\beta E(r))}}_{\text{Microcanonical Ensemble}} \underbrace{\exp(-\beta E(q))}_{\text{Gibbs Measure}} \tag{41}$$

□

Similarly, this corollary shows that statistical mechanics is another special case of the generalized probability measure. By setting the dimension d to 1, choosing the traceless matrix $\mathbf{M}(q)$ to represent the energy of the system, and assuming a uniform initial state $p(q) = 1$, we recover the Gibbs measure and the microcanonical ensemble of statistical mechanics.

The theorem and associated corollaries provides a common framework for understanding the foundations of these theories (e.g. SM, QM and Multivector-valued QM) and highlights the central role of entropy maximization in their construction.

2.1 Obstructions to Multivector amplitudes in 2D

In this section, we apply Theorem 2 to a two-dimensional (2D) space, where the dimension $d = 2$ and the traceless matrix $\mathbf{M}$ is a 2×2 matrix. Although all dimensional configurations except 3+1D contain obstructions, which will be discussed later in this section, the 2D case provides a valuable starting point before addressing the more complex 3+1D case. The distribution in 2D takes the form:

$$\rho(q) = \frac{1}{\sum_{r\in\mathbb{Q}} p(r) \det\exp\left(-\frac{1}{2}\tau\begin{bmatrix} x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q)\end{bmatrix}\right)} \det\exp\left(-\frac{1}{2}\tau\begin{bmatrix} x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q)\end{bmatrix}\right) p(q) \tag{42}$$

To represent this distribution in terms of multivectors, we choose a matrix representation that is group isomorphic to the geometric algebra in 2D over the reals, denoted as $\mathrm{GA}(2) \cong \mathbb{M}(2, \mathbb{R})$:

Definition 4 (Matrix Representation of 2D Multivectors)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \tag{43}$$

where the basis elements of this geometric algebra are defined as:

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{44}$$

A more compact notation for this multivector $\mathbf{u}$ is as follows:

Definition 5 (Compact Notation)**.**

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b} \tag{45}$$

where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

The evolution operator of the distribution can be written as a multivector:

Definition 6 (Evolution Operator).

$$\exp\left(-\frac{1}{2}\tau\left(\begin{smallmatrix} x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q)\end{smallmatrix}\right)\right) = e^{-\frac{1}{2}\tau(\mathbf{x}(q)+\mathbf{b}(q))} \tag{46}$$

We now introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 7 (Multivector conjugate (a.k.a Clifford conjugate)). *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* GA(2)*. The multivector conjugate is defined as:*

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{b} \tag{47}$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 3 (Determinant as a Multivector Self-Product).

$$\mathbf{u}^{\ddagger}\mathbf{u} = \det \mathbf{M}_{\mathbf{u}} \tag{48}$$

Proof. Let $\mathbf{u} = a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$, and let $\mathbf{M}_{\mathbf{u}}$ be its matrix representation $\left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x\end{smallmatrix}\right]$. Then:

$$\begin{aligned}
1: \quad & \mathbf{u}^{\ddagger}\mathbf{u} && (49)\\
& = (a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})^{\ddagger}(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) && (50)\\
& = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) && (51)\\
& = a^2 - x^2 - y^2 + b^2 && (52)\\
2: \quad & \det \mathbf{M}_{\mathbf{u}} && (53)\\
& = \det \left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x\end{smallmatrix}\right] && (54)\\
& = (a+x)(a-x) - (y-b)(y+b) && (55)\\
& = a^2 - x^2 - y^2 + b^2 && (56)
\end{aligned}$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 8 (Multivector Conjugate Transpose). *Let* $|V\rangle\!\rangle \in (\mathrm{GA}(2))^n$*:*

$$|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{57}$$

The multivector conjugate transpose of $|V\rangle\!\rangle$ is defined as first taking the transpose and then the element-wise multivector conjugate:

$$\langle\!\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{58}$$

Definition 9 (Bilinear Form)**.** *Let $|V\rangle\!\rangle$ and $|W\rangle\!\rangle$ be two vectors valued in* GA(2)*. We introduce the following bilinear form:*

$$\langle\!\langle V|W\rangle\!\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{59}$$

The partition function (Equation 42) can be expressed using the bilinear form:

Theorem 4 (Partition Function)**.** $Z = \langle\!\langle V|V\rangle\!\rangle$

Proof.

$$\langle\!\langle V|V\rangle\!\rangle = \sum_{q\in\mathbb{Q}} V(q)^\ddagger V(q) = \sum_{q\in\mathbb{Q}} \det \mathbf{M}_{V(q)} = Z \tag{60}$$

□

Theorem 5 (Invariance of the Partition Function w.r.t. the Evolution Operator)**.**

$$\sum_{q\in\mathbb{Q}} \det(\exp(-\tau(\mathbf{x}(q) + \mathbf{b}(q))/2)) \det \mathbf{M}_{V(q)} = \sum_{q\in\mathbb{Q}} \det \mathbf{M}_{V(q)} \tag{61}$$

Proof. Since the evolution operator is valued in $\mathrm{SL}(2,\mathbb{R})$, it follows that its determinant is 1, therefore the sum indeeds reduces to Z. □

Theorem 6 (Inner Product)**.** *In the even sub-algebra of* GA(2)*, the bilinear form is an inner product.*

Proof.

$$\langle\!\langle V|W\rangle\!\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{62}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$. □

Since the even sub-algebra of GA(2) is closed under addition and multiplication, and the bilinear form constitutes an inner product, it follows that it can be employed to construct a Hilbert space. Furthermore, in the even sub-algebra the distribution $\rho(q)$ becomes a probability distribution. As this leads to a well-defined quantum theory in the familiar sense, we will first focus on the $\mathbf{x} \to 0$ case, then will revisit $\mathbf{x}$ in Section 2.3 on quantum gravity.

We now introduce the wavefunction, which is rotor-valued:

Definition 10 (Rotor-valued Wavefunction)**.** *The rotor-valued wavefunction is defined as follows:*

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n+\mathbf{b}_n)} \end{bmatrix} \tag{63}$$

The rotor wavefunction leads to the (2D) Dirac current:

Definition 11 (Dirac Current)**.** *Given an arbitrary basis* $\mathbf{e}_1$ *and* $\mathbf{e}_2$*, the Dirac current is defined as:*

$$J_1 \equiv \psi(q)^\ddagger \mathbf{e}_1 \psi(q) = \rho(q)\mathbf{e}_1'(q) \tag{64}$$

$$J_2 \equiv \psi(q)^\ddagger \mathbf{e}_2 \psi(q) = \rho(q)\mathbf{e}_2'(q) \tag{65}$$

where $\mathbf{e}_1'$ *and* $\mathbf{e}_2'$ *are a SO(2) rotated frame field.*

The resulting theory is very similar to David Hestenes' geometric algebra formulation of QM[5] and shares its interpretation, but applied to the 2D case. As such, J associates to the probability of finding the particle within the rotated frame field $\mathbf{e}_1$ and $\mathbf{e}_2$ upon measurement.

2.1.1 Obstructions

We identify three obstructions in the 2D case:

1. The Lagrange multiplier requires the proper time τ, but the 2D space considered contains 2 spatial dimensions and 0 time dimensions, leading to an inconsistency.

2. The 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})(a + b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. Consequently, 1+1D would solve the first obstruction at the cost of introducing another.

3. In 2D, the matrices $\hat{\mathbf{x}}_\mu$ are not operators because they are not self-adjoint. Although often used in the context defining the Dirac current, their non-status as observables prevent the construction of the metric tensor as a quantum observable. The benefits of having the basis matrices $\hat{\mathbf{x}}_\mu$ as operators will become obvious in the 3+1D case, where the gamma matrices will be self-adjoint operators. Indeed, in 2D:

$$(\hat{\mathbf{x}}_\mu \mathbf{u})^\ddagger \mathbf{u} = \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu^\ddagger \mathbf{u} = \mathbf{u}^\ddagger(-\hat{\mathbf{x}}_\mu)\mathbf{u} \qquad \neq \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu \mathbf{u} \tag{66}$$

Since $(\hat{\mathbf{x}}_\mu \mathbf{u})^\ddagger \mathbf{u} \neq \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu \mathbf{u}$, it follows that $\hat{\mathbf{x}}_\mu$ is not self-adjoint.

In the following section, we will explore the 3+1D case, which includes the gauge symmetries of the standard model $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and supports quantum gravity, and subsequently investigate obstructions in higher-dimensional configurations. This analysis will demonstrate that the 3+1D multivector quantum theory is the only one that remains obstruction-free.

2.2 Multivector Amplitudes in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. We begin by defining a general multivector in the geometric algebra $\mathrm{GA}(3,1)$.

Definition 12 (Multivector)**.** *Let* $\mathbf{u}$ *be a multivector of* $\mathrm{GA}(3,1)$*. Its general form is:*

$$\begin{aligned}
\mathbf{u} = a & \quad (67)\\
&+ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} + t\hat{\mathbf{t}} \quad (68)\\
&+ f_{01}\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}} \wedge \hat{\mathbf{z}} + f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \quad (69)\\
&+ v_0\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + v_1\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + v_2\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + v_3\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \quad (70)\\
&+ b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \quad (71)
\end{aligned}$$

A more compact notation for $\mathbf{u}$ *is*

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (72)$$

where a *is a scalar,* $\mathbf{x}$ *a vector,* $\mathbf{f}$ *a bivector,* $\mathbf{v}$ *is pseudo-vector and* $\mathbf{b}$ *a pseudo-scalar.*

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation, which establishes a connection between the geometric algebra and matrix algebra.

Definition 13 (Matrix Representation $\mathbf{M}_{\mathbf{u}}$ of $\mathbf{u}$)**.** *In a 3+1-dimensional context, a* 4×4 *real matrix,* $\mathbf{M}$*, can be expressed using the real Majorana representation. Such a matrix has the general form:*

$$\mathbf{M} = \begin{bmatrix} a+x-f_{02}+q & -z-f_{13}+w-b & f_{03}-f_{23}-p-v & t+y+f_{01}+f_{12} \\ -z-f_{13}+w+b & a-x-f_{02}-q & -t+y+f_{01}+f_{12} & f_{03}-f_{23}-p-v \\ f_{03}+f_{23}-p+v & t+y-f_{01}+f_{12} & a+x+f_{02}-q & -z-f_{13}-w+b \\ -t+y+f_{01}-f_{12} & -f_{03}-f_{23}-p+v & -z+f_{13}-w-b & a-x+f_{02}+q \end{bmatrix}, \quad (73)$$

To manipulate and analyze multivectors in $\mathrm{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 14 (Multivector Conjugate (in 4D))**.**

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (74)$$

Definition 15 (3,4 Blade Conjugate). *The 3,4 blade conjugate of* $\mathbf{u}$ *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{75}$$

We can now express the determinant of the matrix representation of a multivector via a self-product[6]:

Theorem 7 (Determinant as a Multivector Self-Product).

$$\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_\mathbf{u} \tag{76}$$

Proof. Omitted due to space constraint. See [6] for a proof. □

Definition 16 (GA(3,1)-valued Vector).

$$|V\rangle\!\rangle = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{f}_1 + \mathbf{v}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{f}_n + \mathbf{v}_n + \mathbf{b}_n \end{bmatrix} \tag{77}$$

These constructions allow us to express the distribution in terms of the multivector self-product.

Definition 17 (Multilinear Form).

$$\langle\!\langle V|V|V|V \rangle\!\rangle = \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{78}$$

Theorem 8 (Partition Function). $Z = \langle\!\langle V|V|V|V \rangle\!\rangle$

Proof.

$$\begin{aligned}
&\langle\!\langle V|V|V|V \rangle\!\rangle && (79)\\
&= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} && (80)\\
&= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 & \dots & \mathbf{u}_n \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^\ddagger \mathbf{u}_n \end{bmatrix} && (81)\\
&= \lfloor \mathbf{u}_1^\ddagger \mathbf{u}_1 \rfloor_{3,4} \mathbf{u}_1^\ddagger \mathbf{u}_1 + \dots + \lfloor \mathbf{u}_n^\ddagger \mathbf{u}_n \rfloor_{3,4} \mathbf{u}_n^\ddagger \mathbf{u}_n && (82)\\
&= \sum_{i=1}^{n} \det \mathbf{M}_{\mathbf{u}_i} && (83)\\
&= Z && (84)
\end{aligned}$$

□

Theorem 9 (Non-negative inner product)**.** *The multilinear form, applied to the even sub-algebra of* GA$(3,1)$ *is awlays non-negative.*

Proof. Let $|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix}$. Then,

$$\langle\!\langle V|V|V|V\rangle\!\rangle \tag{85}$$

$$= \lfloor \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{86}$$

$$= \lfloor \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{87}$$

$$= \lfloor \begin{bmatrix} a_1^2 + a_1\mathbf{f}_1 + a_1\mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1\mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1\mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{88}$$

$$= \lfloor \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{89}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{90}$$

We note that the terms are now complex numbers, which we rewrite as $\mathrm{Re}(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\mathrm{Im}(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} z_1 & \dots & z_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{91}$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_2^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{92}$$

$$= z_1^\ddagger z_1 + \dots + z_n^\ddagger z_n \tag{93}$$

Which is always non-negative. □

We now define the Spin$^c(3,1)$-valued wavefunction, which is valued is the even sub-algebra of GA$(3,1)$:

Definition 18 (Spin$^c(3,1)$-valued Wavefunction)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{4}(a_1+\mathbf{f}_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{4}(a_n+\mathbf{f}_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt[4]{\rho_1}R_1B_1 \\ \vdots \\ \sqrt[4]{\rho_n}R_nB_n \end{bmatrix} \tag{94}$$

where R_i is a rotor and B_i is a phase.

The evolution operator of the partition function becomes:

Definition 19 (Spin$^c(3,1)$ Flow)**.**

$$e^{-\frac{1}{4}\tau(\mathbf{f}(q)+\mathbf{b}(q))} \tag{95}$$

In turn, this leads to a Schrödinger equation by taking the derivative of the wavefunction with respect to the Lagrange multiplier τ:

Definition 20 (Spin$^c(3,1)$ Flow Generating Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\tau) = -\frac{1}{2}(\mathbf{f}+\mathbf{b})\psi(\tau) \tag{96}$$

We will now demonstrate that the theory contains the U(1), SU(2), and SU(3) gauge symmetries, which play a fundamental role in the standard model of particle physics. To demonstrate the conservation of charge density in time, we will utilize the γ_0 basis. First, let us show that within the multilinear form the gamma matrices (which of course includes γ_0) are self-adjoint:

Theorem 10 (Self-Adjointness of the Gamma Matrices)**.**

$$\langle\!\langle \psi(q)|\gamma_\mu\psi(q)|\psi(q)|\gamma_\mu\psi(q)\rangle\!\rangle = \langle\!\langle \gamma_\mu\psi(q)|\psi(q)|\gamma_\mu\psi(q)|\psi(q)\rangle\!\rangle \tag{97}$$

Proof.

$$\begin{aligned}
1:\langle\!\langle \psi(q)|\gamma_\mu\psi(q)|\psi(q)|\gamma_\mu\psi(q)\rangle\!\rangle &= \lfloor \psi(q)^\ddagger\gamma_\mu\psi(q)\rfloor_{3,4}\psi(q)^\ddagger\gamma_\mu\psi(q) && (98)\\
2:\langle\!\langle \gamma_\mu\psi(q)|\psi(q)|\gamma_\mu\psi(q)|\psi(q)\rangle\!\rangle &= \lfloor (\gamma_\mu\psi(q))^\ddagger\psi(q)\rfloor_{3,4}(\gamma_\mu\psi(q))^\ddagger\psi(q) && (99)\\
&= \lfloor \psi(q)^\ddagger(-\gamma_\mu)\psi(q)\rfloor_{3,4}\psi(q))^\ddagger(-\gamma_\mu)\psi(q) && (100)\\
&= \lfloor \psi(q)^\ddagger\gamma_\mu\psi(q)\rfloor_{3,4}\psi(q))^\ddagger\gamma_\mu\psi(q) && (101)\\
&= \langle\!\langle \psi(q)|\gamma_\mu\psi(q)|\psi(q)|\gamma_\mu\psi(q)\rangle\!\rangle && (102)
\end{aligned}$$

□

Theorem 11 (U(1) Invariance)**.** *[7, 8]*

$$\langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \tag{103}$$

Proof.

$$
\begin{aligned}
&\langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle && (104)\\
&\quad = \lfloor \psi(q)^\ddagger e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q) && (105)\\
&\quad = \lfloor \psi(q)^\ddagger \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}}\psi(q) && (106)\\
&\quad = \lfloor \psi(q)^\ddagger \gamma_0 \psi(q)\rfloor_{3,4}\psi(q)^\ddagger \gamma_0 \psi(q) && (107)\\
&\quad = \langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle && (108)
\end{aligned}
$$

□

Theorem 12 (SU(2) Invariance)**.** *[7, 8]*

$$\langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \tag{109}$$

implies $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$*, which generates* SU(2)*.*

Proof.

$$
\begin{aligned}
&\langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle && (110)\\
&\quad = \lfloor \psi(q)^\ddagger e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q) && (111)
\end{aligned}
$$

We can now identify that the condition to preserve the equality reduces to this expression:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \tag{112}$$

We further note that moving the left most term to the right yields:

$$
\begin{aligned}
&e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_2-\theta_3\gamma_0\gamma_3-B_1\gamma_2\gamma_3-B_2\gamma_1\gamma_3-B_3\gamma_1\gamma_2}\gamma_0 e^{\frac{1}{2}\mathbf{f}} && (113)\\
&\quad = \gamma_0 e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_2-\theta_3\gamma_0\gamma_3+B_1\gamma_2\gamma_3+B_2\gamma_1\gamma_3+B_3\gamma_1\gamma_2} e^{\frac{1}{2}\mathbf{f}} && (114)
\end{aligned}
$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $B_1 = B_2 = B_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$:

Finally, we note that $e^{\theta_1\gamma_0\gamma_1+\theta_2\gamma_0\gamma_2+\theta_3\gamma_0\gamma_3}$ generates SU(2). □

Theorem 13 (SU(3) invariance)**.** *[7, 8]*

$$\langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle \mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)|\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)\rangle \tag{115}$$

Proof. From the above relation, we identify that the following expression must remain invariant: $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$. Now, let $\mathbf{f} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$. Then:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \tag{116}$$

The first three terms anticommute with γ_0, while the last three commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2\gamma_1\gamma_3 - B_3\gamma_1\gamma_2)\mathbf{f} \tag{117}$$

This can be written as:

$$\gamma_0(\mathbf{E}-\mathbf{B})(\mathbf{E}+\mathbf{B}) \tag{118}$$

$$= \gamma_0(\mathbf{E}^2+\mathbf{E}\mathbf{B}-\mathbf{B}\mathbf{E}-\mathbf{B}^2) \tag{119}$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{EB} = \mathbf{BE}$. The second requirement means that $\mathbf{E}$ and $\mathbf{B}$ must commute (and thus be isomorphic to three complex numbers), and the first implies:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \tag{120}$$

which are the defining conditions for the SU(3) symmetry group. □

We have now demonstrated that multivector-valued amplitudes offer a powerful framework naturally incorporating SU(3) × SU(2) × U(1) gauge symmetries and associated charge density conservation, and retaining invariance with respect to the SO(3, 1) and $\mathrm{Spin}^{\mathrm{c}}(3,1)$ group.

2.3 Quantum Gravity

The development of the multivector-valued quantum mechanics theory presented in this paper was guided by the pursuit of a consistent interpretation of the mathematical structures emerging from the entropy optimization problem defined by the Lagrange equation:

$$\mathcal{L}(\rho,\lambda,\tau) = -\sum_{q\in\mathbb{Q}} \rho(q)\ln\frac{\rho(q)}{p(q)} + \lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\frac{1}{d}\operatorname{tr}\sum_{q\in\mathbb{Q}}\rho(q)\mathbf{M}(q)\right) \tag{121}$$

where $\rho(q)$ is the probability distribution, d is the dimension of the spacetime, $\mathbf{M}$ is a traceless square matrix, and τ is a Lagrange multiplier representing the proper time.

As shown in the previous section, a reduction to the even sub-algebra ($\mathbf{x} \to 0, \mathbf{v} \to 0$) leads to a consistent quantum theory with non-negative probabilities. However, the price for this reduction is the elimination of the parts of the theory that allow the metric to become a dynamical quantum object. The general solution, however, suggests a wavefunction operating on a much larger group:

Definition 21 ($\mathrm{GL}^+(4,\mathbb{R})$-valued Wavefunction)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{4}(a_1+\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{4}(a_n+\mathbf{x}_n+\mathbf{f}_n+\mathbf{v}_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} e^{\frac{1}{4}a_1}e^{\frac{1}{4}(\mathbf{x}'_1+\mathbf{v}'_1+\mathbf{b}'_1)}e^{\frac{1}{4}\mathbf{f}'_1} \\ \vdots \\ e^{\frac{1}{4}a_n}e^{\frac{1}{4}(\mathbf{x}'_n+\mathbf{v}'_n+\mathbf{b}'_n)}e^{\frac{1}{4}\mathbf{f}'_n} \end{bmatrix} = \begin{bmatrix} Q_1R_1 \\ \vdots \\ Q_nR_n \end{bmatrix} \tag{122}$$

where $e^{\frac{1}{4}a_n}e^{\frac{1}{4}(\mathbf{x}'_n+\mathbf{v}'_n+\mathbf{b}'_n)}e^{\frac{1}{4}\mathbf{f}'_n}$ *is the QR decomposition of* $e^{\frac{1}{4}(a_1+\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1)}$*. The* Q_iR_i *formulation is a notational convenience to represent the* Q *and* R *part of the decomposition.*

Using a $\mathrm{GL}^+(4,\mathbb{R})$-valued wavefunction, the metric becomes a quantum object created by the wavefunction. For instance, its basis elements are simply an adjoint action:

$$\lfloor\psi^\ddagger\rfloor_{3,4}\gamma_\mu\psi \tag{123}$$
$$= e^{-\frac{1}{4}\mathbf{f}'}\underbrace{e^{\frac{1}{4}a}e^{-\frac{1}{4}(\mathbf{x}'+\mathbf{v}'+\mathbf{b}')}\gamma_\mu e^{\frac{1}{4}a}e^{\frac{1}{4}(\mathbf{x}'+\mathbf{v}'+\mathbf{b}')}}_{\mathrm{FX/SO(3,1)}}e^{\frac{1}{4}\mathbf{f}'} \tag{124}$$
$$= \underbrace{e^{-\frac{1}{4}\mathbf{f}'}\mathbf{e}_\mu e^{\frac{1}{4}\mathbf{f}'}}_{\mathrm{SO(3,1)}} \tag{125}$$

Using the adjoint action, the wavefunction applies an $\mathrm{FX/SO}(3,1)$-valued transformation to the frame field, yielding an arbitrary curvilinear basis.

The primary challenge in building a quantum theory around the $\mathrm{GL}^+(4,\mathbb{R})$ group is the non-closure of the group under addition, leading to negative probability values for certain superpositions. Efforts to repair this problem, such as by finding topological obstructions that prevent undesirable superpositions, proved unsatisfactory, either limiting the generation of arbitrary metric transformations or creating new inconsistencies.

The breakthrough came with accepting $\rho(q)$'s role as an object more general than a probability distribution. The insight was bolstered by examining the relationship between $e^{\frac{1}{4}a}$ and the 4-volume density relating to the metric tensor by $\sqrt{-|g|}$. Specifically, to construct a metric, the factor $e^{\frac{1}{4}a}$ must be applied four times to each entry of the metric; twice per basis element $\mathbf{e}_\mu$. The 4-volume density of the metric, given by the square root of the metric determinant $\sqrt{-|g|}$, scales as e^{2a}. Significantly, e^a is the square root of e^{2a}, indicating that the distribution relates to an area rather than a probability.

Consequently, the multivector-valued quantum mechanics theory in 3+1D is interpreted as fundamentally concerning entropy and area. $\rho(q)$ represents a distribution of entropy-bearing areas, with negative "probabilities" understood as differently oriented area segments - valid geometric objects.

In line of this interpretation, the Lagrange multiplier equation is corrected as follows:

Definition 22 (Lagrange Multiplier Equation of Quantum Gravity)**.**

$$\mathcal{L}(A,\kappa) = -\sum_{q\in\mathbb{Q}} A(q)\ln\frac{A(q)}{p(q)} + \kappa\left(\overline{A} - \frac{1}{d}\operatorname{tr}\sum_{q\in\mathbb{Q}} A(q)\mathbf{M}(q)\right) \tag{126}$$

where $A(q)$ is the distribution, d is the number of dimensions, $\mathbf{M}$ *is a $n\times n$ matrix and κ is the Lagrange multiplier.*

We note that we have dropped the normalization constraint $\lambda\left(1-\sum_{q\in\mathbb{Q}} A(q)\right)$ from the equation. As such, $A(q)$ is not a probability distribution, just a distribution. $A(q)$ associates to a total oriented area, which will be invariant with respect to a wide range of geometric transformations related to spacetime. We also note that here $\mathbf{M}$ is an arbitrary matrix, not just a traceless matrix as before.

Theorem 14. *The solution of the Lagrange equation of quantum gravity is:*

$$A(q) = \det\exp\left(-\frac{1}{4}\kappa\mathbf{M}(q)\right)p(q) \tag{127}$$

Proof.

$$\frac{\partial\mathcal{L}(A,\kappa)}{\partial A(q)} = -\ln\frac{A(q)}{p(q)} - \kappa\frac{1}{d}\operatorname{tr}\mathbf{M}(q) \tag{128}$$

$$0 = \ln\frac{A(q)}{p(q)} + \kappa\operatorname{tr}\frac{1}{d}\mathbf{M}(q) \tag{129}$$

$$\implies \ln\frac{A(q)}{p(q)} = -\kappa\operatorname{tr}\frac{1}{d}\mathbf{M}(q) \tag{130}$$

$$\implies A(q) = p(q)\exp\left(-\kappa\operatorname{tr}\frac{1}{d}\mathbf{M}(q)\right) \tag{131}$$

Consequently, the optimal distribution is given by:

$$A(q) = \det\exp\left(-\frac{1}{4}\kappa\mathbf{M}(q)\right)p(q) \tag{132}$$

where $\det\exp M = \exp\operatorname{tr} M$. ☐

Theorem 15 (Area-Entropy Relation)**.** *The entropy* $-\sum_{q\in\mathbb{Q}} A(q)\ln A(q)$ *leads to a thermodynamic law relating the entropy to the area.*

Proof.

$$-k_B\sum_{q\in\mathbb{Q}} A(q)\ln A(q) \tag{133}$$

$$= -k_B\det\exp\left(-\frac{1}{4}\kappa\mathbf{M}(q)\right)p(q)\ln\exp\left(-\frac{1}{4}\kappa\operatorname{tr}\mathbf{M}(q)\right)p(q) \tag{134}$$

$$= k_B\det\exp\left(-\frac{1}{4}\kappa\mathbf{M}(q)\right)p(q)\left(\frac{1}{4}\kappa\operatorname{tr}\mathbf{M}(q)\right) + kBN\det\exp\left(-\frac{1}{4}\kappa\mathbf{M}(q)\right)p(q)\ln p(q) \tag{135}$$

$$= k_B\exp\left(-\kappa a(q)\right)p(q)\left(\kappa a(q)\right) + k_B\exp(-\kappa a(q))p(q)\ln p(q) \tag{136}$$

$$= k_B\kappa\overline{A} + \gamma\ln p(q) \tag{137}$$

Changes in area are thus interpreted as a thermodynamic transformation. ☐

As an example, a surface density equal to $1/4l_p^2$ yields

$$S = k_B \frac{1}{4l_p^2}\overline{A} + \gamma \ln p(q) \tag{138}$$

which is the Bekenstein-Hawking entropy[9] (with a logarithmic correction due to the third law of thermodynamics).

The resulting distribution $A(q)$ is not a statement about probabilities, but rather about surface-size commitments used to describe the states of the wavefunction. What was mistakenly interpreted as negative probabilities are merely differently oriented areas. In situations where all the areas are similarly oriented, or where the evolution is such that they become similarly oriented, the unit vectors of the distribution adheres to the axioms of probability theory and become probability distributions, able to be measured in the quantum mechanical sense.

Let us now further explore the gravity aspects of the solution in more details.

But before we begin, let us state that the results we have previously obtained in the 3+1D section are special cases of this solution. If we take the state of the wavefunction to be the unit vectors of the engendered vector space, then they adhere to the axioms of probability theory and become typical quantum states. The multilinear form is also applicable here. Therefore, the results regarding the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ are also retained in the quantum gravity theory. Specifically:

Corollary 15.1 (Multivector-Valued QM is a special case of Quantum Gravity)**.**

$$A(q)\Big|_{\operatorname{tr}\mathbf{M}(q)\to 0, \kappa\to\tau} = \det\exp\left(-\frac{1}{4}\tau\mathbf{M}(q)\right)p(q) \tag{139}$$

This is equivalent to the multivector amplitudes theory elaborated in the previous section, provided we also introduce a rule to restrict physical states to unit vectors. (this rule replaces the lack of a normalization constraint in the optimization problem).

We are now ready to investigate the quantum gravity aspects.

The construction of the metric tensor as a quantum observable relies on the self-adjointness of the gamma matrices within the multilinear form (Theorem 10):

Theorem 16 (Metric Measurement)**.** *The metric measurement is the expectation value of the γ_μ and γ_ν operators:*

$$\langle g_{\mu\nu}\rangle = \frac{1}{2}\Big(\langle\!\langle\psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle + \langle\!\langle\psi|\gamma_\nu\psi|\psi|\gamma_\mu\psi\rangle\!\rangle\Big) \tag{140}$$

where to improve the legibility, we have dropped the explicit parametrization in (q).

Proof.

$$\frac{1}{2}\langle\!\langle\psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle+\frac{1}{2}\langle\!\langle\psi|\gamma_\nu\psi|\psi|\gamma_\mu\psi\rangle\!\rangle \tag{141}$$

$$=\frac{1}{2}\lfloor\tilde{R}\tilde{Q}\gamma_\mu QR\rfloor_{3,4}\tilde{R}\tilde{Q}\gamma_\nu QR+\frac{1}{2}\lfloor\tilde{R}\tilde{Q}\gamma_\nu QR\rfloor_{3,4}\tilde{R}\tilde{Q}\gamma_\mu QR \tag{142}$$

where $\tilde{R}=e^{-\frac{1}{4}\mathbf{f}}$ and where $\tilde{Q}=e^{\frac{1}{4}a}e^{\frac{1}{4}(-\mathbf{x}+\mathbf{v}+\mathbf{b})}$.

$$=\frac{1}{2}\sqrt{\rho}\tilde{R}\lfloor\tilde{Q}\rfloor_{3,4}\gamma_\mu\gamma_\nu QR+\frac{1}{2}\sqrt{\rho}\tilde{R}\lfloor\tilde{Q}\rfloor_{3,4}\gamma_\nu\gamma_\mu QR \tag{143}$$

because $\lfloor QR\rfloor_{3,4}\tilde{R}\tilde{Q}=e^{\frac{1}{4}a}e^{\frac{1}{4}(\mathbf{x}-\mathbf{v}-\mathbf{b})}e^{\frac{1}{4}\mathbf{f}}e^{-\frac{1}{4}\mathbf{f}}e^{\frac{1}{4}a}e^{\frac{1}{4}(-\mathbf{x}+\mathbf{v}+\mathbf{b})}=e^{\frac{1}{2}a}=\sqrt{\rho}$

$$=\frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu+\mathbf{e}_\nu\mathbf{e}_\mu) \tag{144}$$

$$=g_{\mu\nu} \tag{145}$$

As one can swap γ_μ with γ_ν and obtain the same metric tensor, the multilinear form guarantees that $g_{\mu\nu}$ is symmetric. Finally, since $\langle\gamma_\mu\psi(q)|\psi(q)|\gamma_\nu\psi(q)|\psi(q)\rangle=\langle\psi(q)|\gamma_\mu\psi(q)|\psi(q)|\gamma_\nu\psi(q)\rangle$, then γ_μ and γ_ν are self-adjoint within the multilinear form, entailing the interpretation of $g_{\mu\nu}$ as a quantum observable. □

Let us now look at the dynamics of metric transformations. This is governed by the multivectorial Schrödinger equation. The multivectorial Schrödinger equation is obtained by deriving the general solution with respect to the Lagrange multiplier κ:

Definition 23 (Multivectorial Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\begin{bmatrix}\psi_1(\kappa)\\ \vdots\\ \psi_n(\kappa)\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{x}_n+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_n\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \vdots\\ \psi_n(0)\end{bmatrix} \tag{146}$$

Let us investigate a special case of interest where both the wavefunction and the Schrödinger equation are valued in $\mathbf{x}$. The diffeomorphism-generating part of the Schrödinger equation, where $\mathbf{f},\mathbf{v},\mathbf{b}\to 0$ (leaving only $\mathbf{x}$), bears a strong resemblance to the equation that generates infinitesimal diffeomorphisms from a point p on a manifold X, commonly used in differential geometry:

$$\frac{d}{dt}\varphi_p(t)=\mathbf{x}\varphi_p(0),\text{ with initial condition }\varphi_p(0)=p \tag{147}$$

Specifically, the multivector Schrödinger equation ($\mathbf{f},\mathbf{v},\mathbf{b}\to 0,\mathbf{x}\neq 0$) for state $\psi_i(\kappa)$ reduces to:

$$\frac{d}{d\kappa}\psi_i(\kappa)=-\frac{1}{2}\mathbf{x}\psi_i(0),\text{ with initial condition }\psi_i(0)=e^{\frac{1}{2}\mathbf{x}_i} \tag{148}$$

where $e^{\frac{1}{2}\mathbf{x}_i}$, in geometric algebra, represents a point on the manifold, obtained by applying the exponential map to the vector $\frac{1}{2}\mathbf{x}_i$ in the tangent space at some origin. This $-1/2$ factor is a choice of convention that does not change the meaning of the equation.

Thus, the Schrödinger equation is the generator of active diffeomorphisms. Furthermore, as the probability measure is invariant with respect to the Schrödinger equation, it follows that the theory is invariant under active diffeomorphisms.

In the general case, the multivectorial Schrödinger equation governs the dynamics that enable the active generation of all possible metric transformations, not just diffeomorphisms. In fact, each geometric "block" is represented:

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{f}\psi_i(0), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{f}_i} \tag{149}$$

generates Spin(3,1) transformations.

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{b}\psi_i(0), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{b}_i} \tag{150}$$

generates 4D-handedness reflections

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{v}\psi_i(0), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{v}_i} \tag{151}$$

generates volume shears.

The next theorem provides a general expression for the interference pattern arising from the superposition of the $\mathrm{GL}^+(4,\mathbb{R})$-valued wavefunction, which generalizes the complex interference commonly found in standard quantum mechanics. This interference leads to a sum over geometries within the probability measure:

Theorem 17 (Multivector Superposition and Interference)**.**

Proof. Let $|V\rangle\!\rangle = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{smallmatrix}\right]$. Now suppose an Hadamard transformation yielding $|V'\rangle\!\rangle = \frac{1}{2}\left[\begin{smallmatrix}\mathbf{u}_1+\mathbf{u}_2\\ \mathbf{u}_1-\mathbf{u}_2\end{smallmatrix}\right]$. The general form of geometric interference for two-state system is as follows. Let us take the state $\mathbf{u}_1+\mathbf{u}_2$ as an example (dropping the multiplication scalars for legibility):

$$\begin{aligned}
&\lfloor(\mathbf{u}_1+\mathbf{u}_2)^\ddagger(\mathbf{u}_1+\mathbf{u}_2)\rfloor_{3,4}(\mathbf{u}_1+\mathbf{u}_2)^\ddagger(\mathbf{u}_1+\mathbf{u}_2) && (152)\\
&\quad= \lfloor(\mathbf{u}_1^\ddagger+\mathbf{u}_2^\ddagger)(\mathbf{u}_1+\mathbf{u}_2)\rfloor_{3,4}(\mathbf{u}_1^\ddagger+\mathbf{u}_2^\ddagger)(\mathbf{u}_1+\mathbf{u}_2) && (153)\\
&\quad= \lfloor(\mathbf{u}_1^\ddagger\mathbf{u}_1+\mathbf{u}_1^\ddagger\mathbf{u}_2+\mathbf{u}_2^\ddagger\mathbf{u}_1+\mathbf{u}_2^\ddagger\mathbf{u}_2)\rfloor_{3,4}(\mathbf{u}_1^\ddagger\mathbf{u}_1+\mathbf{u}_1^\ddagger\mathbf{u}_2+\mathbf{u}_2^\ddagger\mathbf{u}_1+\mathbf{u}_2^\ddagger\mathbf{u}_2) && (154)\\
&\quad= \underbrace{\lfloor\mathbf{u}_1^\ddagger\mathbf{u}_1\rfloor_{3,4}\mathbf{u}_1^\ddagger\mathbf{u}_1}_{\rho_1}+\underbrace{\lfloor\mathbf{u}_2^\ddagger\mathbf{u}_2\rfloor_{3,4}\mathbf{u}_2^\ddagger\mathbf{u}_2}_{\rho_2}+\underbrace{\lfloor\mathbf{u}_1^\ddagger\mathbf{u}_1\rfloor_{3,4}\mathbf{u}_1^\ddagger\mathbf{u}_2+13\text{ terms}}_{\text{geometric interference pattern}} && (155)
\end{aligned}$$

□

2.3.1 Fock Space

The elements of a Fock space can be constructed from individual wavefunctions by taking the symmetric or antisymmetric tensor:

$$|\psi_1, \psi_2\rangle\!\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle\!\rangle \otimes |\psi_2\rangle\!\rangle + |\psi_2\rangle\!\rangle \otimes |\psi_1\rangle\!\rangle \right) \qquad \text{Symmetric} \tag{156}$$

$$|\psi_1, \psi_2\rangle\!\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle\!\rangle \otimes |\psi_2\rangle\!\rangle - |\psi_2\rangle\!\rangle \otimes |\psi_1\rangle\!\rangle \right) \qquad \text{Anti-Symmetric} \tag{157}$$

This allows the construction of a Fock space:

$$|\phi\rangle\!\rangle = \alpha_0 |0\rangle\!\rangle + \sum_i \alpha_i |\psi_i\rangle\!\rangle + \sum_{i,j} \alpha_{ij} |\psi_i, \psi_j\rangle\!\rangle + \sum_{i,j,k} \alpha_{ijk} |\psi_i, \psi_j, \psi_k\rangle\!\rangle + \dots \tag{158}$$

where $\alpha_0, \alpha_i, \alpha_{ij}, \alpha_{ijk}, \dots$ are multi-vector valued.
Or with creation and annihilation operators, we get:

$$|\phi\rangle\!\rangle = \alpha_0 |0\rangle\!\rangle + \sum_i \alpha_i \hat{a}_i^\dagger |0\rangle\!\rangle + \sum_{i,j} \alpha_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle\!\rangle + \sum_{i,j,k} \alpha_{ijk} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k^\dagger |0\rangle\!\rangle + ... \tag{159}$$

where $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ (for bosons) or $\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}$ (for fermions).
We expand the metric measurements (Theorem 16) to an operator:

Definition 24 (Metric Operator)**.**

$$\langle \hat{g}_{\mu\nu} \rangle = \frac{1}{2} \Big(\langle\!\langle \phi | \gamma_\mu \phi | \phi | \gamma_\nu \phi \rangle\!\rangle + \langle\!\langle \phi | \gamma_\nu \phi | \phi | \gamma_\mu \phi \rangle\!\rangle \Big) \tag{160}$$

where $|\phi\rangle\!\rangle$ is a element of the Fock space.

Definition 25 (Quantum EFE)**.** *The quantum version of the Einstein Field Equation becomes:*

$$\langle \hat{G}_{\mu\nu} \rangle = \langle \hat{T}_{\mu\nu} \rangle \tag{161}$$

- *where*

$$\langle \hat{G}_{\mu\nu} \rangle = \langle \hat{R}_{\mu\nu} \rangle - \frac{1}{2} \langle \hat{g}_{\mu\nu} \rangle \langle \hat{R} \rangle \tag{162}$$

- *where*

$$\langle \hat{R}_{\mu\nu} \rangle = (1/2) \langle \hat{g}^{\lambda\sigma} \rangle (\partial_\lambda \partial_\nu \langle \hat{g}_{\mu\sigma} \rangle + \partial_\lambda \partial_\mu \langle \hat{g}_{\nu\sigma} \rangle - \partial_\lambda \partial_\sigma \langle \hat{g}_{\mu\nu} \rangle - \partial_\nu \partial_\mu \langle \hat{g}_{\lambda\sigma} \rangle) \tag{163}$$

$$+ \langle \hat{g}^{\lambda\sigma} \rangle \langle \hat{g}^{\rho\tau} \rangle (\langle \hat{\Gamma}_{\lambda\rho\mu} \rangle \langle \hat{\Gamma}_{\sigma\tau\nu} \rangle - \langle \hat{\Gamma}_{\lambda\rho\nu} \rangle \langle \hat{\Gamma}_{\sigma\tau\mu} \rangle) \tag{164}$$

- *where*

$$\langle \hat{\Gamma}_{\lambda\rho\mu} \rangle = (1/2)(\partial_\rho \langle \hat{g}_{\lambda\mu} \rangle + \partial_\lambda \langle \hat{g}_{\rho\mu} \rangle - \partial_\mu \langle \hat{g}_{\lambda\rho} \rangle) \tag{165}$$

- *where*

$$\langle \hat{R} \rangle = \langle \hat{R}_{\mu\nu} \rangle \langle \hat{g}^{\mu\nu} \rangle \tag{166}$$

In many statistical physics theory, fluctuations are often the first prediction to confirm a theory. Consequently, we give the explicit form here. Here, the metric fluctuations are defined using the standard definition of fluctuations in statistical mechanics:

Definition 26 (Metric Fluctuations)**.**

$$\sigma(\hat{g}_{\mu\nu})^2 = \langle \hat{g}_{\mu\nu}^2 \rangle - \langle \hat{g}_{\mu\nu} \rangle^2 \tag{167}$$

which implies a less-than-smooth microscopic spacetime, that reduces to a smooth spacetime in the absence of fluctuations; i.e. when there are no superpositions, or after measurements.

2.3.2 A Geometric Twist on Einstein's Dice

Einstein famously remarked, "God does not play dice." In light of our proposal, it may appear that Einstein was right: God plays with disks, not dice. Specifically, oriented disks.

The entropy in 4D spacetime is associated with oriented area elements, or "disks." This arises from the fact that the determinant of the metric tensor in 4D contains 16 products of $e^{\frac{1}{4}a}$, yielding e^{4a}. The square root of the determinant of the metric tensor, which gives the 4-volume density, scales as e^{2a}. The square root of this 4-volume density scaling, e^a, corresponds to the scaling of an area element and matches the factor found in the multilinear form of the theory. Thus, entropy-bearing oriented disks are the geometric objects that solves the problem of maximizing the entropy of all possible measurements in 4D spacetime.

But the game changes in different dimensions. In 2D space, God trades disks for sticks. The determinant of the metric tensor in 2D contains 4 products of $e^{\frac{1}{2}a}$, yielding e^{2a}. The square root of this expression, e^a, corresponds to the scaling of a line element, matching the factor in the theory's bilinear form in 2D. Therefore, in 2D space, entropy-bearing oriented line elements, or "sticks," solves the problem of maximizing the entropy of all possible geometric measurements.

Moving up to 6D space, God finally picks up the dice. The determinant of the metric tensor in 6D contains 24 products of $e^{\frac{1}{6}a}$, yielding e^{4a}. The 6D hyper-volume scaling is given by the square root of this expression, e^{2a}. The square root of this 6D hyper-volume scaling, e^a, corresponds to the scaling of a

3D volume element, matching the factor in the determinant of a 6x6 matrix in the theory. Thus, in 6D space, entropy-bearing oriented 3D volume elements, or "dice," are the geometric objects that solves the problem of maximizing the entropy of all possible geometric measurements.

In summary, while Einstein was right that God does not play dice in 4D spacetime, the multivector-valued quantum mechanics theory suggests that the divine game varies across dimensions. God plays with sticks in 2D, disks in 4D, and finally rolls the dice in 6D.

2.4 Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to extend the multivector amplitude formalism to dimensions other than 3+1D. We begin by examining the self-products associated with low-dimensional geometric algebras.

Definition 27. *From the results of [6], the self-products associated with low-dimensional geometric algebras are:*

$$\mathrm{CL}(0,1): \qquad \varphi^{\dagger}\varphi \tag{168}$$

$$\mathrm{CL}(2,0): \qquad \varphi^{\ddagger}\varphi \tag{169}$$

$$\mathrm{CL}(3,0): \qquad \lfloor \varphi^{\ddagger}\varphi \rfloor_{3} \varphi^{\ddagger}\varphi \tag{170}$$

$$\mathrm{CL}(3,1): \qquad \lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi \tag{171}$$

$$\mathrm{CL}(4,1): \qquad (\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi)^{\dagger} (\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi) \tag{172}$$

From Theorem 7, and the results obtained in the previous sections, we have seen that in the $\mathrm{CL}(3,1)$ case, the self-product corresponds to the determinant of the matrix representation of the corresponding geometric algebra and can be interpreted as a probability measure associated with many physical phenomena. However, when we investigate other dimensions, we encounter several obstructions that prevent the construction of a consistent and physically meaningful probability measure.

The first obstruction arises in the case of $\mathrm{CL}(0,1)$, $\mathrm{CL}(3,0)$, and higher odd-dimensional geometric algebras, where the determinant of the matrix representation is complex-valued and, consequently, cannot represent a probability.

Theorem 18. *For* $\mathrm{CL}(0,1)$*,* $\mathrm{CL}(3,0)$*, and higher odd-dimensional geometric algebras, the determinant of the matrix representation is complex-valued and, consequently, cannot represent a probability.*

Proof. The probabilities in the POP framework are defined by the determinant of a matrix. 3D geometric algebra is represented by 2x2 complex matrices, and the determinant of such matrices is complex, not real. Hence, the probabilities are complex-valued, not real-valued, making the solution unphysical in 3D. In 0+1D, the GA is isomorphic to the complex numbers, and the determinant of a complex number is the complex number itself. Since odd-dimensional geometric algebras map to complex-valued matrices, this is also the case with 5D geometric algebra and higher odd-dimensional spaces. □

This theorem highlights the fundamental issue with odd-dimensional geometric algebras, where the complex-valued determinant of the matrix representation cannot be interpreted as a physically meaningful probability measure.

The second obstruction concerns the lack of a corresponding geometric algebra formulation for certain matrix dimensions, which limits the ability to define a wavefunction in terms of multivectors, necessary for defining an amplitude.

Theorem 19. *For 1×1, 3×3, or any higher odd-dimensional matrices, there is no corresponding geometric algebra formulation. It is, therefore, not possible to represent the determinant as a self-product of multivectors, which limits the ability to define a wavefunction.*

Proof. All geometric algebras, regardless of signature or dimension, map to even-dimensional square matrices. This means that odd-dimensional square matrices, such as 3x3 matrices, do not have a corresponding geometric algebra formulation and thus cannot define an amplitude. □

This theorem emphasizes the importance of having a geometric algebra formulation for the matrix representation, as it allows for the definition of a wavefunction in terms of multivectors and the construction of an amplitude based on the multivector self-product.

As we move to higher dimensions, we encounter further obstructions that prevent the construction of a consistent probability measure and the satisfaction of observables. In particular, the multivector representation of the norm in 6D fails to extend the self-product patterns found in lower dimensions.

Conjecture 1. *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 27 collapse. The research by Acus et al.[10] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{173}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products.

The full expression [10] is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{174}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{175}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{176}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{177}$$

From Equation 173, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1\mathbf{O}Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2Bg_5(g_4(B)g_3(g_2(B)g_1(B))) = b_1Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2\mathbf{O}Bg_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{178}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory quantum theory (with observables) be constructible in 6D. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a consistent quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2. *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

Finally, we consider the specific case of four dimensions and show that the POP method requires a 3+1D signature to maintain consistency with the previously established results.

Theorem 20. *The POP method in four dimensions specifically requires a 3+1D signature.*

Proof. Starting with 4x4 real matrices as our solution, we are restricted to choosing a geometric algebra isomorphic to it. In 4D, the options are:

1. $\mathrm{GA}(3,1)$ is isomorphic to the algebra of 4×4 real matrices, denoted as $\mathrm{M}(4,\mathbb{R})$.

2. $\mathrm{GA}(1,3)$ is isomorphic to the algebra of 2×2 quaternionic matrices, denoted as $\mathrm{M}(2,\mathbb{H})$ or $\mathbb{H}(2)$.

3. $\mathrm{GA}(4,0)$ is isomorphic to the direct sum of two copies of the algebra of 2×2 real matrices, denoted as $\mathrm{M}(2,\mathbb{R}) \oplus \mathrm{M}(2,\mathbb{R})$.

4. $\mathrm{GA}(2,2)$ is isomorphic to the algebra of 4×4 real matrices, denoted as $\mathrm{M}(4,\mathbb{R})$.

5. $\mathrm{GA}(0,4)$ is isomorphic to the algebra of 2×2 quaternionic matrices, denoted as $\mathrm{M}(2,\mathbb{H})$ or $\mathbb{H}(2)$.

This leaves only the choice of either $\mathrm{GA}(3,1)$ or $\mathrm{GA}(2,2)$ as signatures of interest. □

Conjecture 3 (Obstruction in $\mathrm{GA}(2,2)$). *The maximization problem introduces a single Lagrange multiplier τ, governing the time evolution of systems, leading to possible obstructions when applied to a spacetime with multiple time dimensions, such as* $\mathrm{GA}(2,2)$.

Conjecture 4 (Obstruction in $\mathrm{GA}(4,0)$ and $\mathrm{GA}(0,4)$ and $\mathrm{GA}(2,0)$). *The maximization problem introduces a single Lagrange multiplier τ, governing the time evolution of systems, leading to obstructions when applied to a spacetime with no time dimensions, such as* $\mathrm{GA}(0,4)$, $\mathrm{GA}(4,0)$ *or* $\mathrm{GA}(2,0)$.

Theorem 21 (Obstruction in 1+1D). *We repeat the obstruction found in 1+1D, leading to negative probabilities because the bilinear norm resolves to $a^2 - b^2$.*

These theorems and conjectures provide additional insights into the unique role of the 3+1D signature in the POP method. It suggests a plausible mechanism for the specific dimensional arrangement of the universe deeply linked to the mathematical good behavior of multivector amplitudes.

3 Discussion

3.1 Maximizing the Entropy of Geometric Measurements

The multivector-valued quantum mechanics theory presented in this paper can be interpreted as an optimization problem that maximizes the entropy of all possible geometric measurements of nature. The Lagrange equation of quantum gravity serves as the foundation for this interpretation:

$$\mathcal{L}(A,\kappa) = -\sum_{q\in Q} A(q)\ln\frac{A(q)}{p(q)} + \kappa\left(A - \frac{1}{4}\mathrm{tr}\sum_{q\in Q} A(q)M(q)\right) \tag{179}$$

In this equation, $A(q)$ represents the distribution that maximizes the entropy of all possible geometric measurements in four dimensions. The term $-\sum_{q\in Q} A(q)\ln\frac{A(q)}{p(q)}$ is the relative Shannon entropy, which quantifies the uncertainty associated with the geometric measurements. The term $\kappa\left(A - \frac{1}{4}\mathrm{tr}\sum_{q\in Q} A(q)M(q)\right)$ is the geometric anti-constraint, which shapes the optimization problem and determines the structure of the resulting quantum theory.

By solving this optimization problem, we have obtained the probability measure:

$$A(q) = \det\exp\left(-\frac{1}{4}\kappa M(q)\right)p(q) \tag{180}$$

This probability measure is the least biased distribution consistent with the geometric measurements, as it maximizes the entropy while satisfying the geometric anti-constraint. The method of entropy maximization, which has been rigorously proven in the field of statistical mechanics, guarantees that the derived probability measure is the least biased solution to the optimization problem.

The geometric anti-constraint plays a crucial role in shaping the optimization problem and determining the structure of the resulting quantum theory. It contains the necessary information to describe the fundamental interactions of particles and fields, as well as the geometry of spacetime, without the need for additional assumptions or postulates. The fact that the multivector-valued quantum mechanics theory is derived from a single axiom, the geometric anti-constraint, highlights its parsimony and explanatory power.

Furthermore, the interpretation of the theory as an optimization problem that maximizes the entropy of geometric measurements provides a deep connection between quantum mechanics, spacetime geometry, and thermodynamics. The entropy-area relation that emerges from the theory suggests that changes in area can be interpreted as thermodynamic transformations, linking the geometric properties of spacetime with the laws of thermodynamics.

In conclusion, the multivector-valued quantum mechanics theory can be interpreted as an optimization problem that maximizes the entropy of all possible geometric measurements of nature. The derived probability measure is the least biased solution consistent with the geometric measurements, as guaranteed by the method of entropy maximization. This interpretation provides a unifying framework that connects quantum mechanics, spacetime geometry, and thermodynamics, offering new insights into the fundamental principles governing the universe.

3.2 The Multilinear Form

David Hestenes' work on the representation of the relativistic wavefunction within $\mathrm{GA}(3,1)$ was instrumental in the development of this research. His results served as a milestone, confirming the validity of our approach at various stages. Hestenes' wavefunction, $\psi = e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})} = \sqrt{\rho}Re^{-ib/2}$, contains the same geometric structures as the $\mathrm{Spin}^{\mathrm{c}}(3,1)$ wavefunction in our theory.

However, it is noteworthy that Hestenes' work does not include a fully satisfactory probability measure. He proposes multiplying the wavefunction with its reverse:

$$\tilde{\psi}\psi = \rho\tilde{R}e^{-ib/2}Re^{-b/2} = \rho e^{-ib} \tag{181}$$

The result ρe^{-ib} does contains ρ, but it also includes a phase factor e^{-ib}. As such, it is not a proper probability measure.

Subsequently, Hestenes proposes sandwiching the γ_μ basis to obtain the Dirac current:

$$J = \tilde{\psi}\gamma_\mu\psi = \rho e_\mu \tag{182}$$

This approach eliminates the phase contribution because $e^{-ib/2}\gamma_\mu e^{-ib/2} = \gamma_\mu e^{ib/2}e^{-ib/2} = \gamma_\mu$. Likewise, the Dirac current is not a proper probability measure as it contains a basis e_μ.

The absence of an adapted Born rule that directly yields the probability when applied to the wavefunction raises a question. Why can't we obtain one? One might be tempted to apply the conjugate to ψ in addition to the reverse:

$$\tilde{\psi}^{\ddagger}\psi = \rho\tilde{R}e^{ib/2}Re^{-ib/2} = \rho \tag{183}$$

In this case one indeeds maps ψ to ρ, however, this approach disrupts the definition of the Dirac current: $\tilde{\psi}^{\ddagger}\gamma_\mu\psi = \rho\tilde{R}\gamma_\mu e^{ib/2}Re^{-ib/2} = \rho e_\mu e^{-ib/2} \neq J$.

To correctly incorporate all the necessary features, including both the Dirac current and a probability measure yielding the probability density, the multilinear form must be employed. Transitioning from bilinear forms to multilinear forms involving <u>four</u> self-products of ψ represents a significant conceptual leap. The strength of the entropy maximization problem lies in its ability to automatically reveal the appropriate form to use.

The multilinear form maps ψ to ρ:

$$\begin{aligned}
\lfloor\psi^{\ddagger}\psi\rfloor_{3,4}\psi^{\ddagger}\psi &= \lfloor\sqrt[4]{\rho}\tilde{R}e^{-ib/4}\sqrt[4]{\rho}Re^{-ib/4}\rfloor_{3,4}\sqrt[4]{\rho}\tilde{R}e^{-ib/4}\sqrt[4]{\rho}Re^{-ib/4} && (184)\\
&= \rho\tilde{R}R\tilde{R}Re^{ib/4}e^{ib/4}e^{-ib/4}e^{-ib/4} && (185)\\
&= \rho && (186)
\end{aligned}$$

Furthermore, it gives the Dirac current:

$$\begin{aligned}
\lfloor\psi^{\ddagger}\gamma_\mu\psi\rfloor_{3,4}\psi^{\ddagger}\psi &= \lfloor\sqrt[4]{\rho}\tilde{R}e^{-ib/4}\gamma_\mu\sqrt[4]{\rho}Re^{-ib/4}\rfloor_{3,4}\sqrt[4]{\rho}\tilde{R}e^{-ib/4}\sqrt[4]{\rho}Re^{-ib/4} && (187)\\
&= \rho\lfloor\tilde{R}\gamma_\mu Re^{ib/4}e^{-ib/4}\rfloor_{3,4}\tilde{R}\gamma_\nu Re^{ib/4}e^{-ib/4} && (188)\\
&= \rho\lfloor\tilde{R}\gamma_\mu R\rfloor_{3,4}\tilde{R}R && (189)\\
&= \rho\tilde{R}\gamma_\mu R && (190)\\
&= \rho e_\mu && (191)\\
&= J && (192)
\end{aligned}$$

and in the context of quantum gravity with the $\mathrm{GL}^+(4,\mathbb{R})$-valued wavefunction:

$$\frac{1}{2}\Big(\lfloor\psi^{\ddagger}\gamma_\mu\psi\rfloor_{3,4}\psi^{\ddagger}\gamma_\nu\psi + \lfloor\psi^{\ddagger}\gamma_\nu\psi\rfloor_{3,4}\psi^{\ddagger}\gamma_\mu\psi\Big) = \langle g_{\mu\nu}\rangle \tag{193}$$

leads to the metric measurement, and even to a metric operator over a Fock space:

$$\frac{1}{2}\Big(\lfloor\phi^{\ddagger}\gamma_\mu\phi\rfloor_{3,4}\phi^{\ddagger}\gamma_\nu\phi + \lfloor\phi^{\ddagger}\gamma_\nu\phi\rfloor_{3,4}\phi^{\ddagger}\gamma_\mu\phi\Big) = \langle\hat{g}_{\mu\nu}\rangle \tag{194}$$

Finally the multilinear form is also invariant to U(1) (Theorem 11), SU(2) (Theorem 12) and SU(3) (Theorem 13).

3.3 Density and Continuum

Let us now extend the entropy maximization problem from the discreet Σ to the continuum $\int$, using a Riemann sum:

$$\mathcal{L} = -\lim_{n\to\infty}\left(\sum_{i=1}^{n} A(x_i)\ln\frac{A(x_i)}{p(x_i)} + \kappa\left(\overline{A} - \operatorname{tr}\sum_{i=1}^{n} A(x_i)\frac{1}{\varepsilon(x_i)}\mathbf{M}(x_i)\right)\right)\Delta x \tag{195}$$

where

- n is the number of subintervals,
- $\Delta x = (b-a)/n$ is the width of each subinterval,
- x_i is a point within the i-th subinterval $[x_{i-1}, x_i]$, often chosen to be the midpoint $(x_{i-1}+x_i)/2$.
- $1/\varepsilon(x_i)$ is a factor required to transform the components of the matrix $\mathbf{M}(x_i)$ into a density, required for integration.

which yields an integral:

$$\mathcal{L} = -\int_a^b A(x)\ln\frac{A(x)}{p(x)}\mathrm{d}x + \kappa\left(\overline{A} - \operatorname{tr}\int_a^b A(x)\frac{1}{\varepsilon(x)}\mathbf{M}(x)\mathrm{d}x\right) \tag{196}$$

Solving this optimization problem yields a probability measure parametrized over the continuum.

We can extend this formulation to multivector amplitudes by using the geometric anti-constraint and parametrized over a world manifold X^4:

$$\mathcal{L} = -\int_a^b A(x^\mu)\ln\frac{A(x^\mu)}{p(x^\mu)}\sqrt{-g}\mathrm{d}^4x + \kappa\left(\overline{A} - \operatorname{tr}\int_a^b \frac{1}{4}A(x^\mu)\frac{1}{\varepsilon(x^\mu)}\mathbf{M}(x^\mu)\sqrt{-g}\mathrm{d}^4x\right) \tag{197}$$

The solution to this optimization problem is a distribution density:

$$\frac{\partial\mathcal{L}(A,\kappa,t)}{\partial A} = 0 \implies A(x^\mu) = \underbrace{\exp\left(-\frac{1}{4}\kappa\frac{1}{\varepsilon(x^\mu)}\operatorname{tr}\mathbf{M}(x^\mu)\right)}_{\text{Geometric Born Rule}}\underbrace{p(x^\mu)}_{\text{Initial State}} \tag{198}$$

This formulation extends the multivector amplitude framework to the continuum, allowing for the description of continuous systems while preserving the geometric structure and invariance properties of the theory.

4 Conclusion

In conclusion, this paper advances the 'Prescribed Observation Problem' (POP) into a multivector quantum theory, seamlessly bridging the realms of quantum mechanics and spacetime geometry. Our findings reveal the POP's exceptional ability to generate a mathematically well-behaved theory that generalizes quantum probabilities through the introduction of the multivector probability measure, a generalization of the Born rule. This measure is invariant under a wide range of geometric transformations, including those generated the gauge groups of the standard model, and leading to the metric tensor as a quantum mechanical observables, without the need for additional assumptions beyond the geometric anti-constraint. Remarkably, multivector amplitudes are found to be consistent only with a 3+1D spacetime, encountering obstructions in other dimensional configurations. This finding aligns with the observed dimensionality and gauge symmetries of the universe and suggests a possible explanation for its specificity. This research represents a significant step in reconciling quantum mechanics with general relativity, challenging and expanding conventional methodologies in theoretical physics, and potentially paving the way for new insights in the field.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

- Data Availability Statement: No datasets were generated or analyzed during the current study.

- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

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