A Gravitized Standard Model is Found as the Solution to the Problem of Maximizing the Entropy of All Geometric Measurements

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Abstract

In modern theoretical physics, the laws of physics are represented with axioms (e.g., the Dirac–Von Neumann axioms, the Wightman axioms, and Newton's laws of motion). While axioms in modern logic hold true merely by definition, the laws of physics are entailed by measurements. Motivated by this dissimilarity, we introduce a more suitable foundation than axioms to represent the laws of physics, and then we make the case for its supremacy. Specifically, measurements will be the axioms whose theorems are the laws of physics. Explicitly, we define a maximization problem on the entropy of all geometric measurements; its unique solution is a geometric quantum theory. In 3+1D, its principal symmetry is generated by the exponential map of multivectors $\exp \mathcal{G}(\mathbb{R}^{3,1})$. It is then shown that this symmetry breaks into a $\mathrm{Spin}^c(3,1)$ quantum theory whose Dirac current is invariant in the SU(2)xU(1) and SU(3) groups, and into the quotient bundle $\exp \mathcal{G}(\mathbb{R}^{3,1})/\mathrm{Spin}^c(3,1)$ yielding a theory of gravity for charged fermions. Remarkably, the model fails to admit normalizable observables above 4 dimensions, suggesting an intrinsic limit to the dimensionality of observable geometry.

1 Introduction

The physical laws in modern theoretical physics are expressed as axioms (e.g., the Dirac–Von Neumann axioms, the Wightman axioms, and Newton's laws of motion). The theorems provable by these axioms are the predictions of the theory. If laboratory measurements invalidate the predictions, the postulated laws are deemed falsified, and new laws are postulated.

In this scenario, it is the theorems (predictions) of the theory that are used (in concert with experiments) to invalidate its axioms (laws).

In logic, however, axioms define what is true in a theory. It follows obviously that its theorems cannot invalidate them.

Thus, there is a dissimilarity between using axioms in physics versus their use in logic.

Since the laws of physics require a more complex interplay between axioms, theorems, and their invalidations than the unidirectional entailment between axioms and theorems found in logic, the question of using axioms to express the laws of physics arises.

Motivated by this dissimilarity, we searched for a more appropriate logical formulation of the laws of physics than brute axioms. We intend to show that correcting the axiomatic entailment between the laws and measurements yields a superior and optimized formulation of fundamental physics.

In our proposal, laboratory measurements entail the mathematical expression of those measurements, and it is this expression, not the laws of physics, that will constitute the axioms of our system. The laws of physics will be defined as the solution to a carefully crafted optimization problem based on the entropy of all geometric measurements.

The solution to this optimization problem is a novel and optimized formulation of fundamental physics. In 3+1D, it yields a geometric quantum theory, whose principal symmetry is generated by the exponential map of multivectors $\exp \mathcal{G}(\mathbb{R}^{3,1})$. This map is isomorphic to $\exp \mathbb{M}(4,\mathbb{R})$ and generates (up to isomorphism) $\mathrm{GL}^+(4,\mathbb{R})$, which in turn acts on the frame bundle FX of a world manifold as its structure group. The symmetry breaks into a quantum theory whose Dirac current is invariant in the SU(2)xU(1) and SU(3) gauge groups, and into a theory of gravity of charged fermions defined in the quotient bundle $\mathrm{FX}/\mathrm{Spin}^c(3,1)$. Remarkably, the general solution cannot produce normalizable observables above 4D (the necessary low dimensional coincidences are lacking), suggesting an intrinsic limit to the dimensionality of any geometry observable in the quantum mechanical sense. We interpret this tight configuration as suggestive of the power and efficiency of defining the laws of physics as the solution to a mathematical optimization problem, rather than as brute axioms.

In essence, from laboratory measurements, it is easier to "guess" the correct mathematical expression for all possible (geometric) measurements than to "guess" the right laws of physics. The distance one must travel in "guessing space" is much shorter for the former than the latter, and this reduces the risk of running astray.

Our optimized formulation is unlikely to have been obtained by trial and error or traditional methods, making our optimization problem a key step in the derivation.

Corollaries that follow directly from our solution, such as the mathematical origin of the Born rule, the derivation from first principles of the axioms of quantum physics, an identification of the correct interpretation of quantum mechanics, and the deprecation of the measurement/collapse problem, are also presented.

To define the problem rigorously, we first introduce the key structure that makes our approach possible: the *geometric measurement constraint.* Next, we present its rationale.

The construction of the geometric measurement constraint exploits the connection between geometry and probability via the trace. The trace of a matrix can be understood as the expected eigenvalue multiplied by the vector space dimension, and the eigenvalues as the ratios of the distortion of the linear transformation associated with the matrix[1]. The geometric measurement constraint is defined as follows:

Definition 1 (The geometric measurement constraint)**.** *Let* $\mathbf{u}$ *be a multivector of* $\mathcal{G}(\mathbb{R}^{p,q})$ *(the geometric algebra of* $p+q$ *dimensions, defined over the real field) and let* $\mathbb{Q}$ *be a statistical ensemble. The geometric measurement constraint is:*

$$\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}} = \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{n}\operatorname{tr}\mathbf{u}(q), \tag{1}$$

where $n = p + q$*, and where* $\operatorname{tr}\overline{\mathbf{u}}$ *denotes the expectation eigenvalue of the statistically weighted sum of multivectors* $\mathbf{u}(q)$*, parameterized over ensemble* $\mathbb{Q}$*.*

We note that the trace of a multivector can be obtained by mapping the multivector to its matrix representation (Section 2), and taking its trace.

Now, we discuss its rationale.

Constraints are used in statistical mechanics to derive the Gibbs measure using Lagrange multipliers[2] by maximizing the entropy.

For instance, an energy constraint on the entropy is

$$\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q), \tag{2}$$

which is associated with an energy meter that measures the system's energy and produces a series of energy measurements $E_1, E_2, \ldots,$ convergent to an expectation value $\overline{E}$.

Another common constraint is related to the volume:

$$\overline{V} = \sum_{q\in\mathbb{Q}} \rho(q)V(q), \tag{3}$$

which is associated with a volume meter acting on a system and produces a sequence of measured volumes $V_1, V_2, \ldots,$ converging to an expectation value $\overline{V}$.

Moreover, the sum over the statistical ensemble must equal 1, as follows:

$$1 = \sum_{q\in\mathbb{Q}} \rho(q) \tag{4}$$

Using equations (2) and (4), a typical statistical mechanical system is obtained by maximizing the entropy using the corresponding Lagrange equation. The Lagrange multiplier method is expressed as:

$$\mathcal{L} = -k_B \sum_{q\in\mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \beta \left(\overline{E} - \sum_{q\in\mathbb{Q}} \rho(q) E(q)\right), \quad (5)$$

where λ and β are the Lagrange multipliers.
Therefore, by solving $\frac{\partial\mathcal{L}}{\partial\rho} = 0$ for ρ, we obtain the Gibbs measure as:

$$\rho(q,\beta) = \frac{1}{Z(\beta)} \exp\bigl(-\beta E(q)\bigr), \quad (6)$$

where

$$Z(\beta) = \sum_{q\in\mathbb{Q}} \exp\bigl(-\beta E(q)\bigr). \quad (7)$$

In our method, (2), a scalar measurement constraint, is replaced with $\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}}$, a geometric measurement constraint. Instead of energy or volume meters, we have protractors, and boost, dilation, spin, and shear meters.

As we found, the geometric measurement constraint is compatible with the full machinery of statistical physics. The probability measure resulting from entropy maximization will preserve the expectation eigenvalue of these transformations up to a phase or symmetry group. For instance, based on our entropy maximization procedure, a statistical system measured exclusively using a protractor will carry a local rotation symmetry in the probability of the measured events.

By limiting the definition of constraints to scalar expressions, we believe that statistical physics has failed to capture all measurements available in nature. Our geometric measurement constraint redresses the situation and supports the totality of geometric measurements that are in principle possible.

Finally, it is the relative Shannon entropy (in base e) that we maximize and not the Boltzmann entropy. The resulting probability measure quantifies the information associated with an observer's receipt of a message of measurements. The Shannon entropy does not change the mathematical equation for entropy (minus the Boltzmann constant); only the final interpretation is changed (further details on the interpretation of quantum mechanics resulting from this model are provided in section 5).

The corresponding Lagrange equation is

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau \left(\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}} - \sum_{q\in\mathbb{Q}} \rho(q) \frac{1}{n}\operatorname{tr}\mathbf{u}(q)\right), \quad (8)$$

where we refer to $p(q)$ as the initial preparation. It is sufficient to solve $\frac{\partial\mathcal{L}}{\partial\rho} = 0$ for ρ to obtain the solution, which is our main result.

The manuscript is organized as follows: The Methods section introduces tools using geometric algebra, based on the study by Lundholm et al. [3, 4]. Specifically, we use the notion of a determinant for multivectors and the Clifford conjugate for generalizing the complex conjugate. These tools enable the geometric expression of the results.

The Results section presents four solutions for the Lagrange equation. The first is the recovery of standard non-relativistic quantum mechanics when reducing the matrix from an arbitrary matrix to a representation of the imaginary number. The second and third are the general cases with an arbitrary matrix or multivector, respectively. Finally, the fourth result applies to the continuum ($\sum \to \int$).

We then develop our initial results into a geometric foundation for physics in 2D and 3+1D, consistent with the general solution. We show in the general case that the model is a geometric quantum theory whose principal symmetry is the exponential map of multivectors $\exp \mathcal{G}(\mathbb{R}^{3,1})$. As this map is isomorphic to $\exp \mathbb{M}(4, \mathbb{R})$, it acts (up to isomorphism) on the frame bundle FX of a world manifold. In 3+1D, the symmetry breaks into a quantum theory invariant in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ and $\mathrm{SU}(3)$ gauge groups, and in the quotient bundle $\mathrm{FX}/\mathrm{Spin}^c(3,1)$ into a theory of gravity of charged fermions. Furthermore, we show that the general solution lacks normalizable observables beyond 4D.

Finally, the Discussion section provides an interpretation of quantum mechanics consistent with its newly revealed origin, namely the metrological interpretation. Central to this interpretation is the measure maximizing the Shannon entropy and constrained by geometric measurements, which yields the wavefunction. This interpretation thus considers the information in measurements more fundamental than the now entirely derivable wavefunction. The end product is a theory that deprecates the measurement problem, supersedes it with a theory of instruments, and provides a plausible explanation for the origin of quantum mechanics in nature by connecting it entirely to the entropy of geometric measurements.

2 Methods

2.1 Notation

- Typography:

  Sets are written using the blackboard bold typography (e.g., $\mathbb{L}$, $\mathbb{W}$, and $\mathbb{Q}$) unless a prior convention assigns it another symbol.

  Matrices are in bold uppercase (e.g., $\mathbf{P}$ and $\mathbf{M}$), tuples, vectors, and multivectors are in bold lowercase (e.g., $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{g}$), and most other constructions (e.g., scalars and functions) have plain typography (e.g., a, and A).

The unit pseudo-scalar (of geometric algebra), imaginary number, and identity matrix are $\mathbf{i}$, i, and $\mathbf{I}$, respectively.

- Sets:

  The projection of a tuple $\mathbf{p}$ is $\text{proj}_i(\mathbf{p})$.

  As an example, the elements of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$ are denoted as $\mathbf{p} = (x, y)$.

  The projection operators are $\text{proj}_1(\mathbf{p}) = x$ and $\text{proj}_2(\mathbf{p}) = y$;

  if projected over a set, the corresponding results are $\text{proj}_1(\mathbb{R}^2) = \mathbb{R}_1$ and $\text{proj}_2(\mathbb{R}^2) = \mathbb{R}_2$, respectively.

  The size of a set $\mathbb{X}$ is $|\mathbb{X}|$.

  The symbol $\cong$ indicates an isomorphism, and $\rightarrow$ denotes a homomorphism.

- Analysis:

  The asterisk $z^\dagger$ denotes the complex conjugate of z.

- Matrix:

  The Dirac gamma matrices are γ_0, γ_1, γ_2, and γ_3.

  The Pauli matrices are σ_x, σ_y, and σ_z.

  The dagger $\mathbf{M}^\dagger$ denotes the conjugate transpose of $\mathbf{M}$.

  The commutator is defined as $[\mathbf{M}, \mathbf{P}] : \mathbf{MP} - \mathbf{PM}$, and the anti-commutator is defined as $\{\mathbf{M}, \mathbf{P}\} : \mathbf{MP} + \mathbf{PM}$.

- Geometric algebra:

  The elements of an arbitrary curvilinear geometric basis are denoted as $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ (such that $\mathbf{e}_\nu \cdot \mathbf{e}_\mu = g_{\mu\nu}$), and $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n$ (such that $\hat{\mathbf{x}}_\mu \cdot \hat{\mathbf{x}}_\nu = \eta_{\mu\nu}$) if they are orthonormal.

  A geometric algebra of $m + n$D over field $\mathbb{F}$ is denoted as $\mathcal{G}(\mathbb{F}^{m,n})$.

  The grades of a multivector are denoted as $\langle \mathbf{v} \rangle_k$.

  Specifically, $\langle \mathbf{v} \rangle_0$ is a scalar, $\langle \mathbf{v} \rangle_1$ is a vector, $\langle \mathbf{v} \rangle_2$ is a bivector, $\langle \mathbf{v} \rangle_{n-1}$ is a pseudo-vector, and $\langle \mathbf{v} \rangle_n$ is a pseudo-scalar.

  A scalar and vector such as $\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_1$ form a para-vector; a combination of even grades ($\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_2 + \langle \mathbf{v} \rangle_4 + \ldots$) or odd grades ($\langle \mathbf{v} \rangle_1 + \langle \mathbf{v} \rangle_3 + \ldots$) form even or odd multivectors, respectively.

  Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

  We can formulate a general multivector of $\mathcal{G}(\mathbb{R}^2)$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

  Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set.

  Then, a general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be formulated as $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

2.2 Geometric representation in 2D

Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^2)$ is given as

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}, \tag{9}$$

where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

Each multivector has a structure-preserving (addition/multiplication) matrix representation.

Definition 2 (2D geometric representation)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{10}$$

Thus, the trace of $\mathbf{u}$ is a.

The converse is also true: each 2×2 real matrix is represented as a multivector of $\mathcal{G}(\mathbb{R}^2)$.

In geometric algebra, the determinant[4] of a multivector $\mathbf{u}$ can be defined as:

Definition 3 (Geometric representation of the determinant 2D)**.**

$$\begin{aligned} \det \quad : \quad & \mathcal{G}(\mathbb{R}^2) \longrightarrow \mathbb{R} \\ & \mathbf{u} \longmapsto \mathbf{u}^{\ddagger}\mathbf{u}, \end{aligned} \tag{11}$$

where $\mathbf{u}^{\ddagger}$ is

Definition 4 (Clifford conjugate 2D)**.**

$$\mathbf{u}^{\ddagger} := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2. \tag{12}$$

For example,

$$\begin{aligned} \det \mathbf{u} &= (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) && (13) \\ &= a^2 - x^2 - y^2 + b^2 && (14) \\ &= \det \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} && (15) \end{aligned}$$

Finally, we define the Clifford transpose.

Definition 5 (2D Clifford transpose)**.** *The Clifford transpose is the geometric analog to the conjugate transpose, interpreted as a transpose followed by an element-by-element application of the complex conjugate. Likewise, the Clifford transpose is a transpose followed by an element-by-element application of the Clifford conjugate.*

$$\begin{bmatrix} \mathbf{u}_{00} & \dots & \mathbf{u}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{mn} \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{u}_{00}^{\ddagger} & \dots & \mathbf{u}_{m0}^{\ddagger} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{nm}^{\ddagger} \end{bmatrix} \tag{16}$$

If applied to a vector, then

$$\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{v}_1^{\ddagger} & \dots \mathbf{v}_m^{\ddagger} \end{bmatrix} \tag{17}$$

2.3 Geometric representation in 3+1D

Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be written as:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}, \tag{18}$$

where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

Similarly, each multivector has a structure-preserving (addition/multiplication) matrix representation.

The multivectors of $\mathcal{G}(\mathbb{R}^{3,1})$ are represented as follows:

Definition 6 (4D geometric representation)**.**

$$\begin{aligned} a &+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\ &+ f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 + f_{13}\gamma_1 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 \\ &+ v_t\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + v_x\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v_y\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + v_z\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\ &+ b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \end{aligned}$$

$$\cong \begin{bmatrix} a + x_0 - if_{12} - iv_3 & f_{13} - if_{23} + v_2 - iv_1 & -ib + x_3 + f_{03} - iv_0 & x_1 - ix_2 + f_{01} - if_{02} \\ -f_{13} - if_{23} - v_2 - iv_1 & a + x_0 + if_{12} + iv_3 & x_1 + ix_2 + f_{01} + if_{02} & -ib - x_3 - f_{03} - iv_0 \\ -ib - x_3 + f_{03} + iv_0 & -x_1 + ix_2 + f_{01} - if_{02} & a - x_0 - if_{12} + iv_3 & f_{13} - if_{23} - v_2 + iv_1 \\ -x_1 - ix_2 + f_{01} + if_{02} & -ib + x_3 - f_{03} + iv_0 & -f_{13} - if_{23} + v_2 + iv_1 & a - x_0 + if_{12} - iv_3 \end{bmatrix} \tag{19}$$

Thus, the trace of $\mathbf{u}$ is a.

In 3+1D, we define the determinant solely using the constructs of geometric algebra[4].

The determinant of $\mathbf{u}$ is

Definition 7 (3+1D geometric representation of determinant)**.**

$$\det \quad : \quad \mathcal{G}(\mathbb{R}^{3,1}) \longrightarrow \mathbb{R} \tag{20}$$

$$\mathbf{u} \longmapsto \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}, \tag{21}$$

where $\mathbf{u}^\ddagger$ is

Definition 8 (3+1D Clifford conjugate)**.**

$$\mathbf{u}^\ddagger := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 + \langle \mathbf{u} \rangle_3 + \langle \mathbf{u} \rangle_4, \tag{22}$$

and where $\lfloor \mathbf{u} \rfloor_{\{3,4\}}$ is the blade-conjugate of degrees three and four (the plus sign is reversed to a minus sign for blades 3 and 4)

$$\lfloor \mathbf{u} \rfloor_{\{3,4\}} := \langle \mathbf{u} \rangle_0 + \langle \mathbf{u} \rangle_1 + \langle \mathbf{u} \rangle_2 - \langle \mathbf{u} \rangle_3 - \langle \mathbf{u} \rangle_4. \tag{23}$$

3 Result

3.1 The entropy of complex-phase measurements

In this subsection, which serves as an introductory example, we recover non-relativistic quantum mechanics using the Lagrange multiplier method and a linear constraint on the entropy.

As previously mentioned, the relative Shannon entropy (in base e) is applied instead of the Boltzmann entropy to achieve the aforementioned goal.

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} \tag{24}$$

In statistical mechanics, we use scalar measurement constraints on the entropy, such as energy and volume meters, which are sufficient for recovering the Gibbs ensemble. However, applying such scalar measurement constraints is insufficient to recover quantum mechanics.

A *complex measurement constraint*, an invariant for a complex phase, is used to overcome this limitation. It is defined[1] as

$$\operatorname{tr} \begin{bmatrix} 0 & -\overline{b} \\ \overline{b} & 0 \end{bmatrix} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0, \end{bmatrix} \tag{25}$$

where $\begin{bmatrix} a(q) & -b(q) \\ b(q) & a(q) \end{bmatrix} \cong a(q) + ib(q)$ is the matrix representation of the complex numbers.

[1]We may wonder why we take $n = 1$ (in Equation **??**) if the matrix is 2×2. Here, we only use the imaginary part of the complex numbers $a + ib\mid_{a\to 0} = ib$, making the constraint one-dimensional.

Similar to energy or volume meters, linear instruments produce a sequence of measurements that converge to an expectation value but with phase invariance. In our framework, this phase invariance originates from the trace.

The Lagrangian equation that maximizes the entropy subject to the complex measurement constraint is

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \alpha \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau \left(\operatorname{tr} \begin{bmatrix} 0 & -\overline{b} \\ \overline{b} & 0 \end{bmatrix} - \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{26}$$

This equation is maximized for ρ by imposing the condition $\frac{\partial \mathcal{L}}{\partial \rho(q)} = 0$. The following results are obtained:

$$\frac{\partial \mathcal{L}}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \alpha - \tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{27}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \alpha + \tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{28}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \alpha - \tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{29}$$

$$\implies \rho(q) = p(q) \exp(-1-\alpha) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{30}$$

$$= \frac{1}{Z(\tau)} p(q) \det \exp \left(-\tau \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right), \tag{31}$$

where $Z(\tau)$ is obtained as:

$$1 = \sum_{q\in\mathbb{Q}} p(q) \exp(-1-\alpha) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{32}$$

$$\implies \big(\exp(-1-\alpha)\big)^{-1} = \sum_{q\in\mathbb{Q}} p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{33}$$

$$Z(\tau) := \sum_{q\in\mathbb{Q}} p(q) \det \exp\left(-\tau \begin{bmatrix} 0 & -b(q) \\ b(q) & 0. \end{bmatrix}\right) \tag{34}$$

The exponential of the trace is equal to the determinant of the exponential according to the relation $\det \exp \mathbf{A} \equiv \exp \operatorname{tr} \mathbf{A}$.

Finally, we obtain

$$\rho(\tau,q) = \frac{1}{Z(\tau)}p(q)\det\exp\left(-\tau\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right) \tag{35}$$

$$\cong p(q)|\exp -i\tau b(q)|^2 \qquad \text{Born rule} \tag{36}$$

Renaming $\tau \to t/\hbar$ and $b(q) \to H(q)$ recovers the familiar form of

$$\rho(q) = \frac{1}{Z}p(q)\Big|\exp\big(-itH(q)/\hbar\big)\Big|^2 . \tag{37}$$

or

$$\rho(q) = \frac{1}{Z}\big|\psi(q)\big|^2, \text{ where } \psi(q) = \exp\big(-itH(q)/\hbar\big)\psi_0(q). \tag{38}$$

where $|\psi_0(q)|^2 = p(q)$ is the initial preparation.

We can show that all three Dirac Von–Neumann axioms and the Born rule are satisfied, revealing the possible origin of quantum mechanics as the solution to an optimization problem on the entropy of linear measurements.

From (38), we can identify the wavefunction as a vector of some orthogonal space (here, a complex Hilbert space), and the partition function as its inner product, expressed as:

$$Z = \langle\psi|\psi\rangle . \tag{39}$$

As the result is automatically normalized by the entropy-maximization procedure, the physical states are associated with the unit vectors, and the probability of any particular state is given by

$$\rho(q) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q))^\dagger\psi(q). \tag{40}$$

Finally, any self-adjoint matrix, defined as $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$, will correspond to a real-valued statistical mechanics observable if measured in its eigenbasis, thereby completing the equivalence.

We also note that τ emerges here for the same reason that T, the temperature, emerges in ordinary statistical mechanics — as Lagrange multipliers. Here, τ is the real parameter of the one-parameter group that maps a matrix to a topological group: $\exp\tau\mathbf{M} \to G$. Mathematically, it corresponds to a flow. Thus, we name τ the *entropic flow* (of time).

3.2 The entropy of all linear measurements

Here, we use the linear measurement constraint in its full generality:

$$\frac{1}{n}\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{n}\operatorname{tr}\mathbf{M}(q), \tag{41}$$

where $\mathbf{M}(q)$ is an arbitrary $n \times n$ real matrix.

The Lagrange equation used to maximize the entropy under this constraint is expressed as:

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q)\ln\frac{\rho(q)}{p(q)} + \alpha\left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau\left(\frac{1}{n}\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{n}\operatorname{tr}\mathbf{M}(q)\right), \tag{42}$$

where α and τ are the Lagrange multipliers.

Similarly, we maximize Equation (55) for ρ using the criterion $\frac{\partial\mathcal{L}}{\partial\rho(q)} = 0$ as follows:

$$\frac{\partial\mathcal{L}}{\partial\rho(q)} = -\ln\frac{\rho(q)}{p(q)} - 1 - \alpha - \tau\frac{1}{n}\operatorname{tr}\mathbf{M}(q) \tag{43}$$

$$0 = \ln\frac{\rho(q)}{p(q)} + 1 + \alpha + \tau\frac{1}{n}\operatorname{tr}\mathbf{M}(q) \tag{44}$$

$$\Longrightarrow \ln\frac{\rho(q)}{p(q)} = -1 - \alpha - \tau\frac{1}{n}\operatorname{tr}\mathbf{M}(q) \tag{45}$$

$$\Longrightarrow \rho(q) = p(q)\exp(-1-\alpha)\exp\left(-\tau\frac{1}{n}\operatorname{tr}\mathbf{M}(q)\right) \tag{46}$$

$$= \frac{1}{Z(\tau)}p(q)\det\exp\left(-\tau\frac{1}{n}\mathbf{M}(q)\right) \tag{47}$$

where $Z(\tau)$ is obtained as

$$1 = \sum_{q\in\mathbb{Q}} p(q)\exp(-1-\alpha)\exp\left(-\tau\operatorname{tr}\frac{1}{n}\mathbf{M}(q)\right) \tag{48}$$

$$\Longrightarrow \big(\exp(-1-\alpha)\big)^{-1} = \sum_{q\in\mathbb{Q}} p(q)\exp\left(-\tau\operatorname{tr}\frac{1}{n}\mathbf{M}(q)\right) \tag{49}$$

$$Z(\tau) \coloneqq \sum_{q\in\mathbb{Q}} p(q)\det\exp\left(-\tau\frac{1}{n}\mathbf{M}(q)\right) \tag{50}$$

The resulting probability measure is

$$\rho(q,\tau) = \frac{1}{Z(\tau)} p(q) \det \exp\left(-\tau \frac{1}{n} \mathbf{M}(q)\right), \tag{51}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} p(q) \det \exp\left(-\tau \frac{1}{n} \mathbf{M}(q)\right). \tag{52}$$

Finally, we can pose

$$\rho(q,\tau) = \det \psi(q,\tau), \text{ where } \psi(q,\tau) = \exp\left(-\tau \frac{1}{n} \mathbf{M}(q)\right) \psi_0(q) \tag{53}$$

Here, the determinant acts as a *geometric Born rule*, connecting, in this case, a *geometric amplitude* to a real-valued probability.

The sophistication of the geometric amplitude and determinant acting as a geometric Born rule will provide us with the platform to support fundamental physics.

3.3 The entropy of all geometric measurements

We now use the geometric measurement constraint:

$$\frac{1}{n} \operatorname{tr} \overline{\mathbf{u}} = \sum_{q \in \mathbb{Q}} \rho(q) \frac{1}{n} \operatorname{tr} \mathbf{u}(q), \tag{54}$$

where $\mathbf{u}(q)$ is an multivector of $\mathcal{G}(\mathbb{R}^{p,q})$, where $p+q=n$.

The Lagrange equation used to maximize the entropy under this constraint is expressed as:

$$\mathcal{L} = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \alpha \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right) + \tau \left(\frac{1}{n} \operatorname{tr} \overline{\mathbf{u}} - \sum_{q \in \mathbb{Q}} \rho(q) \frac{1}{n} \operatorname{tr} \mathbf{u}(q)\right), \tag{55}$$

where α and τ are the Lagrange multipliers.

Similarly, we maximize Equation (55) for ρ using the criterion $\frac{\partial \mathcal{L}}{\partial \rho(q)} = 0$. The result is

$$\rho(q,\tau) = \frac{1}{Z(\tau)} p(q) \det \exp\left(-\tau \frac{1}{n} \mathbf{u}(q)\right), \tag{56}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} p(q) \det \exp\left(-\tau \frac{1}{n} \mathbf{u}(q)\right). \tag{57}$$

3.4 Continuum case

In his original paper, Claude Shannon did not derive the differential entropy as a theorem: instead, he posited that the discrete entropy ought to be extended by replacing the sum with the integral:

$$-\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \to -\int_{\mathbb{R}} \rho(x) \ln \rho(x) \, \mathrm{d}x \tag{58}$$

Unfortunately, it was later discovered that the differential entropy is not always positive, and neither is it invariant under a change of parameters. Specifically, it transforms as follows:

$$-\int_{\mathbb{R}} \rho(x) \ln \rho(x) \, \mathrm{d}x \to -\int_{\mathbb{R}} \tilde{\rho}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \ln \left(\tilde{\rho}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}\right) \mathrm{d}x \tag{59}$$

$$= -\int_{\mathbb{R}} \tilde{\rho}(y) \ln \left(\tilde{\rho}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}\right) \mathrm{d}y \tag{60}$$

Furthermore, due to an argument by Edwin Thompson Jaynes[5, 6], it is known not to be the correct limiting case of the Shannon entropy. Rather, the limiting case is the relative entropy:

$$S = -\int_{\mathbb{R}} \rho(x) \ln \frac{\rho(x)}{p(x)} \, \mathrm{d}x \tag{61}$$

where $p(x)$ is the initial preparation.

The relative entropy, unlike the differential entropy, is invariant with respect to a change of parameter:

$$-\int_{\mathbb{R}} \rho(x) \ln \frac{\rho(x)}{p(x)} \, \mathrm{d}x \to -\int_{\mathbb{R}} \tilde{\rho}(y(x)) \frac{dy}{dx} \ln \frac{\tilde{\rho}(y(x)) \frac{dy}{dx}}{\tilde{p}(y(x)) \frac{dy}{dx}} \, \mathrm{d}x \tag{62}$$

$$= -\int_{\mathbb{R}} \tilde{\rho}(y) \ln \frac{\tilde{\rho}(y)}{\tilde{p}(y)} \, \mathrm{d}y \tag{63}$$

Let us also show that the normalization constraint is invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \tilde{\rho}(y(x))\frac{dy}{dx}\,\mathrm{d}x \tag{64}$$

$$= \int_{\mathbb{R}} \tilde{\rho}(y)\,\mathrm{d}y \tag{65}$$

Let us now investigate the differential observable. A differential observable is typically formulated as

$$\overline{O} = \int_{\mathbb{R}} O(x)\rho(x)\,\mathrm{d}x \tag{66}$$

But, this expression is not invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} O(x)\rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \tilde{O}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \tag{67}$$

$$= \int_{\mathbb{R}} \tilde{O}(y)\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}y \tag{68}$$

To correct this, we now introduce the relative (with respect to a reference) observable. For instance, if we stretch space by a factor of 2: $x \to 2x$, then the reference must also be stretched by the same amount for the observable to remain invariant. The consequence is that we observe the ratio between a quantity and its reference:

$$\overline{M/R} = \int_{\mathbb{R}} \frac{M(x)}{R(x)}\rho(x)\,\mathrm{d}x \tag{69}$$

Where R is the reference and the ratio $\overline{O} = \overline{M/R}$ is observable. We now show that it is invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} \frac{M(x)}{R(x)}\rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \frac{\tilde{M}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{R}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\rho(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \tag{70}$$

$$= \int_{\mathbb{R}} \frac{\tilde{M}(y)}{\tilde{R}(y)}\rho(y)\,\mathrm{d}y \tag{71}$$

With these definitions, the Lagrange equation becomes:

$$\mathcal{L} = -\int_{\mathbb{R}} \rho(x)\ln\frac{\rho(x)}{p(x)}\,\mathrm{d}x + \alpha\left(1 - \int_{\mathbb{R}} \rho(x)\,\mathrm{d}x\right) + \tau\left(\frac{1}{n}\,\mathrm{tr}\,\frac{\mathbf{u}}{\mathbf{r}} - \int_{\mathbb{R}} \frac{1}{n}\,\mathrm{tr}\,\frac{\mathbf{u}(x)}{\mathbf{r}(x)}\rho(x)\,\mathrm{d}x\right) \tag{72}$$

Maximizing this equation with respect to ρ gives

$$\rho(x) = \frac{1}{Z(\tau)} p(x) \det \exp\left(-\frac{1}{n}\tau\frac{\mathbf{u}(x)}{\mathbf{r}(x)}\right) \tag{73}$$

where

$$Z(\tau) = \int_{\mathbb{R}} p(q) \det \exp\left(-\frac{1}{n}\tau\frac{\mathbf{u}(x)}{\mathbf{r}(x)}\right) \mathrm{d}x \tag{74}$$

Now $\rho(x)$, including the observables it supports, is invariant with respect to a change of parameter:

$$\frac{\int_a^b p(x) \det \exp\left(-\frac{1}{n}\tau\frac{\mathbf{u}(x)}{\mathbf{r}(x)}\right) \mathrm{d}x}{\int_{\mathbb{R}} p(x) \det \exp\left(-\frac{1}{n}\tau\frac{\mathbf{u}(x)}{\mathbf{r}(x)}\right) \mathrm{d}x} \to \frac{\int_a^b \tilde{p}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \det \exp\left(-\frac{1}{n}\tau\frac{\tilde{\mathbf{u}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{\mathbf{r}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\right) \mathrm{d}x}{\int_{\mathbb{R}} \tilde{p}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \det \exp\left(-\frac{1}{n}\tau\frac{\tilde{\mathbf{u}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{\mathbf{r}}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\right) \mathrm{d}x} \tag{75}$$

$$= \frac{\int_a^b \tilde{p}(y) \det \exp\left(-\frac{1}{n}\tau\frac{\tilde{\mathbf{u}}(y)}{\tilde{\mathbf{r}}(y)}\right) \mathrm{d}x}{\int_{\mathbb{R}} \tilde{p}(y) \det \exp\left(-\frac{1}{n}\tau\frac{\tilde{\mathbf{u}}(y)}{\tilde{\mathbf{r}}(y)}\right) \mathrm{d}y} \tag{76}$$

We can now generalize this result to a manifold.

Let X^4 be a world manifold. We can write the probability density as follows:

$$\rho(x,y,z,t) \mid_a^b = \frac{1}{Z(\tau)} \int_a^b p(x,y,z,t) \det \exp\left(-\frac{1}{4}\tau\frac{\mathbf{u}(x,y,z,t)}{\mathbf{r}(x,y,z,t)}\right) \sqrt{|g|}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}t \tag{77}$$

where $|g|$ is the absolute value of the determinant of the matrix representation of the metric tensor on the manifold.

Finally, we can define a wavefunction

$$\phi(x,y,z,t) = \exp\left(-\frac{1}{4}\tau\frac{\mathbf{u}(x,y,z,t)}{\mathbf{r}(x,y,z,t)}\right) \phi_0(x,y,z,t) \tag{78}$$

where $\det\big(\phi_0(x,y,z,t)\big) = p(x,y,z,t)$.

4 Analysis

This section analyses the main result.

We introduce the *algebra of geometric observables* applicable to the geometric wavefunction. The 2D definition of algebra constitutes a special case reminiscent of the definitions of ordinary quantum mechanics yet includes gravity. The 3+1D case is significantly more sophisticated than the 2D case and is elucidated immediately after the 2D case analysis.

4.1 Axiomatic definition of the algebra in 2D

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^2)$.

A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

A) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the sesquilinear map

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \qquad \mathbb{V} \times \mathbb{V} &\longrightarrow \mathcal{G}(\mathbb{R}^2) \\ \langle \mathbf{u}, \mathbf{v} \rangle &\longmapsto \mathbf{u}^\ddagger \mathbf{v} \end{aligned} \tag{79}$$

is positive-definite such that for $\boldsymbol{\psi} \neq 0$, $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

B) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$. Then, for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q)) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \psi(q)^\ddagger \psi(q) \tag{80}$$

is either positive or equal to zero.

We note the following comments and definitions:

- From A) and B), it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum up to unity:

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q)) = 1 \tag{81}$$

- $\boldsymbol{\psi}$ is called a *physical state*.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$, such that the sum of probabilities remains normalized.

$$\langle \mathbf{T}\boldsymbol{\psi}, \mathbf{T}\boldsymbol{\psi} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{82}$$

are the *physical transformations* of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \in \mathbb{V}$ and $\forall \mathbf{v} \in \mathbb{V}$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{v} \rangle \tag{83}$$

  is called an observable.

- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{84}$$

4.2 Geometric self-adjoint operator in 2D

The general case of an observable in 2D is shown in this section. A matrix $\mathbf{O}$ is observable if it is a self-adjoint operator defined as:

$$\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle \tag{85}$$

$\forall \boldsymbol{\phi} \in \mathbb{V}$ and $\forall \boldsymbol{\psi} \in \mathbb{V}$.

Setup: Let $\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix}$ be an observable.

Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be two two-state multivectors $\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix}$ and $\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}$. Here, the components $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, $\mathbf{o}_{11}$ are multivectors of $\mathcal{G}(\mathbb{R}^2)$.

Derivation: 1. Calculate $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle &= (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger(\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2) \\ &\quad + (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger(\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2) \qquad (86) \\ &= \boldsymbol{\phi_1}^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\phi}_2 \\ &\quad + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\phi}_2 \qquad (87) \end{aligned}$$

2. Next, calculate $\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle &= \boldsymbol{\phi}_1^\ddagger(\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2) + (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger(\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2) + (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 \qquad (88) \\ &= \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\phi}_1 \qquad (89) \end{aligned}$$

To realize $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi}\rangle$, the following relations must hold:

$$\mathbf{o}_{00}^{\ddagger} = \mathbf{o}_{00} \tag{90}$$
$$\mathbf{o}_{01}^{\ddagger} = \mathbf{o}_{10} \tag{91}$$
$$\mathbf{o}_{10}^{\ddagger} = \mathbf{o}_{01} \tag{92}$$
$$\mathbf{o}_{11}^{\ddagger} = \mathbf{o}_{11}. \tag{93}$$

Therefore, $\mathbf{O}$ must be equal to its own Clifford transpose, indicating that $\mathbf{O}$ is an observable if

$$\mathbf{O}^{\ddagger} = \mathbf{O}, \tag{94}$$

which is the geometric generalization of the self-adjoint operator $\mathbf{O}^{\dagger} = \mathbf{O}$ of complex Hilbert spaces.

4.3 Geometric spectral theorem in 2D

The application of the spectral theorem to $\mathbf{O}^{\ddagger} = \mathbf{O}$ such that its eigenvalues are real is shown below:

Consider

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}, \tag{95}$$

Then $\mathbf{O}^{\ddagger}$ is

$$\mathbf{O}^{\ddagger} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}, \tag{96}$$

It follows that $\mathbf{O}^{\ddagger} = \mathbf{O}$

This example is the most general 2×2 matrix $\mathbf{O}$ such that $\mathbf{O}^{\ddagger} = \mathbf{O}$.

The eigenvalues are obtained as:

$$0 = \det(\mathbf{O} - \lambda\mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} - \lambda \end{bmatrix}, \tag{97}$$

This implies that

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12})(a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} + a_{11}) \tag{98}$$
$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2), \tag{99}$$

Finally,

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{100}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\} \tag{101}$$

The roots would be complex if $a^2 - x^2 - y^2 + b^2 < 0$. Since $a^2 - x^2 - y^2 + b^2$ is the determinant of the multivector, the complex case is ruled out for orientation-preserving multivectors. Consequently, it follows $\mathbf{O}^\ddagger = \mathbf{O}$ constitutes an observable with real-valued eigenvalues for orientation-preserving multivectors.

4.4 Invariant transformations in 2D

A left action on the wavefunction $\mathbf{T}\,|\psi\rangle$ connects to the bilinear form as $\langle\psi|\,\mathbf{T}^\ddagger\mathbf{T}\,|\psi\rangle$.

The invariance requirement on $\mathbf{T}$ is

$$\langle\psi|\,\mathbf{T}^\ddagger\mathbf{T}\,|\psi\rangle = \langle\psi|\psi\rangle\,. \tag{102}$$

Therefore, we are interested in the group of matrices that follow

$$\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}. \tag{103}$$

Let us consider a two-state system, with a general transformation represented by

$$\mathbf{T} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}, \tag{104}$$

where u, v, w, x are the 2D multivectors.

The expression $\mathbf{T}^\ddagger\mathbf{T}$ is

$$\mathbf{T}^\ddagger\mathbf{T} = \begin{bmatrix} v^\ddagger & u^\ddagger \\ w^\ddagger & x^\ddagger \end{bmatrix}\begin{bmatrix} v & w \\ u & x \end{bmatrix} = \begin{bmatrix} v^\ddagger v + u^\ddagger u & v^\ddagger w + u^\ddagger x \\ w^\ddagger v + x^\ddagger u & w^\ddagger w + x^\ddagger x \end{bmatrix} \tag{105}$$

For $\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}$, the following relations must hold:

$$v^\ddagger v + u^\ddagger u = 1 \tag{106}$$

$$v^\ddagger w + u^\ddagger x = 0 \tag{107}$$

$$w^\ddagger v + x^\ddagger u = 0 \tag{108}$$

$$w^\ddagger w + x^\ddagger x = 1 \tag{109}$$

This is the case if

$$\mathbf{T} = \frac{1}{\sqrt{v^{\ddagger}v + u^{\ddagger}u}} \begin{bmatrix} v & u \\ -e^{\varphi}u^{\ddagger} & e^{\varphi}v^{\ddagger} \end{bmatrix}, \tag{110}$$

where u, v are the 2D multivectors, and e^{φ} is a unit multivector.

Comparatively, the unitary case is obtained when the vector part of the multivector vanishes, i.e., $\mathbf{x} \to 0$, and we obtain

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta}b^{\dagger} & e^{i\theta}a^{\dagger} \end{bmatrix}. \tag{111}$$

Here $\mathbf{T}$ is the geometric generalization (in 2D) of unitary transformations.

4.5 Gravity in 2D

Roger Penrose argued "that the case for gravitizing quantum theory is at least as strong as that for quantizing gravity"[7].

Gravitizing the quantum (rather than quantizing gravity) is the direction our model leads us in. Indeed, we have attempted no changes to general relativity. Instead, our entropy maximization procedure produced a wavefunction valued in the orientation-preserving general linear group, whose geometric flexibility exceeds the familiar unitary wavefunction. It is within this extra flexibility that we will find gravity.

We will now investigate the quotient bundle associated with the structure reduction from $\mathrm{GL}^+(2,\mathbb{R})$ to $\mathrm{SO}(2)$.

Let X^2 be a smooth orientable real-valued manifold in 2D. We consider its tangent bundle TX and its associated frame bundle FX. Since X^2 is orientable, its structure group is $\mathrm{GL}^+(2,\mathbb{R})$. The action by our wavefunction, valued in $\exp \mathcal{G}(\mathbb{R}^2) \cong \exp \mathbb{M}(2,\mathbb{R})$ generates $\mathrm{GL}^+(2,\mathbb{R})$, and is thus on FX. We now consider a reduction of the structure group of FX to $\mathrm{SO}(2)$.

Let us begin by investigating the cosets of $\mathrm{SO}(2)$ in $\mathrm{GL}^+(2,\mathbb{R})$. Let $g_1 \in \mathrm{GL}^+(2,\mathbb{R})$, $g_2 \in \mathrm{GL}^+(2,\mathbb{R})$ and $s \in \mathrm{SO}(2)$. We now identify the relation $g_2 = g_1 s$. We also note $g_2^T = s^T g_1^T$. Finally, we note the product $g_2 g_2^T = g_1 s s^T g_1^T \implies g_2 g_2^T = g_1 g_1^T$. Since $g_1 g_1^T$ and $g_2 g_2^T$ are symmetric positive-definite 2×2 matrices, one verifies a diffeomorphism between $\mathrm{GL}^+(2,\mathbb{R})/\mathrm{SO}(2)$ and the inner products.

The global section of the quotient bundle $\mathrm{FX}/\mathrm{SO}(2)$ is a tetrad field $h^a_\mu(x)$ and it associates to a Riemannian metric on X^2 via the identity $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$. The connection that preserves the structure $\mathrm{SO}(2)$ across the manifold are the metric connections[8], and with the additional requirement of no torsion, the connections reduce to the Levi-Civita connection. It has been shown recently[9] that the Goldstone fields associated with the quotient bundle have enough degrees of freedom to create a metric and a covariant derivative. Finally, the frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^2[10].

In this work, we have merely maximized the entropy of all possible geometric measurements, and we have arrived, without introducing any other assumptions, at a general linear quantum theory holding in the $\mathrm{GL}^+(2,\mathbb{R})$ group, whose symmetry breaks into the theory of gravity (FX/SO(2)) and into a quantum theory of the special orthogonal group (valued in SO(2)) which we will now investigate.

4.6 Schrödinger equation in $\mathcal{G}(\mathbb{R}^2)$

First, let us recall that the standard Schrödinger equation can be derived as follows.

In the bra-ket notation, we recall that a one-parameter group evolves as follows:

$$\exp(-it\mathbf{H}) \left|\psi(0)\right\rangle = \left|\psi(t)\right\rangle . \tag{112}$$

Thus, an infinitesimal displacement of t is obtained as follows:

$$\exp(-i\delta t\mathbf{H}) \left|\psi(\tau)\right\rangle = \left|\psi(\tau+\delta\tau)\right\rangle . \tag{113}$$

Now, we approximate the exponential into a power series as

$$\exp(-i\delta t\mathbf{H}) \left|\psi(\tau)\right\rangle \approx 1 - i\delta t\mathbf{H} \left|\psi(t)\right\rangle . \tag{114}$$

The process is continued as follows:

$$(1 - i\delta t\mathbf{H}) \left|\psi(t)\right\rangle = \left|\psi(t+\delta t)\right\rangle \tag{115}$$

$$\left|\psi(\tau)\right\rangle - i\delta t\mathbf{H} \left|\psi(t)\right\rangle = \left|\psi(t+\delta t)\right\rangle \tag{116}$$

$$-i\delta t\mathbf{H} \left|\psi(t)\right\rangle = \left|\psi(t+\delta t)\right\rangle - \left|\psi(t)\right\rangle \tag{117}$$

$$-i\mathbf{H} \left|\psi(t)\right\rangle = \frac{\left|\psi(t+\delta t)\right\rangle - \left|\psi(t)\right\rangle}{\delta t} \tag{118}$$

$$-i\mathbf{H} \left|\psi(t)\right\rangle = \frac{d\left|\psi(t)\right\rangle}{dt}. \tag{119}$$

which is the Schrödinger equation.

We now return to our model.

Taking an arbitrary multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, we now perform the elimination $a \to 0, \mathbf{x} \to 0$. The wavefunction, the observables, and the transformation matrix $\mathbf{T}$ become valued in $\langle\mathcal{G}(\mathbb{R}^2)\rangle_2$ (which generates SO(2)), and consequently the Stone theorem on one-parameter groups applies. We obtain

$$\mathbf{T}(\tau)\mid_{a\to 0,\mathbf{x}\to 0} = \exp(\mathbf{i}\tau\mathbf{O}) \tag{120}$$

where

$$(\mathbf{O}^{\ddagger} = \mathbf{O})\mid_{a\to 0,\mathbf{x}\to 0} \implies \mathbf{O}^{\dagger} = \mathbf{O} \tag{121}$$

The result has the same form as the Schrödinger equation (119):

$$-\frac{1}{2}\mathbf{i}\mathbf{O}\left|\psi(\tau)\right\rangle = \frac{d\left|\psi(\tau)\right\rangle}{d\tau}, \tag{122}$$

and the wavefunction is $\psi(\tau) = \exp\left(-\tau\frac{1}{2}\mathbf{i}\mathbf{O}\right)$

Compared to the Schrödinger equation, here $\mathbf{i}$ is not an imaginary unit but a rotor in 2D. We recall that $\mathbf{i} = \hat{\mathbf{x}}_1\hat{\mathbf{x}}_2$ and that rotors $R = \exp\left(\frac{1}{2}\theta\mathbf{i}\right)$ are exponentials of bivectors.

We thus arrived at a quantum theory of the special orthogonal group, where the wavefunction defines the action on a vector, as follows:

$$\begin{aligned}\psi^{\ddagger}(\tau)\hat{\mathbf{x}}_0\psi(\tau) &= \exp\left(\tau\frac{1}{2}\mathbf{i}B\right)\hat{\mathbf{x}}_0\exp\left(-\tau\frac{1}{2}\mathbf{i}B\right) &(123)\\ &= \exp\left(\tau\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B\right)\hat{\mathbf{x}}_0\exp\left(-\tau\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B\right) &(124)\end{aligned}$$

where B is the value of $\mathbf{O}$ at the origin of the vector $\hat{\mathbf{x}}_0$ tangent to the manifold.

The expression $\exp\left(\tau\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B\right)\hat{\mathbf{x}}_0\exp\left(-\tau\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B\right)$ maps $\hat{\mathbf{x}}_0$ to a curvilinear basis $\mathbf{e}_0$ via the application of the rotor and its reverse:

$$\exp\left(\tau\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B\right)\hat{\mathbf{x}}_0\exp\left(-\tau\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B\right) = \mathbf{e}_0(\tau) \tag{125}$$

Finally, in 2D, since Spin(2) = SO(2), and $\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1$ anti-commutes with $\hat{\mathbf{x}}_0$, we can write the above action as a (left) SO(2) action instead of as a joint Spin(2) action:

$$\exp(\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1B)\hat{\mathbf{x}}_0 = \mathbf{e}_0(\tau) \tag{126}$$

4.7 Metric interference in 2D

We now consider a transformation $\mathbf{T}^{\ddagger}\mathbf{T} = \mathbf{I}$ and a wavefunction $|\psi\rangle = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$ such that a multivector $\mathbf{u}$ is mapped to a linear combination of two multivectors. Let us consider this transformation:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{u}+\mathbf{v}\\ \mathbf{u}-\mathbf{v}\end{bmatrix} \tag{127}$$

We can now investigate the probability:

$$\rho(\mathbf{u}+\mathbf{v})=\frac{1}{Z}\det(\mathbf{u}+\mathbf{v}),\ \text{where } Z=\det(\mathbf{u}+\mathbf{v})+\det(\mathbf{u}-\mathbf{v}) \tag{128}$$

We proceed as follows:

$$\begin{aligned}
\det(\mathbf{u}+\mathbf{v}) &= (\mathbf{u}+\mathbf{v})^{\ddagger}(\mathbf{u}+\mathbf{v}) && (129)\\
&= (\mathbf{u}^{\ddagger}+\mathbf{v}^{\ddagger})(\mathbf{u}+\mathbf{v}) && (130)\\
&= (\mathbf{u}^{\ddagger}\mathbf{u}+\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u}+\mathbf{v}^{\ddagger}\mathbf{v}) && (131)\\
&= \det\mathbf{u}+\det\mathbf{v}+\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u} && (132)\\
&= \det\mathbf{u}+\det\mathbf{v}+\mathbf{u}\cdot\mathbf{v} && (133)
\end{aligned}$$

where we have defined the dot product between multivectors as follows:

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u} \tag{134}$$

Since $\det\mathbf{u}>0$ and $\det\mathbf{v}>0$, then $\mathbf{u}\cdot\mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$ (whichever is greater). Therefore, it also satisfies the conditions of an interference term capable of destructive and constructive interference.

In the case $\mathbf{x}\to 0$, the interference pattern reduces to a form identical to the unitary case:

$$\begin{aligned}
\det\Big(\psi_1 e^{\mathbf{b}_1}+\psi_2 e^{\mathbf{b}_2}\Big) &= \det\psi_1+\det\psi_2+2\psi_1\psi_2 e^{\mathbf{b}_1+\mathbf{b}_2} && (135)\\
&= |\psi_1|^2+|\psi_2|^2+2\psi_1\psi_2 e^{\mathbf{b}_1+\mathbf{b}_2} && (136)
\end{aligned}$$

but, unlike the unitary case, here the interference is valued in SO(2).

4.8 A *double-copy* algebra of geometric observables in 4D

In 2D, the determinant can be expressed using only the product $\psi^{\ddagger}\psi$, which can be interpreted as the inner product of two multivectors. This form allowed us to extend the complex Hilbert space to a *geometric* Hilbert space. We then found that the familiar properties of the complex Hilbert spaces were transferable to the geometric Hilbert space, eventually yielding a 2D gravitized quantum theory in the language of geometric algebra.

Although a similar correspondence exists in 4D, it is less recognizable because we need a *double-copy* inner product (i.e., $\rho = \lfloor \phi^\ddagger \phi \rfloor_{3,4} \phi^\ddagger \phi$) to produce a real-valued probability in 4D.

Thus, in 4D, we cannot produce an inner product as in the 2D case. The absence of a satisfactory inner product indicates no Hilbert space in the usual sense of a complete *inner product* vector space.

We aim to find a construction that supports the geometric wavefunction in 4D.

To build the right construction, a double-copy inner product of four terms is devised, superseding the inner product in the Hilbert space, mapping any four vectors to an element of $\mathcal{G}(\mathbb{R}^{3,1})$, and yielding a complete *double-copy* inner product vector space — or simply, a *double-copy* Hilbert space.

We note that the construction will be more familiar than it may first appear. Indeed, the familiar quantum mechanical features (linear transformations, unit vectors, and linear superposition in the probability measure, etc.) will be supported in the construction, and just as it did in 2D, it will also here break into a familiar inner-product Hilbert space and into a theory of gravity for charged fermions.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$.

A subset of vectors in $\mathbb{V}$ forms a double-copy algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, the double-copy inner product form

$$\begin{aligned} \langle \cdot, \cdot, \cdot, \cdot \rangle \quad : \qquad \mathbb{V} \times \mathbb{V} \times \mathbb{V} \times \mathbb{V} &\longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ \langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle &\longmapsto \sum_{i=1}^{m} \lfloor u_i^\ddagger w_i \rfloor_{3,4} y_i^\ddagger z_i \end{aligned} \tag{137}$$

   is positive-definite when $\boldsymbol{\phi} \neq 0$; that is $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle > 0$

2. $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, then for each element $\phi(q) \in \boldsymbol{\phi}$, the function

$$\rho(\phi(q)) = \frac{1}{\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle} \det \phi(q), \tag{138}$$

   is either positive or equal to zero.

We note the following properties, features, and comments:

- From A) and B), it follows that, $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, and the probabilities sum to unity.

$$\sum_{\phi(q) \in \boldsymbol{\phi}} \rho(\phi(q)) = 1 \tag{139}$$

- $\boldsymbol{\phi}$ is called a *physical state*.
- $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle$ is called the *partition function* of $\boldsymbol{\phi}$.
- If $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle = 1$, then $\boldsymbol{\phi}$ is called a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\phi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\phi}$ such as $\mathbf{T}\boldsymbol{\phi} \to \boldsymbol{\phi}'$ makes the sum of probabilities normalized (invariant):

$$\langle \mathbf{T}\boldsymbol{\phi}, \mathbf{T}\boldsymbol{\phi}, \mathbf{T}\boldsymbol{\phi}, \mathbf{T}\boldsymbol{\phi} \rangle = \langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle \tag{140}$$

  are the *physical transformations* of $\boldsymbol{\phi}$.
- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{y} \forall \mathbf{z} \in \mathbb{V}$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{w}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w}, \mathbf{O}\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{O}\mathbf{z} \rangle \tag{141}$$

  is called an observable.
- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle}{\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle} \tag{142}$$

4.9 Geometric observables in 4D

In 4D, an observable must satisfy equation 141. For simplicity, let us take m in equation 153 to be 1. Then,

$$\lfloor (\mathbf{O}u)^{\ddagger} w \rfloor_{3,4} y^{\ddagger} z = \lfloor u^{\ddagger} \mathbf{O} w \rfloor_{3,4} y^{\ddagger} z = \lfloor u^{\ddagger} w \rfloor_{3,4} (\mathbf{O}y)^{\ddagger} z = \lfloor u^{\ddagger} w \rfloor_{3,4} y^{\ddagger} \mathbf{O} z \tag{143}$$

where u_1, w_1, y_1 and z_1 are multivectors.

Let us investigate.

If $\mathbf{O}$ contained a vector, bivector, pseudo-vector, or pseudo-scalar, the equality would not be satisfied as these terms do not commune with the multivectors and cannot be factored out. The equality is satisfied if $\mathbf{O} \in \mathbb{R}$. Indeed, as a real value, $\mathbf{O}$ commutes with all multivectors, and hence, can be factored out to satisfy the equality.

We thus find that the observables are real-valued in the general 4D case.

If we subscribe to the philosophy that real-valued observables are "classical", and operator-valued (or matrix-valued) observables are "quantum", we may prematurely conclude that the 4D case, unlike the 2D, is classical. However, we will see that the "quantumness" of the observables emerges as we break the symmetry.

4.10 Wavefunction in 3+1D

In the David Hestenes' notation[11], the 3+1D wavefunction is expressed as:

$$\psi = \sqrt{\rho e^{-ib}}R, \tag{144}$$

where ρ represents a scalar probability density, e^{ib} is a complex phase, and R is a rotor. In David Hestenes' formulation, τ is a general parametrization and does not appear to be necessarily a one-parameter group.

Comparatively, our wavefunction in $\mathcal{G}(\mathbb{R}^{3,1})$ is:

$$\phi = e^{-\frac{1}{4}\tau(a+\mathbf{x}+\mathbf{f}+\mathbf{v}+\mathbf{b})}\phi_0 \tag{145}$$

To approach David Hestenes' formulation of the wavefunction, it suffices to eliminate the terms $a \to 0$, $\mathbf{x} \to 0$ and $\mathbf{v} \to 0$, and to perform a substitution of the entries of the double-copy inner product (Equation 153), as follows:

$$\mathbf{w} \to \mathbf{u}^\ddagger \tag{146}$$

$$\mathbf{y} \to \mathbf{z}^\ddagger \tag{147}$$

As one of the copies is destroyed by the substitution, the double-copy inner product reduces to an inner product. Furthermore, with the elimination, the blade-3,4 conjugate is also reduced to the blade-4 conjugate, yielding

$$\langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle \to \langle \mathbf{u}, \mathbf{u}^\ddagger, \mathbf{z}^\ddagger, \mathbf{z} \rangle \cong \langle \mathbf{u}, \mathbf{z} \rangle = \sum_{i=1}^{m} \lfloor u_i^2 \rfloor_{2,4}(z_i^2) \tag{148}$$

Consequently, our wavefunction ϕ reduces to

$$\phi^2 = e^{-\frac{1}{2}\tau(\mathbf{f}+\mathbf{b})}\phi_0^2 \tag{149}$$

This shows that the 3+1D wavefunction (comprising a rotor $R(\tau) = e^{-\frac{1}{2}\tau\mathbf{f}}$, a pseudo-scalar $e^{-\frac{1}{2}\tau\mathbf{b}}$ and a prior probability $\phi_0^2 = \sqrt{\rho}$) is a sub-structure of the general $\mathcal{G}(\mathbb{R}^{3,1})$ wavefunction.

In this sub-structure, the observables are satisfied when

$$\lfloor \mathbf{O} \rfloor_{2,4} = \mathbf{O} \tag{150}$$

As we recall, in the double-copy inner product case, the observables satisfied Equation 143 only if they are real-valued. Comparatively, a structure reduction ($a \to 0$, $\mathbf{x} \to 0, \mathbf{v} \to 0$) has increased the quantity of geometry that is observable. In fact, this equation of observables captures the totality of the remaining geometry. The wavefunction is the largest statistical structure in 3+1D that is entirely observable geometrically.

Let us now analyze the symmetry group of this wavefunction.

First, we note that the term $\mathbf{b}$ commutes with $\mathbf{f}$. They can be factored out as

$$e^{-\frac{1}{2}\tau(\mathbf{f}+\mathbf{b})}\phi_0^2 = e^{-\frac{1}{2}\tau\mathbf{b}}e^{-\frac{1}{2}\tau\mathbf{f}}\phi_0^2 \tag{151}$$

Second, the term $\exp \mathbf{f}$ can be understood as the exponential map from the bivectors to the $\mathrm{Spin}_+(3,1)$ group and the term $\exp \mathbf{b}$ to $\mathrm{U}(1)$.

Finally, since $\mathrm{Spin}_+(3,1) \cap \exp \mathbf{b} = \{\pm 1\}$, it must be removed from the group product[12].

We conclude that the wavefunction corresponds to a one-parameter flow (τ) of the following group

$$\mathrm{U}(1) \times (\mathrm{Spin}_+(3,1)/\{\pm 1\}) \cong \mathrm{Spin}^c(3,1) \tag{152}$$

4.11 Algebra of geometric observables in 3+1D (broken symmetry)

Specifically, the substitution Equation 148 yields the following algebra of geometric observables.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$.

A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the inner product form

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \qquad & \mathbb{V} \times \mathbb{V} \longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ & \langle \mathbf{u}, \mathbf{w} \rangle \longmapsto \sum_{i=1}^{m} \lfloor u_i^2 \rfloor_{2,4} w_i^2 \end{aligned} \tag{153}$$

   is positive-definite when $\boldsymbol{\psi} \neq 0$; that is $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

2. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q)) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \det \psi(q), \tag{154}$$

   is either positive or equal to zero.

We note the following properties, features, and comments:

- From A) and B), it follows that, $\forall\boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, and the probabilities sum to unity.

$$\sum_{\psi(q)\in\boldsymbol{\psi}} \rho(\psi(q)) = 1 \tag{155}$$

- $\boldsymbol{\psi}$ is called a *physical state*.
- $\langle\boldsymbol{\psi},\boldsymbol{\psi}\rangle$ is called the *partition function* of $\boldsymbol{\phi}$.
- If $\langle\boldsymbol{\psi},\boldsymbol{\psi}\rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ such as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$ makes the sum of probabilities normalized (invariant):

$$\langle\mathbf{T}\boldsymbol{\psi},\mathbf{T}\boldsymbol{\psi}\rangle = \langle\boldsymbol{\psi},\boldsymbol{\psi}\rangle \tag{156}$$

  are the *physical transformations* of $\boldsymbol{\psi}$.
- A matrix $\mathbf{O}$ such that $\forall\mathbf{u}\forall\mathbf{w} \in \mathbb{V}$:

$$\langle\mathbf{O}\mathbf{u},\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{O}\mathbf{w}\rangle \tag{157}$$

  is called an observable.
- The expectation value of an observable $\mathbf{O}$ is

$$\langle\mathbf{O}\rangle = \frac{\langle\mathbf{O}\boldsymbol{\psi},\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},\boldsymbol{\psi}\rangle} \tag{158}$$

4.12 Gravity and electromagnetism in 3+1D

In 2D, we benefited from a coincidence of low dimensions, where the matrix representation of $\mathcal{G}(\mathbb{R}^2)$ was in $\mathbb{M}(2,\mathbb{R})$. As such, our wavefunction generated $\mathrm{GL}^+(2,\mathbb{R})$ which acted as the structure group of the frame bundle FX, and following a structure reduction from $\mathrm{GL}^+(2,\mathbb{R})$ to SO(2), a tetrad field was associated with the global section of the quotient bundle FX/SO(2) which led to a gravitized quantum theory.

In 4D, unlike in 2D where SO(2) = Spin(2), the geometry of the wavefunction is not in SO but rather in Spin^c. And since Spin^c is not, in general, in GL^+, we cannot benefit from the same coincidences as in 2D.

Typically, to reach $\text{Spin}(p, q)$ from the structure group $\text{GL}(p+q)$, one would reduce $\text{GL}(p+q)$ to $\text{O}(p, q)$, then lift it to $\text{Spin}(p, q)$. Here, however, we will use a different approach to get the spin connection.

Remarkably, 4D admits a coincidence that will allow us to embed the $\text{Spin}^c(3, 1)$ group into the $\text{GL}^+(4, \mathbb{R})$ group, then take its quotient $\text{FX}/\text{Spin}^c(3, 1)$ without having to lift; our model already contains what is necessary to take this quotient.

The coincidence comes from the standard classification of real Clifford algebra[13] and from the fact that $\exp(\mathbf{f} + \mathbf{b}) \cong \text{Spin}^c(3, 1) \subset \exp \mathcal{G}(\mathbb{R}^{3,1})$. The diagram

$$\begin{tikzcd} \mathcal{G}(\mathbb{R}^{3,1}) \arrow[r, "f"] \arrow[d, "exp"] & \mathbb{M}(4, \mathbb{R}) \arrow[d, "exp"] \\ \exp \mathcal{G}(\mathbb{R}^{3,1}) \arrow[r, "f"] & \text{GL}^+(4, \mathbb{R}) \end{tikzcd} \tag{159}$$

commutes by group homomorphisms. Since $\exp(\mathbf{f} + \mathbf{b}) \cong \text{Spin}^c(3, 1) \subset \exp \mathcal{G}(\mathbb{R}^{3,1})$, the map f embeds $\text{Spin}^c(3, 1)$ into $\text{GL}^+(4, \mathbb{R})$. The inclusion of $\text{Spin}^c(3, 1)$ in $\exp \mathcal{G}(\mathbb{R}^{3,1})$ is required to break the symmetry into exactly a theory of gravity of charged fermions and into a $\text{Spin}^c(3, 1)$-valued quantum theory. We are now ready.

Let X^4 be a world manifold.

We first consider the tangent bundle TX along with its associated frame bundle FX. Our wavefunction acts on the frame bundle using the exponential map of multivectors $\exp \mathcal{G}(\mathbb{R}^{3,1}) \cong \exp \mathbb{M}(4, \mathbb{R})$ which generates $\text{GL}^+(4, \mathbb{R})$.

The desired reduction is from $\exp \mathcal{G}(\mathbb{R}^{3,1})$ to the $\text{Spin}^c(3, 1)$ group. With its symmetry reduced, the wavefunction will assign an element of $\text{Spin}^c(3, 1)$ to each event $x \in X^4$, and it "lives" in the 3+1D geometric Hilbert space previously mentioned. The connection that preserves the structure is a $\text{Spin}^c(3, 1)$ preserving connection. It relates to a theory of gravity for charged fermions. We note that since $\text{SO}(3, 1) \times \text{U}(1)$ is a quotient $\text{Spin}^c(3, 1)$, the cosets are further associable with the inner products. Thus, the global section of the quotient bundle $\text{FX}/\text{SO}(3, 1)$ associates with a tetrad field that uniquely determines a pseudo-Riemannian metric. Electromagnetism is also included from the U(1)-bundle. Finally, the frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^4[10].

4.13 Metric interference in 3+1D

A geometric wavefunction would allow a larger class of interference patterns than complex interference. The geometric interference pattern includes the ways in which the geometry of a probability measure can interfere constructively or destructively and includes interference from rotations, boosts, shears, spins, and dilations.

In the case of 4D *metric interference* (shown below), the interference pattern is associated with a superposition of elements of the group $\text{Spin}^c(3, 1)$, whose subgroup $\text{SO}(3, 1)$ associates to a superposition of inner products in the quotient.

It is possible that a *sensitive* Aharonov–Bohm effect experiment on gravity[14] could detect special cases of the geometric phase and interference patterns identified in this section.

An interference pattern follows from a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the application of the determinant:

$$\det(\mathbf{u}+\mathbf{v}) = \det\mathbf{u} + \det\mathbf{v} + \mathbf{u}\cdot\mathbf{v} \tag{160}$$

The determinants $\det\mathbf{u}$ and $\det\mathbf{v}$ are a sum of probabilities, whereas the dot product term $\mathbf{u}\cdot\mathbf{v}$ represents the interference term.

Such can be obtained following a transformation of a wavefunction $|\psi\rangle = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$ such that the multivectors are mapped to a linear combination of two multivectors:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{u}+\mathbf{v}\\ \mathbf{u}-\mathbf{v}\end{bmatrix} \tag{161}$$

The dot product defines a bilinear form.

$$\cdot \quad : \qquad \mathcal{G}(\mathbb{R}^{m,n}) \times \mathcal{G}(\mathbb{R}^{m,n}) \longrightarrow \mathbb{R} \tag{162}$$

$$\mathbf{u}\cdot\mathbf{v} \longmapsto \frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) \tag{163}$$

If $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$, then $\mathbf{u}\cdot\mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$ (whichever is greater). Therefore, it also satisfies the conditions of an interference term.

In 2D, the dot product has this form

$$\begin{aligned}
&\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) && (164)\\
&= \frac{1}{2}\left((\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v}\right) && (165)\\
&= \mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v} - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v} && (166)\\
&= \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} && (167)
\end{aligned}$$

In 3+1D, it has this form.

$$\begin{aligned}
&\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) && (168)\\
&= \frac{1}{2}\left(\lfloor(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})\rfloor_{3,4}(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} - \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\right) && (169)\\
&= \frac{1}{2}\left(\lfloor\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}(\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}) - \ldots\right) && (170)
\end{aligned}$$

$$
\begin{aligned}
&= \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
&\quad + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} - \dots
\end{aligned} \tag{171}
$$

$$
\begin{aligned}
&= \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
&\quad + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}
\end{aligned} \tag{172}
$$

We now consider simpler interference patterns.

Metric interference in 4D:

As seen previously, the substituted double-copy inner product reduces to an inner product (Equation 148). The interference pattern[15] is given as follows:

$$\det(\mathbf{u}+\mathbf{v}) = \lfloor \mathbf{u}+\mathbf{v}\rfloor_{2,4}(\mathbf{u}+\mathbf{v}) \tag{173}$$

$$= \lfloor \mathbf{u}\rfloor_{2,4}(\mathbf{u}+\mathbf{v}) + \lfloor \mathbf{v}\rfloor_{2,4}(\mathbf{u}+\mathbf{v}) \tag{174}$$

$$= \lfloor \mathbf{u}\rfloor_{2,4}\mathbf{u} + \lfloor \mathbf{u}\rfloor_{2,4}\mathbf{v} + \lfloor \mathbf{v}\rfloor_{2,4}\mathbf{u} + \lfloor \mathbf{v}\rfloor_{2,4}\mathbf{v} \tag{175}$$

$$= \det\mathbf{u} + \det\mathbf{v} + \lfloor \mathbf{u}\rfloor_{2,4}\mathbf{v} + \lfloor \mathbf{v}\rfloor_{2,4}\mathbf{u} \tag{176}$$

Now replacing $\mathbf{u} = \rho_u e^{-\frac{1}{2}\tau\mathbf{b}_u} e^{-\frac{1}{2}\tau\mathbf{f}_u}$ and $\mathbf{v} = \rho_v e^{-\frac{1}{2}\tau\mathbf{b}_v} e^{-\frac{1}{2}\tau\mathbf{f}_v}$

$$= |\rho_u|^2 + |\rho_v|^2 + \rho_u\rho_v\left(e^{\frac{1}{2}\tau\mathbf{b}_u}e^{\frac{1}{2}\tau\mathbf{f}_u}e^{-\frac{1}{2}\tau\mathbf{b}_v}e^{-\frac{1}{2}\tau\mathbf{f}_v} + e^{\frac{1}{2}\tau\mathbf{b}_v}e^{\frac{1}{2}\tau\mathbf{f}_v}e^{-\frac{1}{2}\tau\mathbf{b}_u}e^{-\frac{1}{2}\tau\mathbf{f}_u}\right) \tag{177}$$

Due to the presence of $\mathbf{f}$ and $\mathbf{b}$, the geometric richness of the interference pattern exceeds that of the 2D case. The term $\mathbf{f}$ associates with a non-commutative interference effect in the interference pattern, which distinguishes it from complex interference.

4.14 Dirac current

David Hestenes[11] defines the Dirac current in the language of geometric algebra as:

$$\mathbf{j} = \psi^\ddagger(\tau)\gamma_0\psi(\tau) = \rho(\tau)R^\ddagger(\tau)\gamma_0 R(\tau) = \rho(\tau)e_0(\tau) = \rho(\tau)v(\tau) \tag{178}$$

where v is the proper velocity.

In our formulation, this relation also holds; the Dirac current represents the action of the wavefunction on the unit timelike vector in the tangent space on

X^4. Specifically, the Dirac current is a statistically weighted Lorentz action on γ_0:

$$\mathbf{j} = \psi^\ddagger \gamma_0 \psi \tag{179}$$
$$= e^{-\frac{1}{2}\tau\mathbf{f}+\frac{1}{2}\tau\mathbf{b}}\phi_0\gamma_0 e^{\frac{1}{2}\tau\mathbf{f}+\frac{1}{2}\tau\mathbf{b}}\phi_0 \tag{180}$$
$$= \phi_0^2 e^{-\frac{1}{2}\tau\mathbf{f}}\gamma_0 e^{\frac{1}{2}\tau\mathbf{f}} \tag{181}$$
$$= \rho(\tau)e_0(\tau) \tag{182}$$
$$= \rho(\tau)v(\tau) \tag{183}$$

We now have all the tools required to construct particle physics by exhausting the remaining geometry of our model.

4.15 SU(2) × U(1) group

Our wavefunction transforms as a group under multiplication. We now ask, what is the most general multivector $e^{\mathbf{u}}$ which leaves the Dirac current invariant?

$$\psi^\ddagger (e^{\mathbf{u}})^\ddagger \gamma_0 e^{\mathbf{u}} \psi = \psi^\ddagger \gamma_0 \psi \iff (e^{\mathbf{u}})^\ddagger \gamma_0 e^{\mathbf{u}} = \gamma_0 \tag{184}$$

When is this satisfied?

The bases of the bivector part $\mathbf{f}$ of $\mathbf{u}$ are $\gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$. Among these, only $\gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$ commute with γ_0, and the rest anti-commute; therefore, the rest must be made equal to 0. Finally, the base $\gamma_0\gamma_1\gamma_2\gamma_3$ anti-commutes with γ_0 and cancels out.

Consequently, the most general exponential multivector of the form $e^{\mathbf{u}}$ where $\mathbf{u} = \mathbf{f} + \mathbf{b}$ which preserves the Dirac current is

$$e^{\mathbf{u}} = \exp\left(\frac{1}{2}F_{12}\gamma_1\gamma_2 + \frac{1}{2}F_{13}\gamma_1\gamma_3 + \frac{1}{2}F_{23}\gamma_2\gamma_3 + \frac{1}{2}\mathbf{b}\right) \tag{185}$$

We can rewrite the bivector basis with the Pauli matrices

$$\gamma_2\gamma_3 = \mathbf{i}\sigma_x \tag{186}$$
$$\gamma_1\gamma_3 = \mathbf{i}\sigma_y \tag{187}$$
$$\gamma_1\gamma_2 = \mathbf{i}\sigma_z \tag{188}$$
$$\mathbf{b} = \mathbf{i}b \tag{189}$$

After replacements, we obtain

$$e^{\mathbf{u}} = \exp\frac{1}{2}\mathbf{i}(F_{12}\sigma_z + F_{13}\sigma_y + F_{23}\sigma_x + b) \tag{190}$$

The terms $F_{23}\sigma_x + F_{13}\sigma_y + F_{12}\sigma_z$ and b are responsible for SU(2) and U(1) symmetries, respectively[16, 17].

4.16 SU(3) group

The invariance transformation identified by the 3+1D algebra of geometric observables (Equation 156) are $\mathbf{T}^\ddagger\mathbf{T}=\mathbf{I}$, $\mathbf{T}^\dagger\mathbf{T}=\mathbf{I}$ and $\lfloor\mathbf{T}\rfloor_{2,4}\mathbf{T}=\mathbf{I}$. In the first case, the identified evolution is bivectorial rather than unitary.

As we did for the SU(2) $\times$ U(1) case, we ask, in this case, what is the most general bivectorial evolution that leaves the Dirac current invariant?

$$\mathbf{f}^\ddagger\gamma_0\mathbf{f}=\gamma_0 \tag{191}$$

where $\mathbf{f}$ is a bivector:

$$\mathbf{f}=F_{01}\gamma_0\gamma_1+F_{02}\gamma_0\gamma_2+F_{03}\gamma_0\gamma_3+F_{23}\gamma_2\gamma_3+F_{13}\gamma_1\gamma_3+F_{12}\gamma_1\gamma_2 \tag{192}$$

Explicitly, the expression $\mathbf{f}^\ddagger\gamma_0\mathbf{f}$ is

$$\begin{aligned}
\mathbf{f}^\ddagger\gamma_0\mathbf{f}=-\mathbf{f}\gamma_0\mathbf{f}=&\,(F_{01}^2+F_{02}^2+F_{03}^2+F_{13}^2+F_{23}^2+F_{12}^2)\gamma_0 && (193)\\
&+(-2F_{02}F_{12}+2F_{03}F_{13})\gamma_1 && (194)\\
&+(-2F_{01}F_{12}+2F_{03}F_{23})\gamma_2 && (195)\\
&+(-2F_{01}F_{13}+2F_{02}F_{23})\gamma_3 && (196)
\end{aligned}$$

For the Dirac current to remain invariant, the cross-product must vanish:

$$\begin{aligned}
-2F_{02}F_{12}+2F_{03}F_{13}=0 && (197)\\
-2F_{01}F_{12}+2F_{03}F_{23}=0 && (198)\\
-2F_{01}F_{13}+2F_{02}F_{23}=0 && (199)
\end{aligned}$$

leaving only

$$\mathbf{f}^\ddagger\gamma_0\mathbf{f}=(F_{01}^2+F_{02}^2+F_{03}^2+F_{13}^2+F_{23}^2+F_{12}^2)\gamma_0. \tag{200}$$

Finally, $F_{01}^2+F_{02}^2+F_{03}^2+F_{13}^2+F_{23}^2+F_{12}^2$ must equal 1.
We note that we can re-write $\mathbf{f}$ as a 3-vector with complex components:

$$\mathbf{f}=(F_{01}+\mathbf{i}F_{23})\gamma_0\gamma_1+(F_{02}+\mathbf{i}F_{13})\gamma_0\gamma_2+(F_{03}+\mathbf{i}F_{12})\gamma_0\gamma_3 \tag{201}$$

Then, with the nullification of the cross-product and equating $F_{01}^2+F_{02}^2+F_{03}^2+F_{13}^2+F_{23}^2+F_{12}^2$ to unity, we can understand the bivectorial evolution when constrained by the Dirac current to be a realization of the SU(3) group[17].

4.17 Geometric observables in 6D

Let us now investigate what happens in dimensions higher than 4.

First, let us recap.

The observables in 4D must satisfy a more constraining equality relation than in 2D. This reduced the geometric expressivity that such observables could support. Specifically, in 2D, the relation was satisfied for $\mathbf{O}^{\ddagger} = \mathbf{O}$ capturing the full geometry, but was reduced to $\mathbf{O} \in \mathbb{R}$ in 4D, which is a small subset of the available geometry.

What happens if we increase the dimensions even further to 6 and above?

At dimensions of 6 or above, the corresponding observable relation cannot be satisfied. To see why, we look at the results[18] of Acus et al. regarding the 6D multivector norm. The authors performed an exhaustive computer-assisted search for the geometric algebra expression for the determinant in 6D; as conjectured, they found no norm defined via self-products. The norm is a linear combination of self-products.

The system of linear equations is too long to list in its entirety; the author gives this mockup:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{202}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{203}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{204}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{205}$$

The author then produces the special case of this norm that holds only for a 6D multivector comprising a scalar and grade 4 element:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{206}$$

Even in this simplified special case, formulating a linear relationship for observables is doomed to fail. Indeed, the real portion of the observable cannot be extracted from the equation. We find that for any function f_i and g_i, the coefficient b_1 and b_2 will frustrate the equality:

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{207}$$

$$= b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{208}$$

Equations 207 and 208 can only be equal if $b_1 = b_2$; however, the norm $s(B)$ requires both to be different. Consequently, the relation for observables in 6D is unsatisfiable even by real numbers.

Thus, in our framework, the 6D geometry leads to the absence of observables.

Furthermore, since the norms involve more sophisticated systems of linear equations in higher dimensions, this result is likely to generalize to all dimensions above 6.

4.18 Defective probability measure in 3D and 5D

The 3D and 5D cases (and possibly all odd-dimensional cases of higher dimensions) contain a number of irregularities that make them defective to use in this framework. Let us investigate.

In $\mathcal{G}(\mathbb{R}^3)$, the matrix representation of a multivector

$$\mathbf{u} = a + x\sigma_x + y\sigma_y + z\sigma_z + q\sigma_y\sigma_z + v\sigma_x\sigma_z + w\sigma_x\sigma_y + b\sigma_x\sigma_y\sigma_z \tag{209}$$

is

$$\mathbf{u} \cong \begin{bmatrix} a + ib + iw + z & iq - v + x - iy \\ iq + v + x + iy & a + ib - iw - z \end{bmatrix} \tag{210}$$

and the determinant is

$$\det \mathbf{u} = a^2 - b^2 + q^2 + v^2 + w^2 - x^2 - y^2 - z^2 + 2i(ab - qx + vy - wz) \tag{211}$$

The result is a complex-valued probability. Since a probability must be real-valued, the 3D case is defective in our model and cannot be used. In theory, it can be fixed by defining a complex norm to apply to the determinant:

$$\langle \mathbf{u}, \mathbf{u} \rangle = (\det \mathbf{u})^\dagger \det \mathbf{u} \tag{212}$$

However, defining such a norm would entail a double-copy inner product of 4 multivectors, but the space is only 3D, not 4D (so why four?). It would also break the relationship between trace and probability that justified its usage in statistical mechanics.

Consequently, this case appears to us to be defective.

Perhaps, instead of $\mathcal{G}(\mathbb{R}^3)$ multivectors, we ought to use 3×3 matrices in 3D? Alas, 3×3 matrices do not admit a geometric algebra representation because they are not isomorphic with $\mathcal{G}(\mathbb{R}^3)$. Indeed, $\mathcal{G}(\mathbb{R}^3)$ has 8 parameters and 3×3 matrices have 9. 3×3 matrices are not representable geometrically in the same sense that 2×2 matrices are with $\mathcal{G}(\mathbb{R}^2)$.

In $\mathcal{G}(\mathbb{R}^{4,1})$, the algebra is isomorphic to complex 4×4 matrices. In this case, the determinant and probability would be complex-valued, making the case defective. Furthermore, 5×5 matrices have 25 parameters, but $\mathcal{G}(\mathbb{R}^{4,1})$ multivectors have 32 parameters.

4.19 Specialness of 3+1D

Our approach to maximizing the entropy of linear measurements is non-defective in the following dimensions:

- $\mathbb{R}$: This case corresponds to familiar statistical mechanics. The constraints are scalar $\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q)$, and the probability measure is the Gibbs measure $\rho(q) = \frac{1}{Z(\beta)} \exp\big(-\beta E(q)\big)$.

- $\mathbb{C} \cong \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$: This case corresponds to familiar non-relativistic quantum mechanics.
- $\mathcal{G}(\mathbb{R}^2)$: This case corresponds to the geometric quantum theory in 2D. Its $\mathrm{GL}^+(2)$ symmetry breaks into a theory of gravity FX/SO(2) and into a quantum theory valued in SO(2).
- $\mathcal{G}(\mathbb{R}^{3,1})$: This case is valid. Like the 2D case, it also corresponds to a geometric quantum theory. As such, its symmetry will break into a theory of gravity and a relativistic wavefunction. But unlike the 2D case, the wavefunction further admits an invariance with respect to the $\mathrm{SU}(2)\times\mathrm{U}(1)$ and SU(3) gauge groups.

In contrast, our approach is defective in the following dimensions:

- $\mathcal{G}(\mathbb{R}^3)$: In this case, the probability measure is complex-valued.
- $\mathcal{G}(\mathbb{R}^{4,1})$: In this case, the probability measure is complex-valued.
- 6D and above: For $\mathcal{G}(\mathbb{R}^n)$, where $n \geq 6$, no observables satisfy the corresponding observable equation, in general.

We may thus say that 5D fails to normalize, and 6D and above fail to satisfy observables. Consequently, in the general case of our approach, it is the case that normalizable geometric observables cannot be satisfied beyond 4D. This suggests an intrinsic limit to the dimensionality of observable geometry.

5 Discussion

We recovered a geometric quantum theory using the tools of statistical mechanics to maximize the entropy under the effect of a geometric measurement constraint. Important to the interpretation, we replaced the Boltzmann entropy with the relative Shannon entropy to do so. We will now discuss the interpretation of our model in more detail.

Contrary to multiple interpretations of quantum mechanics, the interpretation of statistical mechanics is singular, free of paradoxes, and without a measurement problem; by necessity, this will be inherited by our interpretation of quantum mechanics.

Definition 9 (Metrological interpretation)**.** *There exist instruments that record sequences of measurements on systems. These measurements are unique up to a geometric phase, and the Born rule (including its geometric generalization to the determinant) is the entropy-maximizing measure constrained by the expectation eigenvalue of these measurements.*

The Lagrange multiplier method, which maximizes the entropy subject to the geometric measurement constraint, is the mathematical backbone of this interpretation.

We now discuss the definition of the measuring apparatus entailed by this interpretation.

Integrating formally into physics the notion of an instrument or measuring apparatus has been a long-standing difficulty. One of the pitfalls is attributing too much "detailing" to this instrument (for instance, defining the instrument as a macroscopic system that amplifies quantum information), which increases the risk of capturing only a fraction of all possible instruments in nature. Fractional capture is to be avoided because the instruments are our only "eyes into nature"; consequently, the generality of their definition must be on a level similar to the laws of physics themselves, lest it hampers our chances of deriving the laws of physics from measurements alone.

In statistical mechanics, instruments and their effects on systems are incorporated into mathematical formalism. For instance, an energy or volume meter can produce a sequence of measurements whose average converges towards an expectation value, constituting a constraint on the entropy. However, the generalizability of this definition to all physical systems (including quantum and geometrical) was overlooked. This study capitalized on this definition and extended it appropriately.

The instrument is defined as follows:

Definition 10 (Instrument/Measuring Apparatus)**.** *An instrument, or measuring apparatus, is a device that constrains the entropy of a message of measurements to an expectation eigenvalue (or simply to an expectation value if the instrument is a scalar constraint).*

For instance, a scalar instrument could simply be a rubber balloon constraining a gas to a given expected volume.

However, nature allows for geometrically richer measurements and instrumentations that cannot be expressed with simple "scalar" or "phase-less" instruments. For instance, a protractor or boost meter also admits numerical measurements; however, they also contain geometric phase invariances, such as the rotational or Lorentz invariance, respectively. These invariances must be accounted for by the probability measure.

In the metrological interpretation, the existence of such instruments, not the wavefunction, is taken as axiomatic. The laws of physics are uniquely determined by the geometrical richness (invariance) of the instruments available in nature.

This study interpreted the trace as the expectation eigenvalue of the eigenvalues of a matrix transformation multiplied by the dimension of the vector space. Maximizing the entropy under the constraint of this expectation eigenvalue introduces various phase invariances into the resulting probability measure, consistent with the available measuring apparatuses.

As we have seen, the constraint

$$\operatorname{tr}\begin{bmatrix}0 & -\overline{b}\\ \overline{b} & 0\end{bmatrix} = \sum_{q\in\mathbb{Q}} \operatorname{tr}\rho(q)\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix} \tag{213}$$

induces a complex phase invariance into the probability measure $\rho(q) = \Big|\exp\big(-i\tau b(q)\big)\Big|^2$, which gives rise to the Born rule and wavefunction.

Moreover, the constraint

$$\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}} = \sum_{q\in\mathbb{Q}}\rho(q)\frac{1}{n}\operatorname{tr}\mathbf{u}(q) \tag{214}$$

induces the full geometric phase invariance in the probability measure $\rho(q) = \det\exp\left(-\frac{1}{n}\tau\mathbf{u}(q)\right)$. The resulting probability measure supports a geometric quantum theory.

In each case, we can interpret the constraint as an instrument acting on the system.

In the complex phase case, we interpret the constraint as an incidence counter measuring a particle or photon. Moreover, in the geometric case, we interpret the constraint as a measure that is invariant with respect to natural transformations, such as measurements of the geometry of spacetime events.

The complete correspondence between an ordinary system of statistical mechanics and ours is as follows.

Table 1: Correspondence

Concept	Statistical Mechanics	Geometric Measurements
Entropy	Boltzmann	Shannon
Measure	Gibbs	Born rule
Constraint	Energy meter	Phase-invariant instrument
Micro-state	Energy values	Measurement results
Lagrange multiplier	Temperature	Entropic flow
Experience	Ergodic	Message

In the correspondence, using the Shannon entropy instead of the Boltzmann entropy changes the experience from ergodic to a message (in the sense of the communication theory of Claude Shannon[19]) of measurements. The receipt of such a message by an observer carries information; it is associated to the registration of a "click"[20] on a screen or an incidence counter.

Since the message is received by the observer, the experience is not merely ergodic but actually carries information. As such, we can understand physics in relation to the information as opposed to entropy. That is, physics can be understood as the model that maximizes the <u>information</u> associated with receiving the message. At its most fundamental level, physics is the model of nature that provably renders geometric measurements maximally informative to the observer.

The probabilistic interpretation of the wavefunction via the Born rule is inherited from statistical mechanics and results from maximizing the entropy under a geometric measurement constraint.

The wavefunction is also entailed and consequently, is not considered axiomatic. Instead, it is the receipt of a message about the measurements by an observer, along with the geometric measurement constraint, that is considered axiomatic.

Specifically, the axioms of quantum mechanics are recoverable as theorems from the solution $\frac{\partial \mathcal{L}}{\partial \rho} = 0$ for ρ, where

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau \left(\frac{1}{n} \operatorname{tr} \overline{\mathbf{u}} - \sum_{q\in\mathbb{Q}} \rho(q) \frac{1}{n} \operatorname{tr} \mathbf{u}(q)\right). \tag{215}$$

Due to a failure to produce normalizable observables in most dimensions, the maximizing solution is sensible only in a few dimensions, including 0D (statistical mechanics), 0+1D (non-relativistic quantum mechanics), 2D, and 3+1D. In consideration that there are finitely many sensible cases, we propose the following naming convention:

Table 2: Proposed Naming Convention

Constraint	Entailed Quantum Theory
Linear Measurement Constraint $\mathbb{M}(2,\mathbb{R})$	General Linear Quantum Theory
- with symmetry breaking to SO(2)	Gravitized Quantum Theory
Geometric Measurement Constraint $\mathcal{G}(\mathbb{R}^{3,1})$	Geometric Quantum Theory
- with symmetry breaking to $\mathrm{Spin}^c(3,1)$	Gravitized Standard Model

Using the naming convention, one would say the *general linear quantum theory* breaks to a *gravitized quantum theory* in FX/SO(2), and the *geometric quantum theory* breaks to a *gravitized standard model* in $\mathrm{FX}/\mathrm{Spin}^c(3,1)$.

Now, let us discuss the wavefunction collapse problem.

Specifically, the mathematical foundation of quantum mechanics contains the following axiom: If the measurement of a quantity $\mathbf{O}$ on ψ gives the result o_n, then the state immediately after the measurement is given by the normalized projection of ψ onto the eigensubspace of o_n as

$$\psi \implies \frac{P_n \left|\psi\right\rangle}{\sqrt{\left\langle\psi\right| P_n \left|\psi\right\rangle}} \tag{216}$$

The difficulty of providing a mechanism to explain why this occurs is known as the wavefunction collapse problem.

The measurement-collapse problem is, in our framework, superseded as follows: Before deriving the wavefunction, measurements are assumed to have been registered by an instrument and are associated with a geometric measurement constraint, which is axiomatic. Registering new measurements, in this case, does not mean that a wavefunction has collapsed but implies that we need to adjust the constraints and derive a new wavefunction consistent with the new measurements. Because the wavefunction is derived by maximizing the entropy constrained by the registered measurements, it never updates from an uncollapsed to a collapsed state. The collapse problem is a symptom of attributing an axiomatic status to the wavefunction; however, this status belongs to the instruments and their measurements — not to the wavefunction. As measurements do not update the wavefunction, but rather form the constraints that define it, the measurement postulate is not part of our model.

Since our knowledge of nature comes from the available instruments, postulating these instruments (rather than the wavefunction) to be the axioms of physics makes the mathematics of physics entirely consistent with it being an empirical science.

The full correspondence is also consistent with the general intuition that *random information* must be axiomatic, as, by definition, it cannot be derived from any earlier principles. Ultimately, it is viable to consider *the message of random measurements*, rather than the wavefunction (a derivable mathematical equation), to be the axiomatic foundation of the theory. As shown, the latter can be derived from the former but not vice versa.

5.1 Axioms

We propose that the laws of physics are ultimately entailed by the following minimal axioms related exclusively to measurements and instruments.

Context 1 (Ontology)**.** *The experience of the observer in nature is defined as the receipt of a message* $\mathbf{m}$ *of* n *measurements:*

$$\mathbf{m} = \mathrm{Dom}(O)^n \tag{217}$$

1. *where* $O\colon \mathbb{Q} \to \mathbb{R}$ *is an observable of* $\mathbb{Q}$*,*
2. *and where* $\mathbb{Q}$ *is a statistical ensemble.*

Axiom 1 (Geometricity)**.** *A geometric measuring device constrains the entropy of the elements of a message of measurement according to:*

$$\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}} = \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{n}\operatorname{tr}\mathbf{u}(q) \tag{218}$$

where $\operatorname{tr}\mathbf{u}(q)$ *is an observable (i.e.* $O(q) = \operatorname{tr}\mathbf{u}(q)$*), where* $\operatorname{tr}\overline{\mathbf{u}}$ *is its average, and where* $\mathbf{u}$ *corresponds to a multivector of* $\mathcal{G}(\mathbb{R}^{p,q})$ *such that* $p+q=n$*.*

Theorem 1 (Laws of Physics as a Theorem)**.** *Maximizing the entropy of the elements of a message of measurements constrained by a geometric measuring device yields the theory of physics that maximizes the information acquired by the observer from each such measurement:*

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q) \right) + \tau \left(\frac{1}{n} \operatorname{tr} \overline{\mathbf{u}} - \sum_{q\in\mathbb{Q}} \rho(q) \frac{1}{n} \operatorname{tr} \mathbf{u}(q) \right) \tag{219}$$

Solving for $\partial \mathcal{L} / \partial \rho = 0$ implies

$$\rho(q, \tau) = \frac{1}{Z(\tau)} p(q) \det \exp \left(-\tau \frac{1}{n} \mathbf{u}(q) \right), \tag{220}$$

where

$$Z(\tau) = \sum_{q\in\mathbb{Q}} p(q) \det \exp \left(-\tau \frac{1}{n} \mathbf{u}(q) \right). \tag{221}$$

which, as discussed in this paper, identifies with a general linear quantum theory in 2D, and in 3+1D with a geometric quantum theory whose principal symmetry is generated by $\exp \mathcal{G}(\mathbb{R}^{3,1})$*. The latter breaks into a gravitized standard model in* $\mathrm{FX}/\mathrm{Spin}^c(3,1)$ *associating to a theory of gravity of charged fermions and into* $\mathrm{Spin}^c(3,1)$ *associating to a quantum theory whose Dirac current is invariant for the* $\mathrm{SU}(3)$ *and* $\mathrm{SU}(2) \times \mathrm{U}(1)$ *gauge groups. The theory fails to admit general observables above 4D. We note that no additional assumptions beyond the Geometricity axiom need to be added to obtain this result as a theorem.*

We also note the continuum case given by equation 77.

With this foundation, the pervasive platonic defect of placing laws as axioms, rather than the measurements they are derived from, is now corrected. Theoretical physics is, in this formulation, completely consistent with physics being an empirical science because its laws follow exclusively from the geometric measurements that are in principle possible.

6 Conclusion

We proposed to maximize the entropy under the constraint of a geometric measurement apparatus. The resulting probability measure supports a geometry richer than what could previously be supported in either statistical physics or quantum mechanics. Accommodating all possible geometric measurements entails a geometric wavefunction, for which the Born rule is extended to the determinant. This substantially extends the opportunity to capture all fundamental physics within a single framework. The framework produces models for 2D and 4D in which general observables are normalizable. 4D stands out as the largest geometry that satisfies the conditions for having normalizable observables in the general case. A gravitized standard model results from the frame bundle FX of a world manifold, whose structure group is generated by $\exp \mathcal{G}(\mathbb{R}^{3,1})$ (which is isomorphic to $\exp \mathbb{M}(4,\mathbb{R})$ and as such generates to $\mathrm{GL}^{+}(4,\mathbb{R})$ up to isomorphism), undergoing symmetry breaking to $\mathrm{Spin}^c(3,1)$. The global sections of the quotient bundle $\mathrm{FX}/\mathrm{SO}(3,1)$ identify with a pseudo-Riemannian metric and the natural bundles to general covariant transformations. The connection is a Spin^c-preserving connection. The groups $\mathrm{SU}(2)\times \mathrm{U}(1)$ and $\mathrm{SU}(3)$ are recovered in the broken symmetry and associated with the invariant transformations under the action of the wavefunction on a unit timelike vector of the tangent space, yielding the preservation of the Dirac current for these gauge groups. Finally, an interpretation of quantum mechanics, i.e., the metrological interpretation, is proposed; the existence of instruments and the measurements they produce acquire the foundational role, and the wavefunction is derived as a theorem. In this interpretation, it is considered that an observer receives a message (theory of communication/Shannon entropy) of phase-invariant measurements, and the probability measure, maximizing the information of this message, is the geometric wavefunction accompanied by the geometric Born rule. Finally, as the solution to an optimization problem on entropy, we concluded that physics, distilled to its conceptually simplest expression, is the model of nature that provably makes geometric measurements maximally informative to the observer.

7 Statements and Declarations

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