

# Logic for the Field of Battle\*

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**Abstract:** The truth table method, natural deduction, and the truth tree method, the three validity proving methods standardly taught in an introductory logic course, are too clumsy for the battlefield of real-life. The “short truth table” test is handy at times, but it stumbles at many other times. In this paper, we set up a general method that can beat all the methods mentioned above in a contest of speed. Furthermore, the procedure can be step-by-step paraphrased in a natural language, so that, unlike the other methods, a real-life logical problem can be analyzed and explained in a real-life language too.

**Keywords:** natural deduction, short truth table, truth table, truth tree

## 1. Introduction

Propositional logic has been a well-developed subject, often constituting the first half of an introductory course of logic. The main topic of propositional logic is the proof/disproof of the validity of an argument and standard methods for the establishment of the validity of an argument include truth tables, natural deduction<sup>1</sup>, and truth trees. Each of them provides the student with a set of tools that are sufficient for he or she to assert the validity of a given argument. From a theoretical point of view, this fact alone renders each of them a successful account. However, when it comes to the utility/pragmatic-virtue of an account, there is the matter of efficiency that we need to take into consideration also. In this paper, I adopt the metaphor of a fight to address this issue.

To fight in a UFC<sup>2</sup> match, a fighter needs an extensive amount of trainings in order to be equipped with various martial art techniques that might help him defeat his opponent. However, if he thinks that once he has mastered all those techniques, he can surely win the fight, then he is totally mistaken. In a fight, even if you can execute a punch correctly, so long as your punch is delivered a split second later than that of your opponent, you will be the one that gets punched instead. Metaphorically, establishing the validity of an argument can be likened to the delivering of a punch that is heavy enough to knock down an opponent.

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Nevertheless, to be a real knock-down punch, not only has it to be heavy, it also has to be fast — it had better arrive at the target *before* the opponent's punch does so.

One might object to this metaphor, claiming that one can take time evaluating the validity of an argument in propositional logic: what matters is that the validity of the argument is correctly established, not how fast you can establish it. However, if logic is taught as a general educational course aiming to help the students deal with arguments that they might encounter in their daily life, then the *usefulness* of a validity-discerning method surely depends on how soon it can come up with the proof.

Imagine that a mad scientist locks ten students of logic in a prison cell and gives them an argument in propositional logic — you can assume that the argument consists of five premises and involves at least seven atomic sentences — for them to prove or disprove its validity. The term is that the matter has to be settled in three minutes and the ten of them will be released or detained together depending on whether one of them can produce a right answer in the set period of time. Now, even if the students have taken logic courses from different teachers, and they are capable of employing various methods for the task<sup>3</sup>, it is likely that none of them can finish the task in time.<sup>4</sup> When the mad scientist mocks the usefulness of logic, we simply cannot discard it by claiming that setting a time limit is pointless so far as logic is concerned. An apprentice of logic would naturally expect that the logic he or she is learning is a sharp tool rather than a dull one.

## 2. The inefficiency of existing accounts

As mentioned earlier, we have three candidates, namely, 1) truth tables, 2) natural deduction, and 3) truth trees, for the task in question.<sup>5</sup> They tackle the validity issue from different aspects, but, apparently, the *efficiency/speed* matter does not concern them at all. Recall that an argument  $\alpha, \beta, \gamma / \therefore \delta$  is valid provided that if all of the premises  $\alpha, \beta$ , and  $\gamma$  are true, then  $\delta$  is true. Let us review how the three approaches in question deal with the validity issue, and how they fare insofar as the time issue is concerned, before we introduce a tactically more advantageous approach.

The truth table approach deals with the validity issue by considering *all possible truth-assignments* to the atomic formulas involved in the argument and checking that there is indeed no truth-assignment that renders all the premises true and the conclusion

false. While it captures the essence of logic by requiring us to review all possible cases, the number of rows/possible-worlds grows exponentially with the number of atomic formulas, thus it is by no means a tactics that we would like to adopt in a fight.

Natural deduction grants us a set of useful rules of inference and two powerful schemes of proof, namely, Conditional Proof and Indirect Proof, and it, in effect, releases us from the burden of exponential growth. However, doing the formal, linear derivation can in itself be trivial and line-consuming at times, let alone that one can get stuck in the middle of a proof, as, unlike the truth table method, there is no step-by-step operational guide for the construction of a natural deduction proof. Furthermore, it is powerless when faced with an invalid argument.

In contrast, a truth tree typically takes less time to complete than a truth table, and, it has a step-by-step guide for the entire operation. However, there is a fair amount of unnecessary repetition of formulas — in the process of the decomposition of a sentence into its components, each component is written again in a branch subsequently — and there can be unnecessary branchings due to its indifference to the order of the treatment of sentences. When it comes to the task of proving the validity of an argument in a couple of minutes, the unnecessary repetition and branching can become a matter of winning or losing a contest as they can prolong the process by dozens of seconds.

To avoid the above-mentioned disadvantages, we should first note that from a tactical point of view, every move should contribute to the defeating of the opponent, and every unnecessary move should be skipped to speed up the process — with in mind the metaphor that you are fighting a skillful fighter and every *unnecessary* fancy move of yours increases your chance of being beaten up by him. If your ultimate goal is merely to determine the validity of an argument, then, to save time and energy, your every move should contribute to the finding of a world such that all the premises are true yet the conclusion is false.

Now, observe the following.

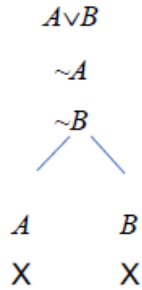
First, if the goal is to find out, as fast as possible, whether it is possible to find a world  $w$  such that  $\alpha$ ,  $\beta$ , and  $\gamma$  are true and  $\delta$  is false, then there seems to be no reason to consider *all* possible worlds in the first place for the task. We should concentrate on worlds meeting those constraints alone. So, the truth table method is by no means an ideal

way to adopt.

Second, to effectively prove the validity of  $\alpha, \beta, \gamma / \therefore \delta$  we should note that the conclusion in question is set to be  $\delta$  already, and there is no point to consider what the truth of  $\alpha, \beta$ , and  $\gamma$  can entail in general and wander around to find a route that leads to the truth of the conclusion in the end. Rather, it suggests us to adopt the Indirect Proof scheme in the first place so that the target is clearly specified. Furthermore, as proving validity involves consideration of the truth values of the premises and the conclusion, there is no intrinsic need to always turn it into a syntactic game of deriving new formulas from given formulas via certain rules of inference, unless by doing so we can beat every alternative method.<sup>6</sup> After all, imposing a suitable truth value to each formula of an argument to see if it is possible to find a world falsifying the validity of the argument is a more straightforward — pertaining to the argument in question — approach than dealing with formulas and the relations between them at an abstract level. Therefore, there stands a chance that we can find a method that can defeat natural deduction in a contest of speed.

Third, the truth tree method starts with  $\alpha, \beta, \gamma, \sim\delta$ , which amounts to looking specifically at the argument itself and concentrate on the possibility that all premises are true and the conclusion is false. So, it bypasses the problems of the two approaches we just mentioned. However, in a truth tree, the truth of a particular sub-formula is represented by rewriting the sub-formula as a stand-alone formula in a branch below the original compound formula. This is a time-consuming repetition that can be avoided. Furthermore, when the decomposition of a compound sentence requires a splitting into two branches, the splitting is to be carried out sooner or later. However, such splitting can actually be avoided, if we are allowed to update the truth of a formula wherever it appears, once we have pinned down its truth value.

More specifically, the truth tree method, in its present form, simply does not grant us a way to assert the truth or falsehood of a formula. For example, to prove the validity of  $A \vee B, \sim A / \therefore B$  with a truth tree, we should do as follows,



And the splitting is inevitable. However, if we are allowed to use the bit of information that  $B$  is false<sup>7</sup> before the splitting, then we can derive that  $A$  has to be true for  $A \vee B$  to be true, and thus obtain a contradiction that  $\sim A$  and  $A$  hold at the same time without enforcing any branching.

### 3. The sharpening of the tool

While the truth tree method can beat the truth table method in speed, and it fares better than natural deduction in that it has a step-by-step guide to obtain a proof, there are four technical issues with it, and we will resolve each of these issues and come up with a new method which not only beats the three general methods mentioned earlier, but also beats the short truth table method which is known for its swiftness in cases not involving any branching.

- 1) (The Assertion) A proposition and the truth of it are two distinct entities.

However, the truth tree method uses ' $P$ ' to stand for the truth of  $P$  and uses ' $\sim P$ ' to stand for the falsehood of  $P$ . This not only conflates mention and use, but also conflates object-language and meta-language. For example, to assert that a sentence in the object language is false, one is supposed to use a metalanguage, but, instead, the truth tree method asks us to simply write down another sentence in the object language. In principle, we need to introduce one predicate in the metalanguage for the truth of  $P$  and another for the falsehood of  $P$ .

This may not complicate our notation too much, because insofar as the truth of a sentence is concerned, we only have two predicates, namely, "is true" and "is false", and we can simply use a dot on top of the main connective of a sentence (or a dot on top of an atomic sentence itself) to stand for 'is true', and use a dot below the main connective of a sentence (or a dot below an atomic sentence itself) to stand for 'is false'. For instance,  $\dot{P}$

stands for ‘ $P$  is true’,  $P$  stands for ‘ $P$  is false’,  $\sim P$  stands for ‘ $\sim P$  is true’, and  $A \vee B$  stands for “ $A \vee B$  is true”.

2) (The Branching) According to the truth tree method, the branch containing  $A \vee B$  *should* be splitted into two branches because the truth of  $A \vee B$  entails that either  $A$  is true or  $B$  is true. However, if we already know that the branch containing  $A \vee B$  contains  $\sim B$  as well, we should be able to eliminate the possibility of  $B$ ’s being true on site. So, for  $A \vee B$  to be true,  $A$  has to be true, and there is no need to go along with the splitting. The truth tree method does not allow us to do so because on the face of it,  $A \vee B$  and  $\sim B$  are two formulas and the truth tree does not implement any rule of inference that leads us from two formulas to a new formula. By adopting the notation introduced in 1), we can, however, go from  $A \vee B$  and  $B$  to  $A$  based on the truth function associated with the disjunctive  $\vee$  alone.

3) (The Ordering) In the truth tree method, all branchings have to be carried out eventually, so there is no need for us to decide which formula is to be dealt with first. However, as we have seen in 2), substituting the known truth (or falsehood) of an atomic formula into all its appearances in the argument can potentially reduce the number of branchings that we need to perform. Therefore, from a tactical point of view, we can order our treatment of formulas in a better way so as to avoid unnecessary splitting of cases and save time. The strategy we will adopt is actually very simple: we should deal with those compound sentences whose truth values fix the truth values of its components first, and return to the treatment of the others only after we have finished these non-branching analyses and have updated the truth values of atomic formulas that had been pinned down earlier.

For instance, to prove the validity of  $A \rightarrow B, B \rightarrow C / \therefore A \rightarrow C$ , we consider the possibility of  $A \rightarrow B, B \rightarrow C$ , and  $A \rightarrow C$ , but we should not deal with the truth analysis of these three formula in turn. We should note that the truth of each of the first two formulas will lead us to a branching, while the truth value (i.e. falsehood) of the last one fixes the truth of its components, namely,  $A$  and  $C$ , so we should start with the last formula first. And the beautiful thing here is that, after we update the truth values of  $A$  and  $C$  to  $\dot{A}$  and  $\dot{C}$  in the first two formulas, and obtain  $\dot{A} \rightarrow B$  and  $B \rightarrow \dot{C}$  respectively, we will find that there is no

longer any need for branching. The statement  $\dot{A} \rightarrow B$  will lead to  $\dot{B}$ , and the statement  $B \rightarrow \dot{C}$  will lead to  $\dot{B}$ , and as  $B$  cannot be both true and false, there is no way for  $A \rightarrow B$ ,  $B \rightarrow C$ , and  $A \rightarrow C$  to hold at the same time, hence the original argument is valid.

4) (The Repetition) As mentioned earlier in 1), the truth tree method conflates the metalanguage with the object language, and a time-consuming consequence of it is that formulas are repetitively written — unnecessarily. To assert the falsehood of  $\alpha$ , a truth tree theorist has to write  $\sim\alpha$ , which in effect rewrite  $\alpha$  again. Furthermore, to analyze the possibility of the falsehood of  $A \rightarrow (B \rightarrow (C \rightarrow D))$ , one would draw a truth tree like

$$\begin{array}{c} \sim(A \rightarrow (B \rightarrow (C \rightarrow D))) \\ A \\ \sim(B \rightarrow (C \rightarrow D)) \\ B \\ \sim(C \rightarrow D) \\ C \\ \sim D \end{array}$$

Notice how ink-consuming, if not time-consuming, this procedure is. In contrast, adopting the notation we suggest in 1), to assert the falsehood of  $\alpha$ , we only need to add a dot below  $\alpha$ , without rewriting  $\alpha$  itself again, and to analyze the possibility of the falsehood of  $A \rightarrow (B \rightarrow (C \rightarrow D))$ , we simply do as follows.<sup>8</sup>

$$\dot{A} \rightarrow (\dot{B} \rightarrow (\dot{C} \rightarrow \dot{D}))$$

Note that every step in the truth tree is reduced to a single dot in our new notation, which reflects the essence of assertion — to assert is not to restate.

Now, before we introduce a swift, nearly automatic procedure of proving the validity of an argument, we should note that, despite of having the afore-mentioned four improvements, there is, in principle, no guarantee that all branchings can be avoided. When facing an unavoidable branching, we have the following two suggestions.

First, to prove the validity of an argument of the form  $\alpha, \beta, \gamma \therefore \sigma \wedge \delta$ , we should consider the possibility of finding a world such that  $\alpha, \beta, \gamma$ , and  $\sigma \wedge \delta$ , where the falsehood

of  $\sigma \wedge \delta$  entails that either  $\sigma$  or  $\delta$ , and if no atomic formula receives a definite truth value from  $\alpha$ ,  $\beta$ , and  $\gamma$ , then this amounts to a splitting that is unavoidable. For this situation, we think the best strategy is to simply regard the argument as two different arguments with the same premises, namely,  $\alpha, \beta, \gamma / \therefore \sigma$  and  $\alpha, \beta, \gamma / \therefore \delta$ , and deal with  $\alpha, \beta, \gamma$ , and  $\sigma$ , and with  $\alpha, \beta, \gamma$ , and  $\delta$ , respectively.

Second, when none of  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  allows us to pin down the truth value of any subformula, we are forced to consider branching. Here we do not suggest having a spatial splitting as that appears in a truth tree, rather we suggest using two distinct symbols to serve as two distinct continuations of the black dot  $\bullet$  and carry on with the truth analysis for each symbol. For instance, we can use a blue dot (represented by  $\circledast$ ) and a red dot (represented by  $\circ$ ) to mark two distinct truth attributions respectively — they can be replaced by other symbols,  $\bar{\cdot}$  and  $\hat{\cdot}$  respectively, say, if you do not have a color pen.

For example, to prove the validity of  $A \rightarrow B, B \rightarrow C, C \rightarrow D / \therefore (A \rightarrow C) \wedge (B \rightarrow D)$ , we can treat it as two arguments (with conclusions  $A \rightarrow C$  and  $B \rightarrow D$  respectively) and deal with them independently, but we can prove it in the following way too.

$$\overset{\circledast}{A} \bullet \rightarrow \overset{\circ}{B}, \overset{\circ}{B} \bullet \rightarrow \overset{\circ}{C}, \overset{\circ}{C} \bullet \rightarrow \overset{\circ}{D} / \therefore (\overset{\circledast}{A} \rightarrow \overset{\circ}{C}) \wedge (\overset{\circ}{B} \rightarrow \overset{\circ}{D})$$

or

$$\bar{A} \rightarrow \hat{B}, \hat{B} \rightarrow \bar{C}, \bar{C} \rightarrow \hat{D} / \therefore (\bar{A} \rightarrow \bar{C}) \wedge (\hat{B} \rightarrow \hat{D})$$

Note that a black dot  $\bullet$  indicates a truth prescription that holds throughout the analysis, but  $\circledast$  and  $\circ$  — alternatively,  $\bar{\cdot}$  and  $\hat{\cdot}$  — need to be read as being associated with two branches that do not interfere with each other. In other words, when you are considering the branch associated with  $\bar{\cdot}$ , you simply ignore all  $\hat{\cdot}$ s, and treat  $\bar{\cdot}$  and  $\cdot$  both as a dot.<sup>9</sup> When a formula ends up with a dot on top of it and a dot below it, we arrive at a contradiction. For example,  $\hat{P}, \bar{P}, \bar{\bar{P}}, \hat{\hat{P}}, \hat{\bar{P}}, \bar{\hat{P}}$  are all contradictions, but  $\bar{\bar{P}}$  and  $\hat{\hat{P}}$  are not. To finish the proof, we need two contradictions, one involving  $\bar{\cdot}$  and the other involving  $\hat{\cdot}$ .<sup>10</sup>

#### 4. A step-by-step guide to proving the validity of an argument



Now, we provide a step-by-step guide to proving the validity of an argument in propositional logic. Given an argument  $\alpha, \beta, \gamma \therefore \delta$ , to prove its validity, do as follows.

Preparatory Steps:

1. (i) If the conclusion  $\delta$  is of the form  $\sigma \wedge \tau$ , divide the argument into two arguments  $\alpha, \beta, \gamma \therefore \sigma$  and  $\alpha, \beta, \gamma \therefore \tau$ , and deal with them respectively. The original argument is valid if and only if the two sub-arguments are both valid.
- (ii) If the conclusion  $\delta$  is of the form  $\sigma \leftrightarrow \tau$ , divide the argument into two arguments  $\alpha, \beta, \gamma \therefore \sigma \rightarrow \tau$  and  $\alpha, \beta, \gamma \therefore \tau \rightarrow \sigma$ , and deal with them respectively. The original argument is valid only when both of the sub-arguments are valid.
2. Add a black dot on top of each premise and a black dot below the conclusion, and obtain something like  $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ , and  $\delta$ . If a formula is an atomic formula, simply place the dot on top of or below it, and if the formula is a compound formula, place the dot on top of or below its main connective.

Truth Derivation and Substitution Steps:

1. (Non-branching) Find a non-splitting statement and proceed adding dots on top of or below some symbols whose truth is uniquely determined by existing dots. Furthermore, whenever an atomic formula is fixed with a dot, its appearances elsewhere should receive the same treatment.<sup>11</sup> More specifically, by “non-splitting statements” and their respective dot-adding rules, we mean the following:
  - (i)  $\sim\alpha \Rightarrow \sim\dot{\alpha}$
  - (ii)  $\neg\alpha \Rightarrow \neg\dot{\alpha}$
  - (iii)  $\sigma \wedge \tau \Rightarrow \dot{\sigma} \wedge \dot{\tau}$
  - (iv)  $\sigma \vee \tau \Rightarrow \dot{\sigma} \vee \dot{\tau}$
  - (v)  $\sigma \rightarrow \tau \Rightarrow \dot{\sigma} \rightarrow \dot{\tau}$
2. (Quasi-non-branching) After one component of a compound formula receives a dot from truth analysis elsewhere, a splitting statement can become non-splitting, allowing us to infer the truth value of the other component from known facts. More specifically, we have the following

- (i)  $\dot{\sigma} \wedge \tau \Rightarrow \dot{\sigma} \wedge \bar{\tau}$  and  $\sigma \wedge \dot{\tau} \Rightarrow \sigma \wedge \dot{\tau}$
- (ii)  $\sigma \vee \tau \Rightarrow \sigma \vee \dot{\tau}$  and  $\sigma \vee \bar{\tau} \Rightarrow \dot{\sigma} \vee \bar{\tau}$
- (iii)  $\dot{\sigma} \rightarrow \tau \Rightarrow \dot{\sigma} \rightarrow \dot{\tau}$  and  $\sigma \rightarrow \bar{\tau} \Rightarrow \sigma \rightarrow \bar{\tau}$
- (iv)  $\dot{\sigma} \leftrightarrow \tau \Rightarrow \dot{\sigma} \leftrightarrow \dot{\tau}$ ,  $\sigma \leftrightarrow \dot{\tau} \Rightarrow \dot{\sigma} \leftrightarrow \dot{\tau}$ ,  $\sigma \leftrightarrow \bar{\tau} \Rightarrow \sigma \leftrightarrow \bar{\tau}$ , and  $\sigma \leftrightarrow \dot{\tau} \Rightarrow \sigma \leftrightarrow \dot{\tau}$
- (v)  $\dot{\sigma} \nrightarrow \tau \Rightarrow \dot{\sigma} \nrightarrow \bar{\tau}$ ,  $\sigma \nrightarrow \dot{\tau} \Rightarrow \sigma \nrightarrow \dot{\tau}$ ,  $\sigma \nrightarrow \bar{\tau} \Rightarrow \sigma \nrightarrow \bar{\tau}$ , and  $\sigma \nrightarrow \dot{\tau} \Rightarrow \sigma \nrightarrow \dot{\tau}$

On the other hand, we can sometimes derive the truth value of a compound formula from the truth value of one component also. Specifically, we have

- (i)  $\sim \dot{\alpha} \Rightarrow \sim \dot{\alpha}$  and  $\sim \alpha \Rightarrow \sim \alpha$
- (ii)  $\sigma \wedge \tau \Rightarrow \sigma \wedge \tau$  and  $\sigma \wedge \bar{\tau} \Rightarrow \sigma \wedge \bar{\tau}$
- (iii)  $\dot{\sigma} \vee \tau \Rightarrow \dot{\sigma} \vee \tau$  and  $\sigma \vee \dot{\tau} \Rightarrow \sigma \vee \dot{\tau}$
- (iv)  $\sigma \rightarrow \tau \Rightarrow \sigma \rightarrow \tau$  and  $\sigma \rightarrow \dot{\tau} \Rightarrow \sigma \rightarrow \dot{\tau}$

Note: Whenever the truth of a premise or the falsehood of the conclusion has incidentally been established during this process, we would no longer need to do any further truth value analysis to it. It can be simply marked as “done”.

3. (Branching) If after carrying out the truth analysis of all non-splitting statements, we still could not establish the truth of all premises and the falsehood of the conclusion, i.e. they have not all been marked as “done” yet, then arbitrarily pick a splitting statement, do the following, and subsequently deal with the  $\bar{\quad}$ -adding and the  $\hat{\quad}$ -adding processes respectively.

- (i)  $\sigma \wedge \tau \Rightarrow \sigma \wedge \bar{\tau}$
- (ii)  $\sigma \vee \tau \Rightarrow \sigma \vee \dot{\tau}$
- (iii)  $\sigma \rightarrow \tau \Rightarrow \sigma \rightarrow \dot{\tau}$
- (iv)  $\sigma \leftrightarrow \tau \Rightarrow \sigma \leftrightarrow \bar{\tau}$
- (v)  $\sigma \nrightarrow \tau \Rightarrow \sigma \nrightarrow \dot{\tau}$

Decision Time:

1. For an argument involving no branching, if, in the process of dot-adding, at any moment a formula has received a dot on top of it and below it, we can stop the process and claim that the argument in question is valid —there is no way for all the premises to be true yet the conclusion be false. There is a fair

chance that the validity of an argument can be established this way.

2. For an argument involving a single branching,  $\neg$  and  $\wedge$  (equivalently,  $\otimes$  and  $\circ$ ), the appearance of a  $\dot{P}$ ,  $\bar{P}$ , or  $\bar{P}$  (alternatively,  $\overset{\bullet}{P}$ ,  $\overset{\circ}{P}$ , or  $\overset{\circ}{P}$ ) counts as a contradiction for the  $\neg$  branch, and the appearance of a  $\dot{P}$ ,  $\hat{P}$ , or  $\hat{P}$  (alternatively,  $\overset{\bullet}{P}$ ,  $\overset{\circ}{P}$ , or  $\overset{\circ}{P}$ ) counts as a contradiction for the  $\wedge$  branch. Only after one secures **both** types of contradiction, namely  $(\cdot, \neg)$  and  $(\cdot, \wedge)$ , (alternatively,  $(\bullet, \otimes)$  and  $(\bullet, \circ)$ ), can he or she claim the validity of the argument in question.
3. With the scheme prescribed above, rarely do we ever need to deal with a second splitting for a real-life argument.<sup>12</sup> However, even if, in the unlikely event, another splitting is required, we can still proceed as follows. For each existing branch, introduce two new types of “dots” for a new branching. For instance,  $\neg$  is to be followed by  $\neg$  and  $\neg$ , and  $\wedge$  is to be followed by  $\wedge$  and  $\wedge$ . And to obtain a contradiction, we now need four contradictions involving  $(\cdot, \neg, \neg)$ ,  $(\cdot, \neg, \neg)$ ,  $(\cdot, \wedge, \wedge)$ , and  $(\cdot, \wedge, \wedge)$  respectively. Further splitting can be dealt with in the same vein, though you most likely would never encounter such cases in real life.
4. If the specific requirement set in Preparatory Step 2 — namely having all premises true and the conclusion false — can be accomplished without obtaining a full contradiction,<sup>13</sup> then the argument is invalid, because we could find a truth assignment such that all the premises are true and the conclusion is false.

## 5. Demonstrations

Some, if not most, textbooks of logic arrange their exercises from easy to difficult, section-wise as well as problem-wise. To demonstrate that our new procedure can elegantly deal with all exercises found in a standard introductory logic textbook, we can pick up a standard textbook, find the last section that completes the introduction of natural deduction, and take the last few problems in its exercise section as our test cases (they are presumably the most difficult ones). Then do the same for another book. The

books we consider here are Kahane (2003), Copi (2002), Gensler (2002) and Smith (2003), which I could reach out and take down from my bookshelf effortlessly.

Note that, while branching can complicate our treatment slightly, the most “difficult” exercises in a standard book may not involve branching at all, hence are deemed “easy” by our standard. Therefore, being able to deal effectively with the most “difficult” exercises in those books does not imply that we can deal with all the exercises therein easily. So, I will handpick a couple of not-so-difficult exercises also for the purpose of illustrating how branching is to be treated.

The following list is what we get.<sup>14</sup> The remarkable thing to observe here is that they are not just ten advanced exercises waiting to be treated, they are the proofs (or disproof) of the validity of these arguments as well. Furthermore, for a reader familiar with the above procedures, it can take less than ten minutes *altogether* to finish these ten problems. (For easy reference, contradictions are highlighted.)

1.  $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) / \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
2.  $\dot{D} \rightarrow (\dot{E} \rightarrow \dot{F}), \dot{H} \rightarrow (\dot{E} \vee \dot{G}), \dot{D} \rightarrow (\dot{G} \rightarrow \sim \dot{H}) / \therefore \sim(\sim F \rightarrow \sim \dot{H}) \rightarrow \sim \dot{D}$
3.  $\dot{N} \rightarrow (\dot{P} \rightarrow \dot{R}), \sim(\dot{R} \rightarrow \dot{S}) \rightarrow \sim \dot{T}, (M \vee \sim \dot{T}) \vee \sim(\dot{P} \rightarrow \dot{S}) / \therefore \sim M \rightarrow \sim(\dot{N} \wedge \dot{T})$
4.  $(H \vee (I \vee J)) \rightarrow (K \rightarrow J), \dot{L} \rightarrow (I \vee (J \vee H)) / \therefore (\dot{L} \wedge K) \rightarrow J$
5.  $(\dot{R} \vee \dot{S}) \rightarrow (T \wedge U), \sim \dot{R} \rightarrow (\dot{V} \rightarrow \sim \dot{V}), \sim T / \therefore \sim \dot{V}$
6.  $\sim \dot{F} \vee \sim(\sim(\dot{G} \wedge \dot{H}) \wedge (\dot{G} \vee \dot{H})), (\dot{G} \rightarrow \dot{H}) \rightarrow ((\dot{H} \rightarrow \dot{G}) \rightarrow \dot{I}) / \therefore \dot{F} \rightarrow (\dot{F} \wedge \dot{I})$
7.  $\sim(A \vee B) \rightarrow (\dot{C} \rightarrow \dot{D}), \sim A \wedge \sim \dot{D} / \therefore \sim B \rightarrow \sim \dot{C}$
8.  $\dot{B} \rightarrow \dot{A}, \sim(\dot{A} \wedge \dot{C}), \dot{B} \vee \dot{C} / \therefore \dot{A} \leftrightarrow \dot{B}$
9.  $/ \therefore \sim(P \leftrightarrow Q) \leftrightarrow ((\sim P \wedge Q) \vee (P \wedge \sim Q))$   
 Sub-Argument 1:  $\sim(\dot{P} \leftrightarrow \dot{Q}) \rightarrow ((\sim \dot{P} \wedge \dot{Q}) \vee (\dot{P} \wedge \sim \dot{Q}))$   
 Sub-Argument 2:  $((\sim \dot{P} \wedge \dot{Q}) \vee (\dot{P} \wedge \sim \dot{Q})) \rightarrow \sim(\dot{P} \leftrightarrow \dot{Q})$
10.  $\dot{P} \wedge (Q \vee \dot{R}), \dot{P} \vee \sim(U \vee \dot{R}), (U \wedge \dot{S}) \rightarrow \sim Q / \therefore (Q \vee \dot{R}) \wedge \dot{S}$  (invalidity: no contradiction is obtained)

Note:

- (i) In 8., as the situation is simple enough, we do not follow the Preparatory Step 1.

Instead, we deal with it directly with a single branching.

- (ii) In 9., the  $\wedge$  part can actually be omitted, by resorting to symmetry in  $P$  and  $Q$  and the fact that the truth functions associated with  $\wedge$ ,  $\vee$ , and  $\leftrightarrow$  are all symmetric in the two components. However, as not every learner of logic is familiar with the notion of symmetry, we shall not introduce this omission into our framework. More advanced reader can resort to symmetry at their discretion.
- (iii) In 10., The  $\wedge$  branching does not yield a contradiction, so the argument is invalid.
- (iv) If one has color pens, Exercise 6 can be presented in a colorful way as well (blue dots and red dots are represented by  $\circ$  and  $\bullet$  respectively here).

$$\sim F \vee \sim (\sim(G \wedge H) \wedge (G \vee H)), (G \rightarrow H) \rightarrow ((H \rightarrow G) \rightarrow I) / \therefore F \rightarrow (F \wedge I)$$

- (v) For an illustration of the efficiency of this new method, the truth table, natural deduction, and the truth tree methods for Problem 1 are listed below for comparison:

a) The truth table method (Table 1)

In principle we need 64 rows for the truth table, but as the truth of  $F$  is evidently irrelevant to the validity of the argument, we simply set its value to be false, and show only the 32 rows associated with it—if  $F$  is true, then the premise  $\sim(EVF)$  is false already, so those 32 rows can be ignored—to save space.

A	B	C	D	E	$\sim A$	$\sim B$	CVD	.....	$\sim(D \wedge \sim(EVB))$	$\sim(EVF)$	$C \rightarrow (EVA)$	$\sim(\sim A \wedge \sim B) \vee \sim(CVD)$
T	T	T	T	T	F	F	T		T	F	T	T
T	T	T	T	F	F	F	T		<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>
T	T	T	F	T	F	F	T		T	F	T	T
T	T	T	F	F	F	F	T		<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>
T	T	F	T	T	F	F	T		T	F	T	T
T	T	F	T	F	F	F	T		<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>
T	T	F	F	T	F	F	F		T	F	T	T
T	T	F	F	F	F	F	F		<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>
T	F	T	T	T	F	T	T		T	F	T	T
T	F	T	T	F	F	T	T		F	T	T	T
T	F	T	F	T	F	T	T		T	F	T	T
T	F	T	F	F	F	T	T		<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>
T	F	F	T	T	F	T	T		T	F	T	T
T	F	F	T	F	F	T	T		F	T	T	T

T	F	F	F	T	F	T	F		T	F	T	T
T	F	F	F	F	F	T	F		T	T	T	T
F	T	T	T	T	T	F	T		T	F	T	T
F	T	T	T	F	T	F	T		T	T	F	T
F	T	T	F	T	T	F	T		T	F	T	T
F	T	T	F	F	T	F	T		T	T	F	T
F	T	F	T	T	T	F	T		T	F	T	T
F	T	F	T	F	T	F	T		T	T	T	T
F	T	F	F	T	T	F	F		T	F	T	T
F	T	F	F	F	T	F	F		T	T	T	T
F	F	T	T	T	T	T	T		T	F	T	F
F	F	T	T	F	T	T	T		F	T	F	F
F	F	T	F	T	T	T	T		T	F	T	F
F	F	T	F	F	T	T	T		T	T	F	F
F	F	F	T	T	T	T	T		T	F	T	F
F	F	F	T	F	T	T	T		F	T	T	F
F	F	F	F	T	T	T	F		T	F	T	T
F	F	F	F	F	T	T	F		T	T	T	T

Table 1. A truth table

As whenever the premises are all true, the conclusion is true, the argument is valid.

b) Natural deduction

1.  $\sim(D \wedge \sim(E \vee B))$
2.  $\sim(E \vee F)$
3.  $C \rightarrow (E \vee A) \quad \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
4.  $\sim(\sim(\sim A \wedge \sim B) \vee \sim(C \vee D))$  AP
5.  $\sim\sim(\sim A \wedge \sim B) \wedge \sim\sim(C \vee D)$  DeM 4
6.  $(\sim A \wedge \sim B) \wedge (C \vee D)$  DN 5
7.  $\sim A \wedge \sim B$  Simp 6
8.  $C \vee D$  Simp 6
9.  $\sim A$  Simp 7
10.  $\sim B$  Simp 7
11.  $\sim D \vee \sim\sim(E \vee B)$  DeM 1
12.  $\sim D \vee (E \vee B)$  DN 11
13.  $(\sim D \vee E) \vee B$  Assoc 12
14.  $\sim D \vee E$  DS 10, 13



Figure 1. A truth tree

d) The new method

$$\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$$

Method d) is clearly more succinct than methods a), b), and c). However, while a martial art master can knock his opponent down with a rapid succession of punches in an MMA<sup>15</sup> match, he may, for teaching purposes, have to demonstrate his moves in slow motion afterwards, so that his pupils can also learn to master that sequence of punches to the extent that their punches are heavy and fast enough to knock down their opponents in a real battle. The following is a step by step illustration of the proof of validity with method d). At each stage, we only add a couple of dots — those highlighted — to the argument.

- Snapshot 1     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 2     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 3     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 4     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 5     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 6     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 7     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 8     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 9     $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$
- Snapshot 10    $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$



Finally, the new method beats the short truth table test also, because the latter is merely a test that works for a restricted set of cases, while the former works generally for all possible cases. Even for those unbranching cases, our notation is much more concise than the short truth table test.

## 6. Logic for everyday use

If a student of logic is learning it for pragmatic purpose, then we had better check whether each of the four methods can indeed help him or her explain to a layperson why the argument in question is valid. Recall that in a standard logic course, students are accustomed to the process of translating an everyday argument into an argument in the language  $L$  of propositional logic and then determining the validity with various techniques. However, to complete the entire process, I think we should translate the treatment back into everyday language as well so that a listener who is not familiar with propositional logic can also understand why the argument is valid or invalid. In other words, we need steps 4 and 5 in the following diagram (Figure 2).

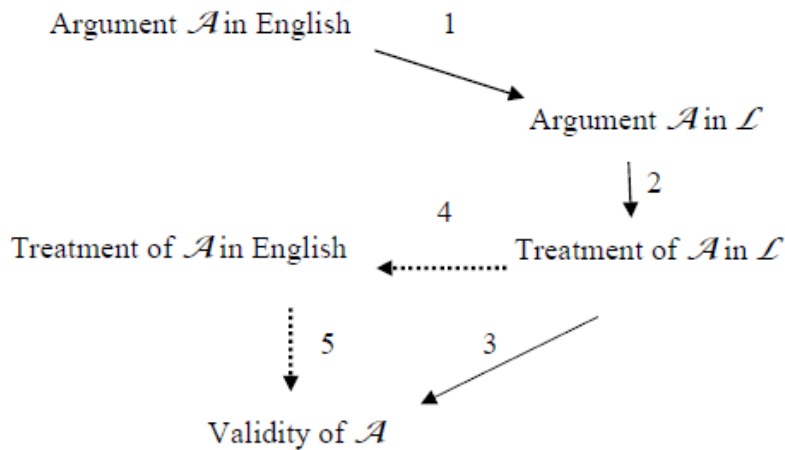


Figure 2. Treatment of a real-life argument

Clearly, the truth table method and the truth tree method in the previous section cannot be properly translated into English. Even if someone comes up with a nice way to translate a truth table or a truth tree into English, it is likely that the language will be

incomprehensible to the untrained ear. In contrast, each step in the natural deduction proof in the previous section can be translated into English. Nevertheless, the nearly two dozen steps involved in the proof still makes it difficult for the reader to follow and grasp the whole deduction in English.

However, we claim that each dot in the new method can be straightforwardly translated into English and, as the derivation is relatively shorter than that involved in a natural deduction proof, it is indeed possible for a listener to understand the “logic” of the argument in English. Consider the following argument, which is an English version of Exercise 1 of the last section.

To say that David will come yet it is not true that Edward or Bill will come, you will be making a false statement. It is not the case that Edward or Frank will come. If Carl comes, Edward or Andy will come. Therefore, it is not the case that both Andy and Bill will not come or it is not the case that Carl or David will come.

Turning the proof  $\sim(D \wedge \sim(E \vee B)), \sim(E \vee F), C \rightarrow (E \vee A) / \therefore \sim(\sim A \wedge \sim B) \vee \sim(C \vee D)$  step by step into English, we obtain the following:

Suppose all premises are true yet the conclusion is false. Then by the falsehood of the conclusion, we would obtain that both Andy and Bill will not come, and Carl or David will come. By the second promise, it is not the case that Edward or Frank will come, so both Edward and Frank will not come. Now, as both Edward and Andy will not come, it is false that Edward or Andy will come, thus, by the third premise, Carl will not come. Furthermore, as both Edward and Bill will not come, it is not the case that Edward or Bill will come, thus, by the first premise, David will not come. But we already know that Carl or David will come and Carl won't come, so David will come. Thus, a contradiction arises. In other words, if all the premises are true then the conclusion has to be true as well, hence the argument is valid.

This is some sort of “argument” that we may encounter in everyday life and most competent users of English will have no difficulty following it. Note that the dots in our method play important roles in guiding us to come up with the proof in English. This closeness to everyday reasoning alone is a welcome feature of the new method, in addition to the effectiveness of the method that we stress in earlier sections.

While we have turned Exercise 1 in the previous section into a real-life argument and deal with it accordingly in this section, some theorists might be reluctant to accept that it is indeed a *real* argument, but regard it as *ad hoc*. For instance, Arthur (2016) prefers dealing with “natural arguments, i.e., ones that have been actually offered by historical agents in real-life argumentative contexts.” (Arthur 2016: xv). In Alec Fisher’s *The Logic of Real Arguments*, it was claimed also that “[n]early all the arguments in this book are arguments which have *actually been used* by someone with a view to convincing others about some matter” (Fisher 2004, p15, original emphasis). In response to this higher standard of realistic demand, we shall, in the rest of the section, consider a handful of examples that are more realistic, but at the same time not too lengthy so that it can be dealt with in the limited space of a journal article.

Recall that we have offered a general method which can decide, for any argument written in  $\mathcal{L}$  — the language of propositional logic — whether it is valid or invalid (Steps 2 and 3 in Figure 2). Furthermore, to deal with an argument used in real life, we need to carry out an extra step, namely Step 1 in Figure 2, to translate the argument into  $\mathcal{L}$ . This is standardly covered in an elementary logic course — we set up the key of symbolization and then do the translation. In addition to the five steps of Figure 2, there is a step prior to Step 1, which is essential for real-life application of logic, but we have so far not touched upon it. This is the recognition or identification of an argument, or even the formulation of an argument. Guidance for the recognition of an argument can be found in practical-logic/rhetoric books such as Fisher (2004) and Arthur (2016). Due to space limit, we shall be content with offering only three examples (all of them involve the pinning down of a conclusion) to show that the dot method outlined in the last section can indeed be applied directly to real-life situations.

### **Example 1.** The Doctor’s Argument

Consider the following advice from a doctor. Is the doctor making an argument? If yes, is the argument valid?

Doctor: Well, both Axone and Bxone can cure this disease. However, Axone is known to have the side-effect of alopecia. I don't suppose you would like to lose your hair while taking the medicine. So, what do you think?

While the doctor is seemingly asking a question, he has actually made the following argument: (this is the step of argument recognition/identification)

*Premise 1* If you take Axone or Bxone, then your disease will be cured.

*Premise 2* If you take Axone then you will lose your hair.

*Conclusion* To have your disease cured without losing your hair, you should take Bxone.

Some, if not most, patients in the clinic might find this argument convincing and follow the doctor's suggestion to take Bxone. However, it is better for us to go through all the steps in Figure 2 to analyze this argument.

Key:

A: You take Axone

B: You take Bxone

C: The disease is cured

D: You lose your hair.

Translation:

$P_1 (A \vee B) \rightarrow C$

$P_2 A \rightarrow D$

$\therefore (C \wedge \sim D) \rightarrow B$

Dot analysis:

$(A \vee B) \rightarrow \dot{C}$

$A \rightarrow \dot{D}$

$\therefore (C \wedge \sim D) \rightarrow B$

Here, we have added all the dots without reaching a contradiction. So, it is possible that both premises are true while the conclusion is false. In other words, the argument is invalid. Now that we have found the truth assignment that make both premises true and the conclusion false, we can explain in English to the patient why the argument is actually flawed:

It is actually possible that your disease is cured without your losing hair or taking Bxone. Simply do not take Axone, as that would cause hair loss. You might wonder how then can the disease be cured. Pay attention to the fact that Premise 1 only says that taking Axone can cure the disease, so is taking Bxone, but it by no means asserts that there is no other way that the disease can be cured. So, the conclusion does not follow from the premises.

**Example 2.** Who is the killer?

Mafia boss Lido Morello was shot dead by the pool of a luxury hotel in Napoli by a professional shooter who acted alone. After a thorough investigation, the police had narrowed down the range to the notorious Sicilian Trio — namely, Alberto, Bruno, and Corrado — but they could not be sure which one of them committed the crime.<sup>16</sup>

The police fetched the three and obtained the following testimonies.

Alberto: *I am innocent. I have never been to Napoli. Bruno was the shooter.*

Bruno: *I did not shoot him. I, for one, have never been to Napoli. All Alberto said were lies.*

Corrado: *Surely, I wasn't the shooter. Actually, both Alberto and Bruno have been to Napoli. One of them killed Lido Morello.*

The problem for the police now is that each one of them could be lying, and there is no way for the police to know which sentence is true and which is false. The only thing they

can be sure of is that, according to the nature of the Sicilian Trio, none of them would ever utter three true sentences in a row.

The translation scheme can be set up as follows:

Key:

*A*: Alberto was the shooter.

*B*: Bruno was the shooter.

*C*: Carrado was the shooter.

*M*: Alberto has never been to Napoli.

*N*: Bruno has never been to Napoli.

Translation:

1.  $(A \wedge (\sim B \wedge \sim C)) \vee ((B \wedge (\sim A \wedge \sim C)) \vee (C \wedge (\sim A \wedge \sim B)))$
2.  $\sim((\sim A \wedge \sim M) \wedge B)$
3.  $\sim((\sim B \wedge \sim N) \wedge ((A \wedge M) \wedge \sim B))$
4.  $\sim((\sim C \wedge (M \wedge N)) \wedge ((A \vee B) \wedge \sim(A \wedge B)))$
5.  $\sim M \rightarrow \sim A$
6.  $\sim N \rightarrow \sim B$

Note that, here we make no attempt to write down a conclusion for these premises. The conclusion will emerge naturally after we finish the dot analysis. The reader can try, alternatively, using the truth table method, natural deduction, or the truth tree method to deal with this problem to see what a mess he or she will be led to.

Dot Analysis:

First, add all the dots before a branching ever occurs

1.  $(A \wedge (\sim B \wedge \sim C)) \vee ((B \wedge (\sim A \wedge \sim C)) \vee (C \wedge (\sim A \wedge \sim B)))$
2.  $\sim((\sim A \wedge \sim M) \wedge B)$
3.  $\sim((\sim B \wedge \sim N) \wedge ((A \wedge M) \wedge \sim B))$
4.  $\sim((\sim C \wedge (M \wedge N)) \wedge ((A \vee B) \wedge \sim(A \wedge B)))$

$$5. \sim M \rightarrow \sim A$$

$$6. \sim N \rightarrow \sim B$$

Continue with branches  $\bar{\quad}$  and  $\circ$ , and the latter is to be followed by branches  $\hat{\quad}$  and  $\check{\quad}$ :

$$1. (\bar{A} \bar{\wedge} (\bar{B} \bar{\wedge} \bar{C})) \check{\vee} ((\hat{B} \hat{\wedge} (\hat{A} \hat{\wedge} \hat{C})) \check{\vee} (\check{C} \check{\wedge} (\check{A} \check{\wedge} \check{B})))$$

$$2. \sim((\bar{A} \hat{\wedge} \hat{M}) \hat{\wedge} \hat{B})$$

$$3. \sim((\hat{B} \hat{\wedge} \hat{N}) \hat{\wedge} ((\bar{A} \bar{\wedge} \bar{M}) \bar{\wedge} \bar{B}))$$

$$4. \sim((\hat{C} \hat{\wedge} (\hat{M} \hat{\wedge} \hat{N})) \hat{\wedge} ((\bar{A} \hat{\wedge} \hat{B}) \hat{\wedge} (\bar{A} \hat{\wedge} \hat{B})))$$

$$5. \bar{M} \rightarrow \bar{A}$$

$$6. \hat{N} \rightarrow \hat{B}$$

Both the branch  $\bar{\quad}$  and the branch  $(\circ, \hat{\quad})$  lead to a contradiction with the dot, leaving us only with the last branch  $(\circ, \check{\quad})$ , which corresponds to  $\hat{A}\hat{B}\check{C}$  (note that the truths of  $M$  and  $N$  can be perceived to be irrelevant). So, the conclusion is that Carrado was the shooter.

**Example 3.** The 2021 MLB wild card race

In a Wall Street Journal article entitled “MLB’s postseason tiebreakers, explained (including the insane AL wild-card race)” written by Matt Bonesteel just before the games on the final day of the 2021 regular season<sup>17</sup>, it reads

Sunday is the final day of Major League Baseball’s regular season, and four teams — the Boston Red Sox, New York Yankees, Seattle Mariners and Toronto Blue Jays — remain alive for the two American League wild-card spots .....

As a matter of fact, before entering that Sunday’s games, the records of the four teams were: the Red Sox, 91-70; the Yankees, 91-70; the Blue Jays, 90-71; the Mariners, 90-71, and they would play their game 162, i.e. the last game of the season, with a team not in the race for a wild-card. There were chances that, even after the final games, a tiebreaker or two were needed to decide which two teams earned the wild-cards. We take this as an

example to illustrate how the dot method of this paper can help us pin down the relation between possible outcomes of the Sunday games and the need of a tiebreaker.

After explaining all possible tiebreak procedures,<sup>18</sup> Bonesteel offers the following summary.

#### What's at stake Sunday

- The Red Sox and Yankees can each clinch a spot in the wild-card game with a win Sunday afternoon. The simplest scenario would involve both teams winning, setting up a wild-card game between the rivals Tuesday at Boston's Fenway Park and removing the need for any AL tiebreaker games.
- If either the Yankees or Red Sox lose (or if both lose), the Blue Jays and Mariners would both have a chance to earn a spot in a wild-card tiebreaker game with a win Sunday. One of the tiebreaker scenarios listed above would go into effect depending on how many teams end up tied.
- If all four teams lose Sunday, the Red Sox would host the Yankees in Tuesday's wild-card game, and no AL tiebreaker games would be needed.

Now, a logic teacher may pose this question: taking as premises these three statements and the fact that no tiebreaker was ever needed after Sunday's games, what conclusion can you make. Note that this is an open question, and the truth-table method, the natural deduction method, and the truth-tree method do not offer us any guidance concerning this kind of problem. However, the dot method can lead us to some conclusion in general.

We can symbolize the statements concerning the need of a tiebreaker game in the following way. As we are only concerned with sentences about the need of a tiebreaker, the first sentence concerning the sufficient condition of securing a wild-card spot will not be listed here — the Red Sox *or* the Yankees' clinching a wild-card spot does not tell us that a tiebreaker is needed nor does it tell us that a tiebreaker is not needed. In contrast, both the Red Sox *and* the Yankees earning a wild-card spot itself would entail that there is no need for a tiebreaker, so the second sentence in the first paragraph quoted will be translated. Similarly, the first sentence of the second paragraph says that in certain condition some teams would have a chance of earning a tiebreaker spot with a win, but



we will only write down that in such and such conditions a tiebreaker would take place, without specifying which teams would play it.

Key:

$R$ : The Red Sox wins

$Y$ : The Yankees wins

$B$ : The Blue Jays wins

$M$ : The Mariners wins

$T$ : A tiebreaker is needed

Translation: (the premises)

1.  $(R \wedge Y) \rightarrow \sim T$
2.  $(\sim R \vee \sim Y) \rightarrow ((B \vee M) \leftrightarrow T)$
3.  $((\sim R \wedge \sim Y) \wedge (\sim B \wedge \sim M)) \rightarrow \sim T$
4.  $\sim T$

Dot analysis:

1.  $(R \wedge Y) \rightarrow \sim T$
2.  $(\sim R \vee \sim Y) \rightarrow ((B \vee M) \leftrightarrow T)$
3.  $((\sim R \wedge \sim Y) \wedge (\sim B \wedge \sim M)) \rightarrow \sim T$
4.  $\sim T$

Here, the  $\sim T$  in premise 4 makes premises 1 and 3 true right away. As for premise 2, each of the branches  $\bar{\quad}$  and  $\hat{\quad}$ , suffices to guarantee its truth. So we immediately reach the conclusion that both the Red Sox and the Yankees win *or* both the Blue Jays and the Mariners lose. In symbolic form, we have that  $P_1, P_2, P_3, \sim T \therefore (R \wedge Y) \vee (\sim B \wedge \sim M)$  is valid.

A logic teacher can train the student to do some hypothetical reasoning also by asking them “Had there been a tiebreaker or two after Sunday’s game, what can you say about the outcomes of Sunday’s games?” The student can then proceed as follows.

Dot analysis:

1.  $(R \wedge Y) \rightarrow \sim \dot{T}$
2.  $(\sim R \dot{\vee} \sim Y) \rightarrow ((\dot{B} \dot{\vee} \dot{M}) \leftrightarrow \dot{T})$
3.  $((\sim R \dot{\wedge} \sim Y) \wedge (\sim \dot{B} \dot{\wedge} \sim \dot{M})) \rightarrow \sim \dot{T}$
4.  $\dot{T}$

Here, the first branching, namely,  $\sim$  and  $\wedge$ , as in  $R \wedge Y$ , converges to a dot in the end, giving us  $\sim R \dot{\vee} \sim Y$ , and, the second branching, namely,  $\vee$  and  $\leftrightarrow$ , as in  $\dot{B} \dot{\vee} \dot{M}$ , which is independent of the first branching, converge to a dot in the end, giving us  $\sim \dot{B} \dot{\wedge} \sim \dot{M}$ . Therefore, we arrive at the conclusion that the Red Sox loses or the Yankees loses *and* the Blue Jays wins or the Mariners wins. In symbolic form, we have that  $P_1, P_2, P_3, T \therefore (\sim R \vee \sim Y) \wedge (B \vee M)$  is valid.

In effect, the two dot-analyses above together prove the validity of  $P_1, P_2, P_3 \therefore T \leftrightarrow ((\sim R \vee \sim Y) \wedge (B \vee M))$ . In some sense, the three statements  $P_1, P_2, P_3$  of Bonesteel, while true, are not “carving nature at its joints”. In particular,  $P_3$  is simply redundant — its truth follows from the truth of  $P_1$  and  $P_2$ . The good news is that the dot method can help us grasp the key ingredients of the tiebreaker condition and group them together in a more natural way. As a simple exercise, the teacher can ask the students to use the dot method to prove that the converse argument  $T \leftrightarrow ((\sim R \vee \sim Y) \wedge (B \vee M)) \therefore (P_1 \wedge P_2) \wedge P_3$  is valid as well.<sup>19</sup> So  $T \leftrightarrow ((\sim R \vee \sim Y) \wedge (B \vee M))$  indeed sums up  $(P_1 \wedge P_2) \wedge P_3$  nicely.

## 7. Conclusion

From a pragmatic point of view — concerning its use in the real-world and concerning its efficiency — logic has to do better than what the three existing validity-proving methods, namely the truth table method, natural deduction, and the truth tree method, have to offer. For a restricted set of cases, the short truth table method can be quick indeed, but for other cases, it can become a total mess. The new method introduced in this paper is general enough to beat these four methods in a contest of speed. Not only so, it captures our everyday reasoning patterns nicely, in the sense that its treatment of an argument can be translated into everyday language effortlessly and the listener has a fair chance of grasping the reasoning process.

Nonetheless, concerning the question of whether this new method is to replace the existing methods or just to supplement them, our answer at present can only be the modest

one: to supplement them. While this method is, in principle, general enough to replace the other methods insofar as propositional logic is concerned, it has not been properly developed into an account for predicate logic or modal logic as yet. So, for the time being, the students are still advised to acquaint themselves with natural deduction and the truth trees etc. so that they can have a smooth passage into the realm of predicate logic and modal logic, because currently most logics are still taught in these terms.

Further work is needed to apply this new method to predicate logic or modal logic, and hopefully such an account shall appear elsewhere in the near future.

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<sup>1</sup> Alternatively, a formal system with axioms and rules of inference.

<sup>2</sup> Short for 'Ultimate Fighting Championship'.

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<sup>3</sup> Natural deduction cannot prove the invalidity of an argument, of course.

<sup>4</sup> It is worth mentioning that the short truth table test which has been briefly- and informally-mentioned in some logic books, see Kahane (2003), pp. 66-71, for example, can be fast at times. But, as most logic teachers know, the authors of such books usually choose straightforward/non-branching examples purposefully to illustrate how that method works, without warning the readers that the so-called “short truth table test” has not been systematically developed into a general method that can deal with arguments involving branching cases, and that branching cases can turn the method into a mess. In a sense, the present paper fixes this problem by developing the method into a general framework that can deal with all types of arguments.

<sup>5</sup> Natural deduction is not capable of disproving the validity of an argument.

<sup>6</sup> It is well known that natural deduction often beats the truth table method in a contest of speed. However, this fact alone does not imply that it is the fastest method conceivable.

<sup>7</sup> Note that ‘*B* is false’ is not the same as ‘ $\sim B$ ’, as the former is a sentence in the meta-language while the latter is in the object language, and the truth tree method simply conflates the two. We shall stress this point further in next section.

<sup>8</sup> For those who need to visualize the order of appearances of the dot, we can of course add some indices to it.

$$\begin{array}{c} \dot{A} \rightarrow (\dot{B} \rightarrow (\dot{C} \rightarrow \dot{D})) \\ 2 \ 1 \ 4 \ 3 \ 6 \ 5 \ 7 \end{array}$$

<sup>9</sup> The case for  $\wedge$  and  $\vee$  can be similarly treated.

<sup>10</sup> For a non-branching case, it suffices to obtain a contradiction of the form  $\dot{\alpha}$ .

<sup>11</sup> This is what we mean by “truth substitution”.

<sup>12</sup> The reader can pick an arbitrary introductory logic textbook and check for himself/herself that this is the case.

<sup>13</sup> That is, one contradiction for a non-branching case, two for a single branching case, and four for a double branching case.

<sup>14</sup> They are from the following sources respectively: 1. Kahane (2003), the last problem, Problem (15), in Exercise 4-13 (before the introduction of CP and IP); 2. Kahane (2003), the last but one problem, Problem 14, in Exercise 5-5; 3. Kahane (2003), the last

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problem, Problem (15), in Exercise 5-5; 4. Copi (2002) p368, Problem 10\*; 5. Copi (2002) p368, Problem 20\*; 6. Copi (2002) p368, Problem 17; 7. Gensler (2002) p91, Problem 8; 8. Gensler (2002) p91, Problem 3; 9. Gensler (2002) p108, Problem V(8); 10. Smith (2003) pp165-66, Example C.

<sup>15</sup> It stands for Mixed Martial Arts.

<sup>16</sup> A version of this story was told, with no analysis, in Tsai (2003) to hint at the inefficiency, even ineffectiveness, of existing logical methods in dealing with real-life problems.

<sup>17</sup> It was updated on Sunday, October 3, 2021 at 6:53 p.m. EDT after the games in question. See Bonesteel (2021).

<sup>18</sup> For a four-way tie; a three-way tie for two wild-card spots; a three-way tie for the second wild-card spot; a two-way tie for the second wild-card spot; and, finally, a two-way tie for the first wild-card spot. For all but the last possibility, tiebreaker(s) would be needed.

<sup>19</sup> For instance, one can first prove the validity of  $P_1, P_2 \therefore P_3$ , then prove the validity of  $T \leftrightarrow ((\sim R \vee \sim Y) \wedge (B \vee M)) \therefore P_1 \wedge P_2$ .