A Formal Framework for Future Contingents

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Abstract

In this article, I present a formal semantic framework that renders explicit how to reconcile the condition that a proposition about a contingent future event is true at a moment t_0 with the idea that at t_0, this proposition is 'truth-maker indeterminate': a state of affairs making it true will obtain later on, though no such state of affairs obtains at t_0. The semantics I formulate employs 'open temporal models'. They represent the passage of time by a specific component termed *time-resource*, which acts on durations construed as *model-external* inputs. A model does not by itself specify which course of events gets actualized in a given duration depending on the latest moment that has already got actualized. A time-resource merely represents schematically the dependence between a moment t and a course of events that gets actualized in a time-span of a given length counted from t; until that much time has indeed passed, it is not fixed which course of events actually extends t. Further, I introduce *evaluations* as a fine-grained tool for studying truth-conditions of tensed formulas, and I use this tool to define the notion of truth-maker. I define what it means that a truth-maker will obtain but does not, and what it means for a truth-maker to be determinate. It is proven that my semantic analysis retains the desirable link between determinacy and historical necessity—namely, a truth-maker of a proposition being determinate entails that the proposition is historically necessary.

1. Modeling the passage of time

This article is a follow-up to the paper 'The Truth of Future Contingents: An Analysis of Truth-Maker Indeterminacy',[1] to which I will refer as 'the background paper'. The reader may wish to get acquainted with the background paper first, in order to appreciate the motivation for the

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framework developed here. However, the present paper is self-contained insofar as the description of the framework and the study of its properties are concerned.

I will operate with two metaphysical background assumptions:

> *Objective indeterminism:* At any given time t_0 and for any duration n, there are normally several possible future courses of events of length at most n (or n-maximal temporal progressions beginning at t_0) such that none of them is at t_0 metaphysically prior to any other.[2]

> *Temporal becoming:* At the end of any non-empty time span from t_0, a certain course of events of the corresponding length will have got fixed, and as a consequence some of the 'histories' (or maximal courses of events) that were possible at t_0 no longer are so.

It should be noted that the content of the thesis of temporal becoming is not merely that a transition from a moment t_1 to a possible later moment t_2 normally rules out certain possible histories in the sense that the set $H(t_2)$ of histories passing through t_2 is a proper subset of the set $H(t_1)$ of histories passing through t_1. This much is entailed already by the thesis of objective indeterminism combined with the assumption that the relation of temporal precedence (the causal ordering) induces a tree-like structure on moments of time.[3] What temporal becoming in effect means is that among those moments t_2 that are later than t_1 and in particular are endpoints of n-maximal temporal progressions beginning at t_1, exactly one gets spontaneously selected by the passage of time, if only we let n time units pass. The passage of time takes care by itself of singling out a certain moment $t_2 \succ t_1$ in such a way that some histories h that were possible at t_1 no longer are so at t_2.[4] If, however, we consider a moment $s_2 \succ t_1$ on some such *counterfactual* history h (counterfactual, that is, from the viewpoint of t_2), we may correctly say that *had* things gone so that s_2 had become actual, some possibilities open at t_1—

[2] For the notion of 'n-maximal temporal progression', cf. Section 4 of the background paper. If t_0 is a moment in $(T, \prec)$ and $\prec$ is discrete, a *temporal progression beginning at t_0* is a set $\{t_0,\ldots,t_n\}$ such that t_{i+1} is an immediate $\prec$-successor of t_i for all $0 \leq i < n$. The temporal progression $\{t_0,\ldots,t_n\}$ is *m-maximal*, if either $n = m$, or else $n < m$ and the passage of time ends with t_n.

[3] For the notion of causal ordering, cf. Section 1 of the background paper.

[4] Here, '$\succ$' stands for the converse of the relation of temporal precedence (i.e., it stands for the *later than* relation among moments).

say, time passing through t_2—*would* no longer be open at s_2. What the passage of time does, considered from the vantage point of a moment t_1 through which time is passing, is that it picks out, for every duration n, one moment among moments whose distance from t_1 is (at most) n and that are later than t_1. More precisely, for every duration n, the passage of time selects a moment s_n that is the endpoint of an n-maximal temporal progression beginning at t_1. Furthermore, it does this in a 'cumulative' fashion, so that the set of moments consisting of the unique past of t_1 followed by the moments s_n with $n \in \mathbb{N}$ is a history according to the causal ordering of 'our world'.[5]

1.1 Temporal frames

A *temporal frame* is a pair $\boldsymbol{T} = (T, \prec)$, where T is a non-empty set of *moments* and $\prec$ is an irreflexive and transitive binary relation on T (referred to as *causal ordering relation*) that satisfies the following further conditions:[6]

(i) *Discreteness:* For every t and t' with $t \prec t'$, we have: (a) there is s with $t \prec s \preccurlyeq t'$ such that s is an immediate successor of t (i.e., there is no x with $t \prec x \prec s$), and (b) there is r with $t \preccurlyeq r \prec t'$ such that r is an immediate predecessor of t' (i.e., there is no x with $r \prec x \prec t'$).[7]

(ii) *Finite composition:* For any t and t' with $t \prec t'$, the set $\{x : t \preccurlyeq x \preccurlyeq t'\}$ is finite.[8]

(iii) *Finite branching:* For any $t \in T$, the set of immediate successors of t is finite.

(iv) *Backwards linearity:* For all $t \in T$, all predecessors of t are comparable in terms of $\prec$. That is, if $s \prec t$ and $s' \prec t$, then $s = s'$ or $s \prec s'$ or $s' \prec s$.

(v) *Historical connectedness:* For all $t, t' \in T$ there is s such that $s \preccurlyeq t$ and $s \preccurlyeq t'$. More specifically, there is $s \in T$ such that $s = \inf_{\preccurlyeq}\{t, t'\}$).[9]

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[5] Note that the set $\{s_n : n \in \mathbb{N}\}$ is finite if one of the moments s_i, and therefore every moment s_j with $j \geq i$, is an end of the passage of time.

[6] I use the symbol '$\preccurlyeq$' to stand for the relation $\prec \cup \{(t, t) : t \in T\}$.

[7] Both conditions are needed. E.g., the ordinal $\omega + \omega$ has the property that its every element having a successor has an immediate successor, but the element ω has predecessors without having an immediate predecessor. That is, (a) does not entail (b).

[8] Note that an irreflexive transitive discrete order is not automatically finitely composed. E.g., any linear order of order type $\omega + \omega^*$ offers a counterexample (between its minimum and its maximum there are infinitely many elements).

[9] If S is a subset of T, the *infimum* (or *greatest lower bound*) of S with respect to the order $\preccurlyeq$, denoted $\inf_{\preccurlyeq} S$, equals $\max\{x : \text{for all } s \in S, x \preccurlyeq s\}$. If t and t' are incomparable w.r.t. $\preccurlyeq$, then $\inf_{\preccurlyeq}\{t, t'\} \prec t$ and $\inf_{\preccurlyeq}\{t, t'\} \prec t'$. Otherwise $\inf_{\preccurlyeq}\{t, t'\} = \min_{\preccurlyeq}\{t, t'\}$.

I require that the causal ordering relation satisfy the conditions (i), (ii), and (iii) merely to somewhat simplify the exposition. If these three conditions are dropped, my definition of causal ordering coincides with that of Belnap and his collaborators.[10] Given the above definition, if (T, $\prec$) is a temporal frame, the set T contains at least one element and may but need not be finite. There may but need not be a uniform upper bound on the number of immediate successors of a moment. The set T may have a minimal element, and if it does, the element is uniquely determined (and called the *root* of the frame). The set T may have any finite number of maximal elements, and even a countable infinity of them. (If there are such elements, they are called the *leaves* of the frame.) However, the set T need not have extrema in either direction. The set T may easily be uncountable: even if it has a root and each moment has only two immediate successors, it has the cardinality of the continuum (i.e., 2^κ with $\kappa = \aleph_0$).

To facilitate reasoning about temporal frames, I introduce the notions of course of events, history, and partial history—already employed without precise definitions above and in the background paper.

A non-empty subset S of T is a *course of events* in frame $\boldsymbol{T}$ iff[11] (a) any distinct moments s and s' in S are comparable in terms of the relation $\prec$ (that is, satisfy either $s \prec s'$ or $s' \prec s$); and (b) S is inward-closed (i.e., if $s, s' \in S$ and $s \prec s'$, then for all x with $s \prec x \prec s'$, we have $x \in S$). The causal ordering of the frame induces a *linear* order on any course of events. By the assumption of backwards-linearity, the set of predecessors of any moment is a course of events. Due to branching, the set of successors of a moment, again, is normally *not* a course of events.

A course of events S is a *history* in $\boldsymbol{T}$ iff S is *maximal*: there is no S' with $S \subset S'$ such that S', too, is a course of events. Every course of events S in a temporal frame has at least one maximal extension h in the frame, i.e., a history h such that $S \subseteq h$. If t is moment, I write $H(t)$ for the set of all histories h in $\boldsymbol{T}$ that pass through t (i.e., satisfy $t \in h$).[12] A course of events S is a *partial history* in $\boldsymbol{T}$ iff there is $t \in T$ such that $S = \{x : x \preccurlyeq t\}$. If g is a partial history and its last element is t, I say that g *leads to* t. I write $[t]$ for the

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[10] See Belnap & Green (1994), p. 371; Belnap *et al.* (2001), pp. 178–189.

[11] Here and henceforth, ‘iff’ stands for ‘if and only if’.

[12] Because the causal ordering relation of a temporal frame is discrete, finitely composed, and backwards-linear, each history is either finite or has one of the following three order types: ω^* (end but no beginning), ω (beginning but no end) or $\omega^* + \omega$ (no beginning, no end).

unique partial history that leads to t, and $H_p(s)$ for the set of all partial histories g passing through s (i.e., satisfying $s \in g$). For every $g \in H_p(s)$, there is t_g such that $g = [t_g]$, whence $s \preccurlyeq t_g$. Exactly those courses of events are partial histories that have a last element and are downwards-closed (contain all moments of T that are earlier than the last element of this partial history). And exactly those histories are partial histories that have a last element (which in that case is an end of time, i.e., a moment with no successor in T).

1.2 Static structures versus open frames

Temporal frames are supposed to provide a way of modeling objective indeterminism. We must still find a way to model temporal becoming (passage of time). It has been proposed in the literature that a way of accomplishing this might be to associate each moment t with a specific history h_t, referred to as the *thin red line* through t.[13] This is supposed to be *the* history that actually will unfold, supposing that t has already got actualized. As Belnap and his collaborators have argued, it would be reasonable to assume that the map $t \mapsto h_t$ satisfies the following conditions (1) and (2):[14]

(1) For all $t \in T$, we have $t \in h_t$.
(2) For all $t, s \in T$, if $t \prec s$ and $s \in h_t$, then $h_t = h_s$.

In this setting, it may or may not happen that the thin red lines h_t cover all histories of the frame. (It is not excluded that there are histories h such that $h \neq h_t$ for all $t \in T$.)

Now, if a model really provides, for a given time t, a history h_t understood as representing the passage of time through t, then according to the model it is entirely fixed how time will unfold from t—a consequence that makes the assumption of objective indeterminism collapse. Such a model depicts things from a highly unrealistic God's-eye viewpoint. If objective indeterminism is correct, the information encoded in the model would be available only if *not merely* the actual world history had got realized in its entirety (already an utterly illicit idealization), but it would even be settled for *all counterfactual*

[13] See Belnap & Green (1994), Belnap *et al.* (2001), ch. 6, Øhrstrøm & Hasle (2015). Thin red lines are supposed to represent the 'real' or 'true' future—the history that the actual passage of time singles out.

[14] Belnap *et al.* (2001), p. 165, postulate (14) and p. 166, postulate (17).

moments t how the world history would have got realized had time passed through t. Now, as long as indeterminism persists and the passage of time continues indefinitely in the direction of future, a fully actualized world history is merely an ideally existing limit, not completed at any moment. In short, adding to a temporal frame a family of histories $\langle h_t \rangle_{t \in T}$ yields far too much information, if the goal is to model time under the assumption of objective indeterminism and temporal becoming.

What is needed is that we rethink the notion of model. What I propose is that the full machinery of semantic evaluation must resort to information *external* to the model itself. If we try to pack all information about temporal becoming in the model itself, we end up misrepresenting 'our indeterminist world', supposing that temporal becoming is taken to be one of its features. We cannot represent the passage of time through t by a full-fledged history h_t—or, what amounts to the same thing, we cannot represent the way the future unfolds from t on by a function f_t mapping non-negative durations to moments, so that $f_t(n)$ = the n-th successor of t on h_t. The point is that for any positive duration n and any moment t, there will be a fixed moment $f_t(n)$ that will have become actualized when n time units counted from t have passed. Crucially, however, it is not determined at t *which* moment the moment $f_t(n)$ will be (among the m-th successors of t with $m \leq n$).[15] In fact, at t, the function f_t is not defined on any positive argument. When n time units have passed after t, the function f_t is defined on arguments $0,\ldots,n$ but not on any argument $n' > n$. It is this phenomenon—the domain of the function f_t increasing with time, and its corresponding values being gradually fixed—that we must get a grip on. Generally, we can just say that there is a functional dependence between the length of a time span whose origin lies at a given fixed moment t and a later moment that gets actualized at the end of that time span. The passage of time from t spontaneously creates a correlation between a duration n and a moment later than t. Retrospectively, the function f_t acquires a value for the argument n. However, the function f_t simply has no value for the argument n until n time units have passed after t. Therefore, it strictly speaking does not even make sense to speak of 'the' function f_t.

In order to facilitate comprehension, I formulate two ways of attempting to model temporal becoming: with thin red lines (the option I reject) and

[15] It was noted in Section 4 of the background paper that if we opt to use functional expressions such as '$f_t(n)$' here, they should be read schematically, not as literally employing a full-fledged function f_t defined on all natural numbers n.

without thin red lines (the option I adopt). A *static temporal structure* is a triple $\boldsymbol{S} = (T, \prec, \langle h_t \rangle_{t \in T})$, where $(T, \prec)$ is a temporal frame, and $\langle h_t \rangle_{t \in T}$ is a collection of thin red lines subject to the above conditions (1) and (2). An *open temporal frame* is a triple $\boldsymbol{F} = (T, \prec, \text{TIME})$, where $(T, \prec)$ is still a temporal frame, but TIME is a resource that acts on external information: if a partial history g and a duration $n \in \mathbb{N}$ are provided as inputs, TIME produces, in n time units, a partial history TIME(g, n) that extends the partial history g by a course of events consisting of (at most) n moments. I refer to TIME as the *time-resource* of $\boldsymbol{F}$. The resource TIME is assumed to satisfy the conditions (a) and (b):

(a) For all partial histories g, we have:

(i) $\text{TIME}(g, 0) = g$

(ii) For all $n \in \mathbb{N}$, either $\text{TIME}(g, n + 1) = [s]$ for some immediate successor s of the endpoint of $\text{TIME}(g, n)$, or else $\text{TIME}(g, n + m) = \text{TIME}(g, n)$, for all $m \geq 1$.

(b) For all partial histories g and all $n, n' \in \mathbb{N}$:

$\text{TIME}(\text{TIME}(g, n), n') = \text{TIME}(g, n + n')$.

Condition (a) means that in zero duration a partial history is not extended at all, and in a finite positive duration n it is extended by at most n moments. If it is not extended by exactly n moments, the passage of time has come to an end, and no further duration will extend it either. The formulation of condition (a.ii) leaves open that the passage of time could end with TIME(g, n) not only because the endpoint of TIME(g, n) has no successor in the set T, but even when according to the causal ordering the endpoint of TIME(g, n) indeed has successors. This allows modeling cases in which the passage of time comes to an end but could have gone on. (In a mere tree-structure, the end of time must be represented by a maximal element. This blocks the representational possibility that there still are later moments, albeit not ones that get realized by temporal becoming.) Condition (b) is a sort of stability condition regarding iterated applications of TIME: consecutive applications of TIME to shorter durations amount to the same as a single application of TIME to the sum of those durations.

We may observe that for all partial histories g and all $n, n' \in \mathbb{N}$, the resource TIME satisfies:

(1) $\text{TIME}(g, n) \subseteq \text{TIME}(g, n + n')$
(2) $g \subseteq \text{TIME}(g, n)$
(3) If $g' = \text{TIME}(g, n)$, then $\text{TIME}(g', n') = \text{TIME}(g, n + n')$.

Feature (1) means that TIME produces cumulative partial histories: when extending a given g, the partial history corresponding to a longer duration contains all partial histories corresponding to shorter durations as subsets. Feature (2) is an analogue of property (1) of thin red lines. In particular, if $t = \max(g)$ and therefore $g = [t]$, we have $t \in \text{TIME}([t], n)$. Feature (3) is a reformulation of the above condition (b). It is an analogue of property (2) of thin red lines: Write $g' = \text{TIME}(g, n)$ and suppose $n > 0$. Let $t = \max(g)$ and $s = \max(g')$, whence $t \prec s$. We may compare partial histories that are determined by $\text{TIME}(g, x)$ as x grows with those that are determined by $\text{TIME}(g', y)$ as y grows. Feature (3) guarantees that $\text{TIME}(g, n + y) = \text{TIME}(g', y)$ for all durations y. Thus, it cannot happen that TIME yields from g an extended partial history g' in n time units, and from g' it yields an extended partial history g'' in n' time units, but still TIME would have yielded in $n + n'$ time units from g an extended partial history distinct from g''. Generation from the generated yields nothing that could not have been directly generated.

The resource TIME represents both 'factual' and 'counterfactual' temporal becoming. Note that if $t \prec t'$ and $t = \max(g)$, it can well happen that there is no duration n such that t' lies on $\text{TIME}(g, n)$. The passage of time generates one extension of g, but t may have incomparable successors (and normally does) . Besides, even if the set of moments later than t was linearly ordered, it might happen that the passage of time through t comes to an end without actualizing all moments that are later than t according to the causal order. Observe, however, that there can be no moment that in an absolute sense remains outside the passage of time: if $s \in T$ is any moment, TIME is defined at least on the partial history $[s]$. But it can happen that there is no moment $s' \prec s$ such that s is attained from s' along the passage of time through s'—i.e., that there is a duration n such that s is the endpoint of $\text{TIME}([s'], n)$. And it can also happen that the passage of time through s leads to no further moment—i.e., that $\text{TIME}([s], n) = [s]$ for all durations n.

Static structures contain much more information than open frames. Indeed, a static structure specifies for every moment t an entire history h_t, the thin red line through t. By contrast, an open frame does not *by itself* contain any more information than the corresponding plain frame $(T, \prec)$. Its extra component TIME is merely a mechanism that alone provides no information at all, but in combination with an external input (a duration) n associates a partial history g with a possibly longer partial history. In order to get anything out of the resource TIME (in order to use it), an *internal input* (a partial history) must be combined with an *external input* (an elapsed duration). The value TIME(g, n) gets determined when n time units have passed since the realization of g. The component TIME allows us to speak of a functional dependence between the length of a time span originating at g and a partial history that gets actualized at the end of that time span—without pretending that the value TIME(g, n) is anticipatedly determined.

It would be an interpretational mistake to view open frames as static structures. Technically, we could transform an open frame into a static structure if the following preliminary procedure could be carried out: for each moment t, consider successively all positive integers n and apply TIME successively to each pair (g, n), where $g = [t]$ (thus, apply it infinitely many times) to yield a sequence $\langle t_1, t_2,\ldots\rangle$, where t_n equals the endpoint of TIME(g, n), for all $n > 0$. Then define $h_t = [t] \cup \{t_1, t_2,\ldots\}$. The properties (a) and (b) of TIME guarantee that the histories h_t, thus defined, satisfy the conditions (1) and (2) of thin red lines. Needless to say, in any circumstance in which we actually can find ourselves wishing to evaluate temporal propositions, we cannot carry out such a preliminary task, which would consist of going through possibly infinitely many moments, and for each of them considering possibly infinitely many durations. Given how the resource TIME is to be construed, even for a single moment the requisite part of the procedure could take literally an eternity (running through all durations corresponding to positive integers). An open structure is, so to say, a static structure in a dormant form: it can never by itself produce any information about the future, but it allows to represent a potential for extending already actualized partial histories by ever longer courses of events. A static structure would only result, if all such potentiality could at one stroke be actualized—if the time-consuming use of our resource could *per impossibile* be carried out instantaneously.

2. Temporal-modal language *L*

Let **prop** be a set of propositional atoms, denoted p, q, r etc. *Positive literals* are simply propositional atoms. *Negative literals* are expressions of the form $\neg p$, where $p \in$ **prop**. The set **lit** of *literals* consists of positive and negative literals. I will consider a propositional temporal-modal language, to be called

L, built on the set **lit** and syntactically closed under the binary connectives ∨ (disjunction) and ∧ (conjunction), as well as under the unary connectives P (*it was the case that*), H (*it has always been the case that*), F (*it will be the case that*), G (*it will always be the case that*), $\Diamond$ (*it is historically possible that*), and $\Box$ (*it is settled that*). We may, then, recursively define a function ~ from L to L by the following clauses: $\sim(\phi \wedge \psi) \coloneqq (\sim\phi \vee \sim\psi)$, $\sim(\phi \vee \psi) \coloneqq (\sim\phi \wedge \sim\psi)$, $\sim(F\phi) \coloneqq G\sim(\phi)$, $\sim(G\phi) \coloneqq F\sim(\phi)$, $\sim(P\phi) \coloneqq H\sim(\phi)$, $\sim(H\phi) \coloneqq P\sim(\phi)$, $\sim(\Box\phi) \coloneqq \Diamond\sim(\phi)$, $\sim(\Diamond\phi) \coloneqq \Box\sim(\phi)$, $\sim(p) \coloneqq \neg p$, $\sim(\neg p) \coloneqq p$. Then, we may use in a metalanguage the symbol '→' so that '$(\phi \rightarrow \psi)$' abbreviates '$(\sim\phi \vee \psi)$'. I opt for a syntax in which the negation symbol ¬ appears only in front of atomic formulas and in which every generally applicable connective has its dual in the syntax; the fact that the syntax is thus formulated will facilitate certain metatheoretical considerations. I proceed to formulate a semantics for L, the goal being that the semantics is compatible with both objective indeterminism and temporal becoming. I provide two alternative semantics: one taking inspiration from thin red lines (models based on static temporal structures), another in terms of durations construed as model-external inputs (models based on open temporal frames). I am advocating the latter semantics and not the former, but the former being an 'idealized' version of the latter, it can help appreciate the general semantic ideas underlying both formulations.

2.1 Models based on static structures

Let $\mathcal{S} = (\boldsymbol{S}, V)$, where $\boldsymbol{S} = (T, \prec, \langle h_t \rangle_{t \in T})$ is a static temporal structure, and V is a *valuation function* such that $V(p) \subseteq T$ for all $p \in$ **prop**. Relative to such *static models* $\mathcal{S}$ and pairs (s, h_t) of moments s and histories h_t with $s \in h_t$, we may recursively define the semantics of our language:

- $\mathcal{S}, s, h_t \vDash p$ iff: $s \in V(p)$.
- $\mathcal{S}, s, h_t \vDash \neg p$ iff: $s \notin V(p)$.
- $\mathcal{S}, s, h_t \vDash (\phi \vee \psi)$ iff: there is $\chi \in \{\phi, \psi\}$ such that $\mathcal{S}, s, h_t \vDash \chi$.
- $\mathcal{S}, s, h_t \vDash P\phi$ iff: there is x with $x \prec s$ (and $x \in h_t$) such that $\mathcal{S}, x, h_t \vDash \phi$.
- $\mathcal{S}, s, h_t \vDash F\phi$ iff: there is x with $s \prec x$ and $x \in h_t$ such that $\mathcal{S}, x, h_t \vDash \phi$.
- $\mathcal{S}, s, h_t \vDash \Diamond\phi$ iff: there is x with $s \preccurlyeq x$ such that $\mathcal{S}, s, h_x \vDash \phi$.

The clauses for the 'duals' ∧, H, G, and $\Box$ of ∨, P, F, *resp*. $\Diamond$ can be defined in an obvious way. We may observe that the clauses respect the requirement that the value of the 'moment parameter' must be an element of the value of the 'history parameter': semantic clauses never lead from a context satisfying this condition to one not satisfying it.[16]

The histories h_t correspond to what are termed thin red lines in the literature, but their role for the present semantics is not the same as the role of thin red lines is usually taken to be. Thin red lines are usually used for eliminating any need for an explicit history parameter. The semantics has recourse to a thin red line given by the model whenever needed: a thin red line h_t is used to evaluate a future-tense operator at t. I do not wish to eliminate the history parameter. I do not require that $F\phi$ evaluated at x use h_x as the history component of evaluation. Indeed, in the above clause for F, it can happen that $s \prec t$, in which case it is compatible with my assumption about thin red lines that $h_s \neq h_t$. In fact, in the above semantics, thin red lines are employed to interpret the modal operator $\Diamond$ (and its dual $\Box$). As a matter of fact, the semantic clauses for all connectives except for $\Diamond$ and $\Box$ are exactly like in the Ockhamist semantics.[17] In the clause for $\Diamond$, I wish to avoid explicit existential quantification over histories passing through s. Instead, quantification is over all moments x later than or identical to s. Each such moment determines a history, the thin red line h_x through x. Thus, indirectly we end up quantifying over the histories of the family $\langle h_t\rangle_{t\in T}$. Unless this family covers the totality of histories in the frame $(T, \prec)$, this results in quantifying over fewer histories than in the Ockhamist semantics. Still, among the h_t there are 'arbitrarily good approximations' of all frame histories: for every initial segment of every frame history (no matter how long), there is a history belonging to the family $\langle h_t\rangle_{t\in T}$ with that same initial segment.

As an example of semantic evaluation, consider the formula $q \rightarrow PFq$ (that is, the formula $\neg q \vee PFq$), refutable according to the usual formulation of thin red line semantics that employs moments of evaluation and drops histories of evaluation.[18] For, suppose there are just three moments t_0, t_1, t_2, the causal ordering being $\prec = \{(t_0, t_1), (t_0, t_2)\}$. Suppose $V(q) = \{t_2\}$. Finally,

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[16] The semantic relation $\vDash$ is quaternary; if we have $\mathcal{S}, s, h_t \vDash \phi$, I say that the second term s is the value of the 'moment parameter' and the third term h_t the value of the 'history parameter'. Cf. footnote 19.

[17] For the Ockhamist semantics, see, e.g, Øhrstrøm & Hasle (2015), Subsection 5.1.

[18] For the use of thin red lines in semantics, see Øhrstrøm & Hasle (2015), Subsection 5.3. The only semantic clauses that in effect make recourse to thin red lines are the clauses for F and G. In particular, $F\phi$ is taken to be true at t iff there is t' that lies on the thin red line TLR(t) associated with t such that $t \prec t'$ and ϕ is true at t'.

suppose the thin red line TRL(t_0) associated with t_0 equals $\{t_0, t_1\}$. Now, according to thin red line semantics, the formula PFq is false at t_2, since Fq is false at t_0—false, since there is no time later than t_0 belonging to the privileged history TRL(t_0) at which q would be true. Still q is true at t_2, so the material implication is false at t_2. By contrast, according to my static model semantics, the formula $q \rightarrow PFq$ is true in all static models at all non-minimal moments s relative to all histories h_t with $s \in h_t$. While it is perfectly possible that $h_s \neq h_t$, the thin red line h_s will not be employed in evaluating the formula $q \rightarrow PFq$. Generally, the history component can only get changed via the evaluation of a modal operator. So, suppose that q is true at s in the history h_t with $s \in h_t$ (that is, $\mathcal{S}, s, h_t \vDash q$). Since s is non-minimal, there is $s' \prec s$. Now, $\mathcal{S}, s', h_t \vDash Fq$, because indeed there is x with $s' \prec x$ and $x \in h_t$, namely $x := s$. Thus, $\mathcal{S}, s, h_t \vDash PFq$. It can happen that from the viewpoint of s', both moments s and t are counterfactual (in the sense that $s \notin h_{s'}$ and $t \notin h_{s'}$). But this fact has no bearing, as long as h_t remains the value of the history parameter: as long as the evaluation is relative to the history h_t.

Interpretatively, the static-model semantics is highly problematic for my purposes. It operates shamelessly with entire histories h_t, each of which is introduced at the outset as a representation of the ultimate result of temporal becoming passing through t, whether t be from our perspective an actual or counterfactual moment. I wish to keep the spirit of the semantics, but to reformulate it so that it becomes intelligible when evaluation is considered as taking place *in time*, instead of being observed from a viewpoint lying outside time—the only type of viewpoint that makes instantaneously available all histories of the temporal frame. As a result, we cannot operate with entire histories but merely partial histories. Statements about the future must rely on durations acting as an external input to the semantic evaluation. Statements about modalities at t are in terms of 'eventualities'—moments s with $t \preccurlyeq s$ causally possible relative to the time of evaluation t.

2.2 Models based on open frames

Let $\mathcal{M} = (T, \prec, V, \text{TIME})$, where $(T, \prec, \text{TIME})$ is an open temporal frame and $V(p) \subseteq T$ for all $p \in$ **prop**. I refer to the structure $(T, \prec, V)$ as an *event structure*, and to $\mathcal{M}$ itself as an *open temporal model*. (Moments—elements of T—satisfying specified propositional atoms as indicated by V can be referred to as *eventualities*.) Formulas of L are evaluated on open temporal models $\mathcal{M}$ relative to pairs (s, g) of moments s and *partial* histories g with $s \in g$. Given a *circumstance of evaluation* $(\mathcal{M}, s, g)$, I say that s is the value of the *moment parameter* and g the value of the *history parameter*.[19] The idea is that the value g of the history parameter keeps track of moments 'already actualized'. It is always of the form $g = [\max(g)] = [t] = \{x : x \preccurlyeq t\}$, where t is the moment most recently actualized (either factually or counterfactually). Consequently, the value of the history parameter is normally not a fully fledged history but merely a partial history. The value s of the moment parameter must lie on the partial history g corresponding to the history parameter—but may be arbitrarily far away from the endpoint $\max(g)$ of that partial history.

If at (s, g) we are to evaluate a construction that regards future, it may but need not happen that we must call attention to how the partial history g unfolds. It may be enough to consider moments t that already lie on g but are later than s, but in general this does not suffice. Due to objective indeterminism, the temporal frame itself does not yield a unique continuation of any length to the partial history g. Then again, the thesis of temporal becoming entails that there is in fact a unique way in which g unfolds in any given number of time units—though the way in which it does is merely determined *ex post facto* when a duration of the corresponding length has passed. The partial history g itself does not determine its own continuation. The best we can say in general terms is that for any duration $n > 0$ there will be a correlation between a certain set of moments $\{t_1,\ldots,t_n\}$ and the pair (g, n) with the property that $g \cup \{t_1,\ldots,t_n\}$ is the partial history to which g in fact will lead in n time units—but the identity of the moments t_i is revealed only at the end of a time span of length n beginning at $\max(g)$. The correlation can be established only retroactively, not when merely the moments in g have got actualized. In fact, the passage of time does precisely that: establishes spontaneously such a correlation for ever longer durations n measured from the endpoint of a given partial history g. It is this dynamics—determining how the temporal becoming will continue, if it has already reached a given moment—that is meant to be modeled by the resource TIME. By my modeling-hypothesis, this resource interacts with reality external to the model; the resource is used by letting a finite duration determine the continuation of a partial history.

By insisting on the fact that the resource TIME requires external information (a duration) as an input, I wish to stress that the right way of understanding my semantic model is *not* to construe TIME as a

[19] We say so, even if g is a *partial* history, not a history. There should be no risk of serious misuderstanding.

function whose values are determined for all pairs of partial histories and durations by the temporal model itself. Doing that would lead us immediately to the above static semantics. Instead, the only way I can imagine to model the effects of the passage of time in semantics is to accept viewing TIME as a resource that can be consulted locally regarding this or that particular argument-pair, but not globally, in the sense that one could construct entire frame histories by consulting the resource for *all* combinatorially possible relevant argument-pairs before proceeding to evaluate anything at all.

At the level of temporal experience, consulting TIME regarding a pair (g, n) corresponds to the case where g consists of all moments already actualized, the endpoint of g being the moment we experience as present, and we proceed to wait n time units to see how g spontaneously evolves during the ensuing period of length n. At the end of a time span of n units, a certain longer partial history has got actualized—a partial history that in no way was determined already at g but is nevertheless determined at the end of the period. Indeed, in a concrete evaluation situation, there is only one direct way of consulting TIME: waiting. Actual consultation can be replaced by hypothetical consultation whenever we can recognize beforehand that the outcome of our evaluation is independent of how the history evolves. This is the case of propositions that are not historically contingent—i.e., either hold or fail always for logical or semantic reasons, or in fact invariably hold or invariably fail in virtue of suitable causal laws or perhaps for metaphysical reasons due to the natures of things.

If $(T, \prec)$ is a temporal frame and $t \preccurlyeq s$, I write $\Delta(t, s) = n$ iff $s \in succ^n(t)$, and I say that $\Delta(t, s)$ is the *distance* of t from s (in this order).[20] Relative to open temporal models $\mathcal{M}$, and pairs (s, g) of moments s and partial histories g with $s \in g$, the semantics of L is defined as follows:

- $\mathcal{M}, s, g \vDash p$ iff: $s \in V(p)$.
- $\mathcal{M}, s, g \vDash \neg p$ iff: $s \notin V(p)$.
- $\mathcal{M}, s, g \vDash (\phi \vee \psi)$ iff: there is $\chi \in \{\phi, \psi\}$ such that $\mathcal{M}, s, g \vDash \chi$.
- $\mathcal{M}, s, g \vDash P\phi$ iff: there is x with $x \prec s$ (and $x \in g$) such that $\mathcal{M}, x, g \vDash \phi$.

[20] We may stipulate that the map Δ is defined on all pairs of moments (t, s) lying on one and the same partial history—and not merely on those satisfying $t \preccurlyeq s$—by setting $\Delta(t, s) = -\Delta(s, t)$, if $t \succ s$. Thus, Δ expresses a 'directed' distance, the value $\Delta(t, s)$ being negative if t is later than s.

- $\mathcal{M}, s, g \vDash F\phi$ iff: there is $n > 0$ such that $\mathcal{M}, s', g' \vDash \phi$, where s' is the unique moment satisfying $\Delta(s, s') = n$ and
  — $g' = g$, if $\Delta(s, \max(g)) \geq n$; and
  — $s' = \max(g')$ and $g' = \mathrm{TIME}(g, n^*)$
  with $n = \Delta(s, \max(g)) + n^*$, otherwise.
- $\mathcal{M}, s, g \vDash \Diamond\phi$ iff: there is x with $s \preccurlyeq x$ such that $\mathcal{M}, s, [x] \vDash \phi$.

The clauses for the 'duals' $\wedge$, H, G, and $\Box$ of $\vee$, P, F, *resp*. $\Diamond$ can, again, be defined in an obvious way. If the condition $\mathcal{M}, s, g \vDash \phi$ holds, I say that ϕ is 'true at s on g', otherwise I say that ϕ is 'false at s on g'.

The semantic clause of F covers two cases, depending on whether a suitable duration n witnessing the truth of ϕ can be found among the moments already actualized on g (in which case the distance between the moment of evaluation s and the endpoint of g is at least n), or whether the passage of time must actualize further moments in order for n to be a duration witnessing the truth of ϕ. Note that the requirement $\Delta(s, s') = n$ entails that in the latter case (i.e., the case in which the distance $\Delta(s, \max(g))$ is less than n), the passage of time must actually extend g. To be sure, it might happen that TIME does not extend g, so that $\mathrm{TIME}(g, x) = \mathrm{TIME}(g, 0)$ for all positive integers x. However, this is not compatible with $F\phi$ being true at s on g according to the second alternative. Namely, if the passage of time failed to extend g, the partial history g' would equal g, so that we would have $s' = \max(g') = \max(g)$, and consequently $n = \Delta(s, s') = \Delta(s, \max(g)) < n$, which is impossible. Whether or not the evaluation needs to have recourse to TIME depends, in particular, on the position of the value of the moment parameter in relation to the value of the history parameter. Consider, for example, the model described above in connection with the formula $q \to PFq$. Let $g = \{t_0\}$ and $g' = \{t_0, t_2\}$, and suppose that $\mathrm{TIME}(g, 1) = \{t_0, t_1\}$. Then the formula Fq is false relative to (t_0, g), since the passage of time extends g to t_1 at which q is false. Yet Fq is true relative to (t_0, g'), since indeed there is a moment later than t_0 in g' (namely, t_2) at which q is true.

Modal operators shift the value of the history parameter. Relative to a moment s, they quantify over 'eventualities'—moments t satisfying $s \preccurlyeq t$. I allow the identity $s = t$, for otherwise the truth of ϕ at a moment with no successors would not entail $\Diamond\phi$. When the value of the history parameter is changed, eventual tense operators in the scope of the modal operator are evaluated relative to this new value. For example, let t_0, t, s, s' be four pairwise distinct moments, and consider the frame $(\{t_0, t, s, s'\}, \prec)$ with $\prec$ =

$\{(t_0, t), (t_0, s), (s, s'), (t_0, s')\}$. Suppose $V(q) = \{s'\}$. Finally, suppose TIME($\{t_0\}$, 1)= $\{t_0, t\}$ and TIME($\{t_0, s\}$, 1) = $\{t_0, s, s'\}$. Then $\neg Fq$ is true at t_0 relative to the partial history $\{t_0\}$, because all positive durations n will yield the partial history TIME($\{t_0\}$, n) = TIME($\{t_0\}$, 1) = $\{t_0, t_1\}$, and q is false at t. Yet $\Diamond Fq$ is true at t_0 on $\{t_0\}$. For example, $\Diamond$ is witnessed by s with $t_0 \prec s$. Namely, indeed Fq is true at t_0 on the partial history $\{t_0, s\}$, since there is a duration $n \coloneqq 2$ such that q is true at the endpoint s' of the partial history $\{t_0, s, s'\}$ = TIME($\{t_0, s\}$, 1), where $1 = 2 - 1 = n - \Delta(t_0, s)$. Alternatively, $\Diamond$ is witnessed directly by s' (which satisfies $t_0 \prec s'$), since Fq is true at t_0 on $\{t_0, s, s'\}$. In this latter evaluation, no (counterfactual) recourse to TIME is needed, because q is true at a moment later than t_0 (namely s') that lies directly on the partial history $\{t_0, s, s'\}$ relative to which the evaluation is being effected.

Superficially, the clauses for F and P do not resemble each other very much. This can actually be seen as being promising, since the corresponding truth-conditions indeed are far from being on a par metaphysically: the semantics of F makes generally allusion to moments not yet realized (which is why in connection with F we quantify over durations and not over moments), whereas the semantics of P can profit from the fact that the history of the moment of evaluation is uniquely determined by this moment of evaluation itself. Then again, nothing of course prevents us from formulating the semantics of P, too, in terms of durations:

- $\mathcal{M}, s, g \vDash P\phi$ iff: there is $n > 0$ such that $\mathcal{M}, s', g \vDash \phi$, with $\Delta(s', s) = n$.

Here, the value g of the history parameter remains unchanged and the moment parameter is changed to a moment that is n units in the past of s on the very partial history g (i.e., to the n-th predecessor of s on g). Still of course the clauses for F and P are not simple mirror images of each other (unlike in the standard Ockhamist semantics), because I take temporal becoming more seriously than those propagating the Ockhamist semantics do. In order for $F\phi$ to be true at s on g, a recourse to the resource TIME may be needed, though in special cases this is avoided. However, if such a fortunate special case does not present itself, the passage of time through s either does not render ϕ true at all, or renders it true at a moment that is yet to be actualized. And the evaluation can move beyond the endpoint of g only in virtue of TIME. By contrast, the evaluation of $P\phi$ can under no circumstances appeal to TIME; it makes use of the partial history already realized.

The semantic evaluation will inevitably employ partial histories, but if we so wish, we can introduce a (seemingly) history-independent truth-relation, as follows.

Definition 2.1 (Truth relative to a moment only) If ϕ is an L-formula, $\mathcal{M} = (T, \prec, V, \text{TIME})$ is an open temporal model, and $t_0 \in T$, then ϕ is said to be *true in* $\mathcal{M}$ *at* t_0 (in symbols $\mathcal{M}, t_0 \vDash \phi$) iff $\mathcal{M}, t_0, g_0 \vDash \phi$, where $g_0 = [t_0]$. ▮

That is, a formula is true simply at t_0 iff it is true at t_0 relative to the partial history whose endpoint is t_0. This merely moment-relative notion of truth is of some use, because our main interest is usually to evaluate a formula relative to a moment construed as being present, and it is only the further semantic processing that may lead us to circumstances in which the moment component is earlier than the endpoint of the history component. And if t_0 is construed as present, then the history component, which keeps track of all by-now actualized moments, will not contain any moments later than t_0.

Having fixed the semantics of our language L, in particular the semantics of formulas that utilize future-tense operators, I will move on to explicate how this semantics allows us to make precise sense of the idea that the truth-maker of a proposition about a future event will obtain but does not.

3. Truth analyzed: evaluations

I wish to explain what it means that a proposition is true while its truth-maker merely will obtain but does not yet do so. To accomplish this task, we need to have a firm and fine-grained understanding of what it takes for a proposition to be true at a time t on a partial history g. To this end, I introduce the notion of *evaluation*. Relative to a fixed open temporal model $\mathcal{M}$, evaluations are finite sequences of triples (ψ, s, f), where ψ is a formula of L, s is a moment, and f is a partial history passing through s. In terms of such sequences, we can articulate in a very precise fashion what the truth of a formula at a moment on a partial history amounts to. If $\sigma = \langle \pi_1, \ldots, \pi_n \rangle$ is a sequence whose members are the π_i with $1 \leq i \leq n$, and π_{n+1} is any object, I write $\sigma \wedge \pi_{n+1}$ for the sequence $\langle \pi_1, \ldots, \pi_n, \pi_{n+1} \rangle$ that results from extending σ by π_{n+1}.

Definition 3.1 (Evaluation position, evaluation) Let $\mathcal{M} = (T, \prec, V, \text{TIME})$ be an open temporal model. An *evaluation position* on $\mathcal{M}$ is any triple (ψ, s, f), where ψ is a formula, $s \in T$, and $f \in H_\text{p}(s)$. If $t \in T$ and $g \in H_\text{p}(t)$, an *evaluation* of formula ϕ on the circumstance of evaluation $(\mathcal{M}, t, g)$ is any minimal set $\mathcal{E}$ of sequences of triples containing the unit sequence $\langle(\phi, t, g)\rangle$ and satisfying the following closure condition that specifies whether and how a sequence belonging to $\mathcal{E}$ can be extended depending on the last member of this sequence. If $\sigma \in \mathcal{E}$ and its last member is (ψ, s, f), then:

- If $\psi = (\chi_1 \vee \chi_2)$, we have $\sigma \,\hat{}\, (\chi_i, s, f) \in \mathcal{E}$ for exactly one $i \in \{1, 2\}$.
- If $\psi = (\chi_1 \wedge \chi_2)$, we have $\sigma \,\hat{}\, (\chi_i, s, f) \in \mathcal{E}$ for all $i \in \{1, 2\}$.
- If $\psi = P\chi$ and s has predecessors on f, we have $\sigma \,\hat{}\, (\chi, u, f) \in \mathcal{E}$ for exactly one u with $u \prec s$ (and $u \in f$).
- If $\psi = H\chi$, we have $\sigma \,\hat{}\, (\chi, u, f) \in \mathcal{E}$ for all u such that $u \prec s$ (and $u \in f$).
- If $\psi = F\chi$ and s has successors (not necessarily on f but) in $\cup_{m>0}\mathrm{TIME}(f, m)$, then for exactly one $n > 0$, we have:
  — if $\Delta(s, \max(f)) \geq n$ and u is the n-th successor of s on f, then $\sigma \,\hat{}\, (\chi, u, h) \in \mathcal{E}$, where $h = f$; and
  — if $\Delta(s, \max(f)) < n$ and $n = \Delta(s, \max(f)) + n^*$ and u is the endpoint of $\mathrm{TIME}(f, n^*)$, then $\sigma \,\hat{}\, (\chi, u, h) \in \mathcal{E}$, where $h = \mathrm{TIME}(f, n^*)$.
- If $\psi = G\chi$, then for all $n > 0$, we have:
  — if $\Delta(s, \max(f)) \geq n$ and u is the n-th successor of s on f, then $\sigma \,\hat{}\, (\chi, u, h) \in \mathcal{E}$, where $h = f$; and
  — if $\Delta(s, \max(f)) < n$ and $n = \Delta(s, \max(f)) + n^*$ and u is the endpoint of $\mathrm{TIME}(f, n^*)$, then $\sigma \,\hat{}\, (\chi, u, h) \in \mathcal{E}$, where $h = \mathrm{TIME}(f, n^*)$.
- If $\psi = \Diamond\chi$, we have $\sigma \,\hat{}\, (\chi, s, [u]) \in \mathcal{E}$ for exactly one u with $s \preccurlyeq u$.
- If $\psi = \Box\chi$, we have $\sigma \,\hat{}\, (\chi, s, [u]) \in \mathcal{E}$ for all u with $s \preccurlyeq u$.

The triple (ϕ, t, g) is the *initial evaluation position*. Maximal sequences belonging to an evaluation $\mathcal{E}$ are called *evaluation sequences*. (A sequence $\sigma \in \mathcal{E}$ is 'maximal' if there is no triple π such that $\sigma \,\hat{}\, \pi \in \mathcal{E}$.) ∎

The following observations can be made regarding evaluations as just defined.[21] (1) Elements of evaluations are sequences of evaluation positions,

[21] Readers familiar with Hintikka-style *game-theoretical semantics* will notice that if the above semantics employing a time-resource is formulated in terms of semantic games, evaluation sequences are *plays* of the semantic game associated with ϕ relative to $(\mathcal{M}, t, g)$; initial segments of evaluation sequences are its *partial plays*; and evaluation positions are *positions* in this game. An arbitrary element of an evaluation is either a play or a partial play. For every evaluation $\mathcal{E}$, there is a *strategy* of the 'initial verifier' such that $\mathcal{E}$ is the set of all plays that can result when the 'initial verifier' employs this strategy against some sequence of moves by the 'initial falsifier'. For semantic games, cf., e.g., Väänänen (2007), ch. 5.

not just sequences of arbitrary triples of formulas, moments, and partial histories. Namely, thanks to the way in which the initial clause and the clauses of the closure condition are formulated, in fact any triple (ψ, s, f) appearing as a member of an element of an evaluation always satisfies $f \in H_p(s)$, or in other words, satisfies $s \in f$. (2) All sequences belonging to an evaluation are finite sequences of evaluation positions. In fact, if the maximum number of nested tokens of connectives in ϕ equals n (taking into account tokens of temporal and modal operators as well as tokens of conjunctions and disjunctions), then the maximum number of members of a sequence belonging to an evaluation of ϕ equals n. (3) The cardinality of an evaluation (the number of sequences in it) can well be infinite, if the temporal frame is infinite and the formula contains tokens of the operators $\Box$, H or G. In particular, the number of *evaluation sequences* in an evaluation (i.e., *maximal* sequences belonging to an evaluation) can be infinite. (4) The last position of an evaluation sequence has always one of the following forms: (q, s, g) or ($\neg q$, s, g) for some $q \in$ **prop**; or (Oχ, s, g) for some O $\in \{P, H\}$ and some s that has no predecessor; or (Oχ, s, g) for some O $\in \{F, G\}$ and some s to which the time-resource provides no successor. In these cases, a sequence already formed cannot be extended, either because the formula component of the last position reached is already a literal (which as such does not admit of further decomposition); or because the formula component begins with a past-tense operator but the moment component is minimal with respect to the frame relation $\prec$; or because the formula component begins with a future-tense operator but the moment component s is the endpoint of the partial-history component g, and moreover the time-resource does not extend the partial-history component any further.[22] (5) The last position of a maximal sequence cannot be of the form (Oχ, s, g) for some O $\in \{\Diamond, \Box\}$, for even if s had no successor, the sequence would allow at least an extension by the position (χ, s, [s]), given that the moment t chosen to interpret a modal operator must only satisfy $s \preccurlyeq t$, not necessarily $s \prec t$. (6) In the clauses for past-tense operators, the fact that we have $s \in f$ and $u \prec s$ actually entails that $u \in f$, because every partial history (thus, f) contains all predecessors of its all members. So it is redundant to separately require that $u \in f$. This is why this clause was put in parentheses, but still kept visible to facilitate comparison with other clauses of the closure condition.

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[22] This *may* be because $s = \max(g)$ is maximal with respect to the frame relation $\prec$. However, it may also happen that max(g) has successors in T, but nevertheless the time-resource does not extend g—i.e., TIME(g, n) = g for all $n \in \mathbb{N}$.

Definition 3.2 (Truth-evaluation) An evaluation $\mathcal{E}$ of ϕ on $(\mathcal{M}, t, g)$ is a *truth-evaluation*, if every evaluation sequence Σ belonging to $\mathcal{E}$ satisfies: the last position of Σ is either of the form (q, s, g) with $s \in V(q)$, or of the form $(\neg q, s, g)$ with $s \notin V(q)$, or of the form $(H\chi, s, g)$ with $s = \min(g)$, or of the form $(G\chi, s, g)$ with $s = \max(g)$ and $\mathrm{TIME}(g, n) = g$ for all $n \in \mathbb{N}$, with TIME being the time-resource of $\mathcal{M}$. ▮

It is possible to characterize the property of a formula being true in a circumstance of evaluation in terms of truth-evaluations.[23]

Fact 3.3 Let $\mathcal{M}$ be an open temporal model. Let ϕ be a formula, let t be a moment, and let $g \in H_{\mathrm{p}}(t)$. Then we have:

$\mathcal{M}, t, g \models \phi$ iff there is a truth-evaluation of ϕ on $(\mathcal{M}, t, g)$.

Proof. By induction on the complexity of ϕ. For the direction from left to right, the Axiom of Choice is needed, because of the unicity requirement inbuilt into the clauses for F, P, and $\Diamond$; see Hodges (1983), p. 94; cf. Hodges (2013), Section 3. ▮

Consequently, the property of a formula of being false in a circumstance is likewise characterizable by using truth-evaluations: $\mathcal{M}, t, g \not\models \phi$ iff there is no truth-evaluation of ϕ on $(\mathcal{M}, t, g)$ iff for every evaluation $\mathcal{E}$ of ϕ on $(\mathcal{M}, t, g)$, there is an evaluation sequence in $\mathcal{E}$ such that its last position is either of the form (q, s, g) with $s \notin V(q)$, or of the form $(\neg q, s, g)$ with $s \in V(q)$, or of the form $(P\chi, s, g)$ with $s = \min(g)$, or of the form $(F\chi, s, g)$ with $s = \max(g)$ and $\mathrm{TIME}(g, n) = g$ for all $n \in \mathbb{N}$. Evaluations allow us to discuss the semantics of our temporal-modal language in a much more fine-grained fashion than would be the case if we simply proceeded from the declaration that a formula is true or false in a given circumstance of evaluation. The following definition will prove useful.

Definition 3.4 (Moment set of an evaluation) Suppose $\mathcal{E}$ is an evaluation and the common initial position of all evaluation sequences in $\mathcal{E}$ is (ϕ, t_0, g_0). If Σ is an evaluation sequence in $\mathcal{E}$, then its *moment set*, denoted by $\mathrm{M}(\Sigma)$, consists of all those moments that either belong to the initial partial history g_0

[23] Evaluations allow us to employ the fine-grained level of analysis familiar from game-theoretical semantics without getting overtly ludic. Cf. footnote 21.

or appear as the second member u of some position (χ, u, h) of Σ. The moment set of $\mathcal{E}$, denoted by $\mathrm{M}(\mathcal{E})$, is the union of all moment sets $\mathrm{M}(\Sigma)$, with Σ being an evaluation sequence in $\mathcal{E}$. ∎

Note that it could happen that the endpoint of the initial partial history g_0 does not appear as the moment component of any position appearing in a given evaluation sequence Σ, though automatically the endpoint of any other partial history appearing in a position of Σ indeed appears as a moment component of some position in Σ. By including g_0 in $\mathrm{M}(\Sigma)$, we take care that no explicitly introduced moment in any way constitutive of Σ remains *above* all moments in $\mathrm{M}(\Sigma)$. The set $\mathrm{M}(\mathcal{E})$—the union of all sets $\mathrm{M}(\Sigma)$ with $\Sigma \in \mathcal{E}$—may well be infinite. Already the moment set $\mathrm{M}(\Sigma)$ of a single evaluation sequence Σ may be infinite (due to the initial partial history g_0 being possibly infinite in the direction of the past), but $\mathrm{M}(\Sigma)$ may not contain infinitely many moments later than the initial moment of evaluation t_0. However, jointly all evaluation sequences in $\mathcal{E}$ can indeed involve infinitely many such moments.

4. States of affairs and truth-makers

4.1 States of affairs

I take a *momentary state of affairs* to be a moment exemplifying specified features—features that can be described in terms of propositional atoms. Generally, I take a *state of affairs* (SOA) to be a structure of momentary SOAs in which these momentary SOAs are arranged by a causal ordering, and in connection with which it is possible to speak of the passage of time. Open temporal models represent SOAs in this sense. In particular, a momentary SOA is represented by an open model $(\{t\}, \varnothing, V_t, \mathrm{TIME}_t)$, with $\mathrm{TIME}_t(\{t\}, 0) = \{t\}$, and $V_t(p) = \varnothing$ or $V_t(p) = \{t\}$ for all $p \in \mathbf{prop}$. ‘Our indeterminist world’ is itself one enormous SOA.

In what follows, I use the standard notion of substructure in connection with event structures $(T, \prec, V)$. I refer to such structures as ‘static substructures’ in order to stress the fact that they involve no resource representing temporal becoming.

Definition 4.1 (Static substructure of an event structure) If $\mathfrak{S} = (T, \prec, V)$ is an event structure, and $\varnothing \neq T' \subseteq T$, then the *static substructure* of $\mathfrak{S}$ determined by T' is the structure $(T', \prec', V')$, where $\prec' = \prec \cap (T' \times T')$ and

$V'(p) = V(p) \cap T'$ for all $p \in$ **prop**. I write $\mathfrak{S}' \Subset \mathfrak{S}$ to indicate that $\mathfrak{S}'$ is a static substructure of $\mathfrak{S}$ (determined by some non-empty subset of the domain of $\mathfrak{S}$). If $\mathfrak{S}'$ is a static substructure of $\mathfrak{S}$, I say conversely that $\mathfrak{S}$ is a *static extension* of $\mathfrak{S}'$. ▮

The following auxiliary notion will be useful.

Definition 4.2 (Static substructure generated by a set of moments) Let $\mathfrak{S} = (T, \prec, V)$ be an event structure. Suppose $\varnothing \neq S \subseteq T$. Let T' be the smallest subset of T that includes S and satisfies the following closure conditions:

- *Inward closure:* For all $s, t \in S$ and all x with $s \preccurlyeq x \preccurlyeq t$, we have: $x \in T'$.
- *Closure under formation of minimal paths between incomparables:* For all $s, t \in S$ that are not comparable with respect to the relation $\prec$, and for all x such that $\inf\{s, t\} \preccurlyeq x \preccurlyeq s$ or $\inf\{s, t\} \preccurlyeq x \preccurlyeq t$, we have: $x \in T'$.[24]

The static substructure of $\mathfrak{S}$ *generated by S* is the unique substructure $\mathfrak{S}'$ of $\mathfrak{S}$ whose domain is T'. ▮

Note that in the generated static substructure $\mathfrak{S}'$, the interrelations of the elements of S are the same as they were in the model $\mathfrak{S}$: if a moment $s \in S$ was in $\mathfrak{S}$ an n-th successor (or the n-th predecessor) of t_0, then s is even in $\mathfrak{S}'$ an n-th successor (respectively, the n-th predecessor) of t_0. And if moments $s, s' \in$ S were incomparable in $\mathfrak{S}$ and their greatest lower bound was the n-th predecessor of s and the m-th predecessor of s', then s and s' are incomparable in $\mathfrak{S}'$, as well, and in $\mathfrak{S}'$, too, their greatest lower bound is the n-th predecessor of s and the m-th predecessor of s'. Finally, observe that for each partial history g' of $\mathfrak{S}'$, there is a unique partial history g in $\mathfrak{S}$ such that g' is a 'final segment' of g: either g' = g or else there is s such that $s = \min(g')$ and $g' = \{t : t \in g \text{ and } s \preccurlyeq t\}$.

Definition 4.3 (Dynamic and static substructures of a SOA) Let $\mathcal{M}_1 = (\mathfrak{S}_1, \text{TIME}_1)$ and $\mathcal{M}_2 = (\mathfrak{S}_2, \text{TIME}_2)$ be SOAs, with $\mathfrak{S}_1 = (T_1, \prec_1, V_1)$ and $\mathfrak{S}_2 = (T_2, \prec_2, V_2)$ being event structures. It is said that $\mathcal{M}_1$ is a *static*

[24] I formulate the two conditions separately for the sake of clarity. It would be sufficient to use the following single more general condition: For all s, $t \in S$ and for all x such that $\inf\{s, t\} \preccurlyeq x \preccurlyeq s$ or $\inf\{s, t\} \preccurlyeq x \preccurlyeq t$, we have: $x \in T'$.

substructure of $\mathcal{M}_2$ (conversely: $\mathcal{M}_2$ is a *static extension* of $\mathcal{M}_1$) iff the event structure $\mathfrak{S}_1$ a static substructure of the event structure $\mathfrak{S}_2$. Further, it is said that $\mathcal{M}_1$ is a *dynamic substructure* of $\mathcal{M}_2$ (conversely: $\mathcal{M}_2$ is a *dynamic extension* of $\mathcal{M}_1$), in symbols $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$, iff $\mathcal{M}_1$ is a static substructure of $\mathcal{M}_2$, and for all partial histories g of $\mathcal{M}_1$, we have:

* $\mathrm{TIME}_1(g, 1) = \mathrm{TIME}_2(g, 1)$, if $\mathrm{TIME}_2(g, 1) \subseteq T_1$;
* $\mathrm{TIME}_1(g, 1) = g$, otherwise.

If $X \subseteq T_2$ and $\mathfrak{S}_X$ is the static substructure of $\mathfrak{S}_2$ generated by X, then the *dynamic substructure* of $\mathcal{M}_2$ *generated* by X is the unique dynamic substructure $(\mathfrak{S}, \mathrm{TIME})$ of $\mathcal{M}_2$ such that $\mathfrak{S} = \mathfrak{S}_X$. ▮

For the above definition, note that if g is a partial history of $\mathcal{M}_1$, there are two separate reasons why we may have $\mathrm{TIME}_1(g, 1) = g$. Either already in $\mathcal{M}_2$ the history g is not extended by the time-resource of $\mathcal{M}_2$, so that $\mathrm{TIME}_1(g, 1) = \mathrm{TIME}_2(g, 1) = \mathrm{TIME}_2(g, 0) = g$. Or else the time-resource $\mathcal{M}_2$ indeed extends g in a duration of length 1 to a partial history of $\mathcal{M}_2$, but this partial history is not a partial history of $\mathcal{M}_1$. In both cases, the time-resource of $\mathcal{M}_1$ simply does not extend g at all. Further, the former option admits of two subcases: even if in $\mathcal{M}_2$ the passage of time from g ends with g—with $\mathrm{TIME}_2(g, 1) = \mathrm{TIME}_2(g, 0) = g$—still the model $\mathcal{M}_2$ may or may not contain moments later than the endpoint of g. Now, if indeed there are moments in $\mathcal{M}_2$ later than $\max(g)$, those moments might belong to $\mathcal{M}_1$, as well. Note also that if g is a partial history of $\mathcal{M}_1$ (and therefore of $\mathcal{M}_2$), it precisely need not happen that $\mathrm{TIME}_2(g, 1)$, too, is a partial history of $\mathcal{M}_1$.

Some, but not all, ‘fragments’ (dynamic substructures) of a SOA are SOAs, too. If $(T', \prec', V', \mathrm{TIME}')$ is a SOA and the subset T of T' consists for example of only two moments t_1 and t_2, and these moments are incomparable in terms of $\prec'$, then the causal order $\prec$ of the substructure determined by T is empty, whence this substructure is not a SOA, as it violates the condition of historical connectedness required of temporal frames. However, if $\mathcal{M}'$ is a SOA and $\mathcal{M}$ is its dynamic substructure *generated* by a set S of moments, then $\mathcal{M}$ is itself a SOA. All moments of $\mathcal{M}$ are indeed arranged in terms of the causal order of $\mathcal{M}'$ into a single ‘constellation’ upon which the resource TIME' induces a notion of passage of time, each moment of $\mathcal{M}$ being historically connected to each further moment of $\mathcal{M}$ in the sense of ‘historical connectedness' defined in Subsection 1.1. In what follows, I will make use of the following notion.

Definition 4.4 (Centered state of affairs) If $\mathcal{M}$ is a SOA and t_0 belongs to the set of moments of $\mathcal{M}$, the pair $(\mathcal{M}, t_0)$ is a centered state of affairs. ∎

4.2 What exactly is a truth-maker?

In Section 3 of the background paper, I adopted a preliminary understanding of truth-makers, according to which all truth-makers are circumstances of evaluation, and more specifically, a truth-maker of proposition *p* is a circumstance of evaluation in which *p* is true (i.e., is a 'realization' of *p*). I further noted that metaphysicians tend to view truth-makers as *minimal* in some sense. Metaphysicians' prejudices must certainly not be taken as a standard by which to judge semantic notions, but the idea is indeed appealing that there might be something superfluous in a circumstance in which a proposition is true, in the sense that the proposition would have been true even if certain aspects of the circumstance had not been there. Accordingly, I agreed to impose a certain 'minimality condition' that a realization must meet in order to count as a truth-maker. This, of course, does not force us to maintain that for any circumstance in which *p* is true, there is a unique 'minimal fragment' of this circumstance in which it is true. It just means that generally, if *p* is true in *k*, *p* would even be true in *k′* which is in some sense 'smaller' than *k*. There might, however, be many alternative transitions from *k* to a 'minimal' *k′* that preserve the truth of *p*. These remarks can be made more precise by utilizing the notions of evaluation and state of affairs as defined above.

Recall that circumstances of evaluation are structures $(\mathcal{M}, t_0, g_0)$, while centered states of affairs are structures $(\mathcal{M}, t_0)$. Recall also that in Subsection 2.2, I noted that we may consider *not* explicitly relativizing the truth of a proposition to a partial history, provided that by speaking of the truth of a proposition at a time, we really mean its truth at a time relative to the unique partial history leading to that time (see Definition 2.1). I will define truth-makers as centered SOAs $(\mathcal{M}, t_0)$ that are in a certain sense 'minimal'. This is equivalent to defining truth-makers as circumstances of evaluation $(\mathcal{M}, t_0, g_0)$ with $g_0 = [t_0]$ subject to the relevant 'minimality' condition. The following variant of the notion of evaluation (as introduced in Definition 3.1) will be employed.

Definition 4.5 (Evaluation on a centered SOA) Let ϕ be a formula, and let $(\mathcal{M}, t_0)$ be a centered SOA. An *evaluation of* ϕ *on* $(\mathcal{M}, t_0)$ is by stipulation an evaluation of ϕ on the circumstance of evaluation $(\mathcal{M}, t_0, g_0)$, where $g_0 = [t_0]$. ∎

In the following sense, evaluations produce SOAs out of SOAs.

Definition 4.6 (Evaluation-induced SOA) Let ϕ be a formula and let $(\mathcal{M}, t_0)$ be a centered SOA. Any evaluation $\mathcal{E}$ of ϕ on $(\mathcal{M}, t_0)$ induces a SOA—namely, the dynamic substructure of $\mathcal{M}$ generated by the moment set $\mathrm{M}(\mathcal{E})$. I denote this dynamic substructure by $\mathcal{M}[t_0, \mathcal{E}]$, and say that it is a SOA *induced by the evaluation* $\mathcal{E}$. ∎

Note that evaluations of several formulas on $(\mathcal{M}, t_0)$ may induce the same SOA; all that counts is that such evaluations have the same moment set. Thus, on a given centered SOA $(\mathcal{M}, t_0)$, for example every evaluation $\mathcal{E}$ of the formula Fq induces a SOA that is likewise induced by an evaluation $\mathcal{E}'$ of the formula $\Diamond Fq$ (though the converse does not hold). Namely, suppose the unique evaluation sequence Σ in $\mathcal{E}$ is $\langle (Fq, t_0, [t_0]), (q, s, [s]) \rangle$, with s obtained by applying the time-resource of $\mathcal{M}$. Whenever x and y are moments such that $t_0 \preccurlyeq x$ and $t_0 \prec y$ and y can be obtained from x by the time-resource of $\mathcal{M}$, it is possible to generate an evaluation of $\Diamond Fq$ at t_0 whose unique evaluation sequence is $\langle (\Diamond Fq, t_0, [t_0]), (Fq, t_0, [x]), (q, y, [x]) \rangle$, with x corresponding to $\Diamond$ and y corresponding to F. In particular, such an evaluation sequence is obtained by letting $x \coloneqq t_0$ and $y \coloneqq s$. Now, let $\mathcal{E}'$ be the evaluation of $\Diamond Fq$ at t_0 whose unique evaluation sequence is $\langle (\Diamond Fq, t_0, [t_0]), (Fq, t_0, [t_0]), (q, s, [s]) \rangle$. We end up having, then, $\mathrm{M}(\mathcal{E}) = \mathrm{M}(\mathcal{E}')$, even if the evaluation sequences Σ and Σ' are differently generated. Observe, further, that we can have $\mathrm{M}(\mathcal{E}) = \mathrm{M}(\mathcal{E}')$ even when one of the evaluations $\mathcal{E}$ and $\mathcal{E}'$ employs the time-resource but the other does not. Indeed, let $\mathcal{E}$ be as above, but consider the evaluation of $\Diamond Fq$ at t_0 whose unique evaluation sequence Σ' equals $\langle (\Diamond Fq, t_0, [t_0]), (Fq, s, [s]), (q, s, [s]) \rangle$. Here, s is an 'eventuality' at t_0 (since $t_0 \preccurlyeq s$) and therefore it can be used to interpret $\Diamond$. Then, the evaluation may be continued by selecting s for F (since $t_0 \prec s$), and here the time-resource is *not* needed for accessing s from t_0, since s is directly available on the previously triggered partial history $[s]$. Even in this case, we have $\mathrm{M}(\mathcal{E}) = \mathrm{M}(\mathcal{E}')$. Regarding evaluation-induced SOAs, the following holds.

Fact 4.7 Suppose $\mathcal{E}$ is an evaluation of ϕ on $(\mathcal{M}, t_0)$. Then:

(a) $\mathcal{E}$ is an evaluation of ϕ on $(\mathcal{M}[t_0, \mathcal{E}], t_0)$.
(b) $(\mathcal{M}[t_0, \mathcal{E}])[t_0, \mathcal{E}] = \mathcal{M}[t_0, \mathcal{E}]$.

Proof. If $\mathcal{E}$ is an evaluation of ϕ on $(\mathcal{M}, t_0)$ and $\mathcal{M} = (T, \prec, V, \mathrm{TIME})$, then $\mathcal{M}[t_0, \mathcal{E}]$ is the dynamic substructure of $\mathcal{M}$ generated by the moment set $\mathrm{M}_T(\mathcal{E})$ with $\mathrm{M}_T(\mathcal{E}) \subseteq T$. Thus, letting D be the smallest subset of T that includes $\mathrm{M}_T(\mathcal{E})$ and is subject to the two closure conditions of Definition 4.1, by definition $\mathcal{M}[t_0, \mathcal{E}]$ is the unique substructure of $\mathcal{M}$ whose domain is D. Therefore, to arrive at $\mathcal{M}[t_0, \mathcal{E}]$ from $\mathcal{M}$, nothing has been removed from $\mathcal{M}$ that could prevent the construction of all evaluation sequences of the evaluation $\mathcal{E}$, when the evaluation takes place on $(\mathcal{M}[t_0, \mathcal{E}], t_0)$. This is why (a) holds.

As to (b), since the domain of $\mathcal{M}[t_0, \mathcal{E}]$ is D, the structure $(\mathcal{M}[t_0, \mathcal{E}])[t_0, \mathcal{E}]$ is by definition the dynamic substructure of $\mathcal{M}[t_0, \mathcal{E}]$ generated by the moment set $\mathrm{M}_D(\mathcal{E})$ with $\mathrm{M}_D(\mathcal{E}) \subseteq D \subseteq T$. Letting D' be the smallest subset of D that includes $\mathrm{M}_D(\mathcal{E})$ and is subject to the two closure conditions of Definition 4.1, we have that $\mathcal{M}[t_0, \mathcal{E}]$ is the dynamic substructure of $\mathcal{M}$ determined by D'. Write $(D, \prec_D)$ for the frame of $\mathcal{M}[t_0, \mathcal{E}]$. Now, $D' \subseteq D$. To show that even the converse holds, suppose that $x \in D$, whence there are $s, t \in \mathrm{M}_T(\mathcal{E})$ such that $s \preccurlyeq x \preccurlyeq t$ or $\inf\{s, t\} \preccurlyeq x \preccurlyeq s$ or $\inf\{s, t\} \preccurlyeq x \preccurlyeq t$. Since $\mathrm{M}_D(\mathcal{E}) = \mathrm{M}_T(\mathcal{E}) \subseteq D$ and $\prec_D = \prec \cap (D \times D)$, it follows that these moments s, t, and x satisfy: $s, t \in \mathrm{M}_D(\mathcal{E})$, and s $\preccurlyeq_D x \preccurlyeq_D t$ or $\inf\{s, t\} \preccurlyeq_D x \preccurlyeq_D s$ or $\inf\{s, t\} \preccurlyeq_D x \preccurlyeq_D t$. Thus, $x \in D'$. We may, then, conclude that $D = D'$. It follows that $(\mathcal{M}[t_0, \mathcal{E}])[t_0, \mathcal{E}]$ is the dynamic substructure of $\mathcal{M}[t_0, \mathcal{E}]$ determined by the set D. However, D is the domain of $\mathcal{M}[t_0, \mathcal{E}]$. Thus, $(\mathcal{M}[t_0, \mathcal{E}])[t_0, \mathcal{E}]$ is the result of restricting the SOA $\mathcal{M}[t_0, \mathcal{E}]$ by its own domain, wherefore $(\mathcal{M}[t_0, \mathcal{E}])[t_0, \mathcal{E}] = \mathcal{M}[t_0, \mathcal{E}]$. ∎

From Fact 4.7(a), it follows:

Fact 4.8 Let $\mathcal{E}$ be an evaluation of ϕ on $(\mathcal{M}, t_0)$. Then: $\mathcal{E}$ is a truth-evaluation for ϕ on $(\mathcal{M}, t_0)$ iff $\mathcal{E}$ is a truth-evaluation for ϕ on $(\mathcal{M}[t_0, \mathcal{E}], t_0)$.

Proof. If $\mathcal{E}$ is a truth-evaluation for ϕ on $(\mathcal{M}, t_0)$, then by Fact 4.7, $\mathcal{E}$ is at least an *evaluation* for ϕ on $(\mathcal{M}[t_0, \mathcal{E}], t_0)$. But whether an evaluation is a *truth-evaluation* depends only on the last positions of its evaluation sequences.

Thus, $\mathcal{E}$ is a truth-evaluation for ϕ on $(\mathcal{M}[t_0, \mathcal{E}], t_0)$. Conversely, if $\mathcal{E}$ is both an evaluation for ϕ on $(\mathcal{M}, t_0)$ and a truth-evaluation for ϕ on $(\mathcal{M}[t_0, \mathcal{E}], t_0)$, then by the reason just mentioned $\mathcal{E}$ is in fact a truth-evaluation for ϕ on $(\mathcal{M}, t_0)$. ▮

The above fact allows us to get a grip on the sense of 'minimality' I wish to impose on realizations that qualify as truth-makers. In what follows, I suppose that 'our indeterminist world' is fixed, represented by a certain large state of affairs to be denoted by $\mathcal{M}_w$.

Definition 4.8 (Truth-makers) Let ϕ be a formula.

(a) Suppose $\mathcal{M}$ and $\mathcal{N}$ are SOAs and t_0 is a moment in $\mathcal{M}$. Now, $(\mathcal{M}, t_0)$ is a *truth-maker of* ϕ in $\mathcal{N}$ iff $\mathcal{M} \sqsubseteq \mathcal{N}$ and there is a truth-evaluation $\mathcal{E}$ of ϕ in $(\mathcal{N}, t_0)$ such that $\mathcal{N}[t_0, \mathcal{E}] = \mathcal{M}$.
(b) $(\mathcal{M}, t_0)$ is a *real truth-maker* of ϕ iff $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{M}_w$.
(c) $(\mathcal{M}, t_0)$ is an *intrinsic truth-maker* of ϕ iff $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{M}$ itself—that is, iff there is a truth-evaluation $\mathcal{E}$ of ϕ in $(\mathcal{M}, t_0)$ such that $\mathcal{M}[t_0, \mathcal{E}] = \mathcal{M}$. ▮

It can be checked that the above definition respects the basic requirement that a truth-maker of a formula be its realization: the formula is true in its truth-maker!

Fact 4.10 Let ϕ be a formula.

(a) If $(\mathcal{M}, t_0)$ is an intrinsic truth-maker of ϕ, then $\mathcal{M}, t_0 \vDash \phi$.
(b) If $\mathcal{N}$ is a SOA such that $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{N}$, then $\mathcal{M}, t_0 \vDash \phi$.
(c) If $\mathcal{N}$ is a SOA, then every truth-maker of ϕ in $\mathcal{N}$ is an intrinsic truth-maker of ϕ.

Proof. For (a), note that if $(\mathcal{M}, t_0)$ is an intrinsic truth-maker of ϕ, then in particular there is a truth-evaluation of ϕ in $(\mathcal{M}, t_0)$, whence by Fact 3.3, we have $\mathcal{M}, t_0 \vDash \phi$. For (b) and (c), suppose that $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{N}$. There is, then, a truth-evaluation $\mathcal{E}$ of ϕ in $(\mathcal{N}, t_0)$ such that $\mathcal{N}[t_0, \mathcal{E}] = \mathcal{M}$. The claim (b) follows, since by Fact 4.8, the evaluation $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{N}[t_0, \mathcal{E}], t_0) = (\mathcal{M}, t_0)$ and so, by Fact 3.3, we have $\mathcal{M}, t_0 \vDash \phi$. To prove (c), it remains to show that $(\mathcal{N}[t_0, \mathcal{E}], t_0)$ is an intrinsic truth-maker of ϕ. Now, as just noted, $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{N}[t_0, \mathcal{E}], t_0)$. What is more, by Fact 4.7(b), we have $\mathcal{N}[t_0, \mathcal{E}] = (\mathcal{N}[t_0, \mathcal{E}])[t_0, \mathcal{E}]$. Thus, $(\mathcal{M}, t_0) = (\mathcal{N}[t_0, \mathcal{E}], t_0)$ is an intrinsic truth-maker of ϕ. ∎

While a truth-maker of a formula in a SOA is always its intrinsic truth-maker, an intrinsic truth-maker of a formula need not be a truth-maker of this formula in a larger SOA. For example, suppose $\mathcal{N}$ is a linearly ordered SOA in which t_0 has infinitely many predecessors, and in which the time-resource yields exactly 10 successors to t_0, q being true at all predecessors $t_{-1}, t_{-2},\ldots$ of t_0, as well as at the first and third successor of t_0 (i.e., t_1 and t_3). Let $\mathcal{M}$ be the dynamic substructure of $\mathcal{N}$ whose domain is the set $\{t_0, t_{-1}, t_{-2}\}$. Then, $(\mathcal{M}, t_0)$ is an intrinsic truth-maker of the formula Hq (since q is true at both moments preceding t_0 in $\mathcal{M}$), but $(\mathcal{M}, t_0)$ is not a truth-maker of Hq in $\mathcal{N}$, because the moment set of the one and only evaluation of Hq in $(\mathcal{N}, t_0)$ comprises all the infinitely many predecessors of t_0, and such an infinite evaluation cannot induce the finite truth-maker $(\mathcal{M}, t_0)$. In fact, the dynamic substructure of $(\mathcal{N}, t_0)$ whose domain is the set of all moments t_i with $i \leq 0$ is a truth-maker of Hq in $\mathcal{N}$, actually its only truth-maker in $\mathcal{N}$. On the other hand, if $\mathcal{M}'$ is the dynamic substructure of $\mathcal{N}$ whose domain is the set $\{t_0, t_1\}$, then not only is $(\mathcal{M}', t_0)$ an intrinsic truth-maker of the formula Fp, but it is even a truth-maker of Fp in $\mathcal{N}$. It is possible to construct an evaluation for Fp without employing moments other than t_0 and t_1. Incidentally, there is even another truth-maker for Fp in $\mathcal{N}$, namely the centered SOA $(\mathcal{M}'', t_0)$, where $\mathcal{M}''$ is the dynamic substructure of $\mathcal{N}$ whose domain is the set $\{t_0, t_1, t_2, t_3\}$.

Note that if $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{N}$, there is, by definition, a truth-evaluation $\mathcal{E}$ that 'witnesses' the truth of ϕ in the centered SOA $(\mathcal{N}, t_0)$. Furthermore, $\mathcal{M}$ is 'minimal' in the sense that the dynamic substructure of $\mathcal{N}$ containing all and only moments needed for constructing the evaluation $\mathcal{E}$ is exactly $\mathcal{M}$. Indeed $\mathcal{N}[t_0, \mathcal{E}] = \mathcal{M}$. Insofar as $\mathcal{E}$ is concerned, there is nothing superfluous in $\mathcal{M}$. Observe, however, that this is not to say that there could not be a 'smaller' truth-maker $\mathcal{M}' = \mathcal{N}[t_0, \mathcal{E}']$ for ϕ, satisfying $\mathcal{M}' \subsetneq \mathcal{M}$, generated by another truth-evaluation $\mathcal{E}'$ for ϕ in $\mathcal{N}$. Observe also that the feature of a centered SOA of being a truth-maker is evidently a relational property, indeed a property relative to a formula: the same centered SOA may, of course, be a truth-maker of one formula without being a truth-maker of another formula.

Fact 4.11 (For every truth there is a truth-maker) Let ϕ be a formula and let $(\mathcal{N}, t_0)$ be a centered SOA. We have: If $\mathcal{N}, t_0 \vDash \phi$, then there is $\mathcal{M}$ such that $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{N}$.

Proof. Suppose $\mathcal{N}, t_0, g_0 \vDash \phi$ with $g_0 = [t_0]$. By Fact 3.3, there is a truth-evaluation, $\mathcal{E}$, for ϕ in $(\mathcal{N}, t_0, g_0)$. Thus, $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{N}, t_0)$. Now, $\mathcal{N}[t_0, \mathcal{E}] \sqsubseteq \mathcal{N}$, so by definition, $(\mathcal{N}[t_0, \mathcal{E}], t_0)$ is a truth-maker of ϕ in $\mathcal{N}$. We may, then, let $\mathcal{M} \coloneqq \mathcal{N}[t_0, \mathcal{E}]$. ▮

From the above fact we may, by Fact 4.10(c), infer that every formula ϕ true in a centered SOA $(\mathcal{N}, t_0)$ has an *intrinsic* truth-maker $(\mathcal{M}, t_0)$ in which ϕ is true, as well. Here, $\mathcal{M} = \mathcal{N}[t_0, \mathcal{E}]$ is a dynamic substructure of $\mathcal{N}$. The intrinsic truth-maker $(\mathcal{M}, t_0)$ is itself a centered SOA, and it is a 'minimal' realization of ϕ in the sense that a certain truth-evaluation of ϕ (namely, $\mathcal{E}$) induces $\mathcal{M}$ from $\mathcal{M}$ itself: $\mathcal{M} = \mathcal{M}[t_0, \mathcal{E}]$.

5. Typology of centered states of affairs

The remaining task in the semantic analysis of our temporal-modal language is to explain what it means for a proposition to be true without having a truth-maker that obtains. True future contingent sentences are supposed to lead to cases of this kind. I begin by clarifying what it means to say that a truth-maker *obtains*, or that it *will obtain*, or that it *is a process of coming to obtain*, or that it is *modal*. Actually, these are more generally features of a centered SOA—a centered SOA has or lacks such a feature independently of any specific proposition that might be true in it. I will also formulate an analysis of what it means for a truth-maker to be determinate or indeterminate, and I will study the conditions under which different types of truth-makers may be determinate. Determinacy is not simply a property of a centered SOA: the same centered SOA may be a determinate truth-maker for one proposition, indeterminate truth-maker for another proposition, and not a truth-maker at all for a still third proposition. Determinacy and indeterminacy are, then, features relative to a fixed proposition.

The analyses I will give are not meant as arbitrary nominal definitions of expressions such as 'obtaining truth-maker' or 'truth-maker that will obtain' or 'determinate truth-maker', but as ways of making explicit what is meant, when in our less-than-fully-articulated philosophical parlance we attempt qualifying a truth-maker, say, as determinate, or as obtaining, or as something that will obtain.

Suppose, now, that $\mathcal{M}$ is a SOA with $\mathcal{M} = (T, \prec, V, \text{TIME})$ and $t_0 \in T$. Centered SOAs are divided into those that are *factual* and those that are *modal*. Those that are factual are divided into those that *obtain*, those that *will obtain*, and those that *are processes of coming to obtain*; among modal centered SOAs, again, I distinguish two types: those that are *historically modal* and those that are *presently modal*. Note that I consider centered SOAs with a designated moment t_0—instead of considering mere SOAs. Here, the designated moment t_0 should be understood as indicating the location of a hypothetical now-point. The position of this now-point in the overall structure of the SOA, on the one hand, and the passage of time represented by the time-resource of the SOA, on the other hand, jointly determine the status of the SOA itself as factual or modal. Likewise, they determine its specific type of factuality or modality.

(I) **'Is factual'**: A centered SOA $(\mathcal{M}, t_0)$ is factual iff (a) all moments $t \in T$ are comparable with t_0, and (b) for all $t \succ t_0$ there is $n_t > 0$ such that t is the endpoint of $\text{TIME}([t_0], n_t)$. Regarding this case, note:

1. Clause (a) guarantees that there is no moment earlier than t_0 with incomparable successors according to the causal ordering $\prec$.
2. Clause (b), again, entails that the set of moments later than t_0 is linearly ordered by the causal ordering.
3. The joint effect of (a) and (b) is, then, that the causal ordering $\prec$ imposes a linear order on the set T of moments.
4. In particular, clause (b) entails that only those moments lie in the future of t_0 that are actualized by the passage of time in some finite duration as counted from t_0, the passage of time being represented by the resource TIME.

(II) **'Is modal'**: A centered SOA $(\mathcal{M}, t_0)$ is *modal* iff $(\mathcal{M}, t_0)$ is *not* factual. Here, we may observe:

1. Either clause (a) or clause (b) fails to hold: there is a moment $t \in T$ such that either t is not comparable with t_0, or else $t \succ t_0$ but for all $n > 0$, $t \notin \text{TIME}([t_0], n)$.

2. In the former case, T is not linearly ordered by the causal ordering.
3. In the latter case, in fact there is a moment s and a duration $n_0 \geq 0$ such that $t_0 \preccurlyeq s \prec t$ and s is the endpoint of TIME($[t_0]$, n_0) and TIME($[t_0]$, $n_0 + 1$) = TIME($[t_0]$, n_0). That is, there is at least one moment (t) later than t_0 not attained by the passage of time. The passage of time ends at s, before reaching this causally possible moment.

Let us, then, look at subdivisions of factuality and modality. There are three mutually exclusive and jointly exhaustive sources of factuality of a state of affairs:

(I.i) **'Obtains'**: There is no $t \in T$ such that $t_0 \prec$ t. In this case, ($\mathcal{M}$, t_0) is said to *obtain*.

- The moment t_0 is the maximum of the whole set T with respect to the relation $\prec$, i.e., the unique maximal moment in T. Since T is linearly ordered by $\prec$, it cannot have several maximal elements.
- In fact, $T = [t_0] = \{t : t \preccurlyeq t_0\}$.

(I.ii) **'Will obtain'**: The set $\{x : t_0 \prec x\}$ is non-empty and has a maximum. When this condition holds, it is said that ($\mathcal{M}$, t_0) *will obtain*.

- The moment $t^* \coloneqq \max\{x : t_0 \prec x\}$ is actually the maximum of the whole set T with respect to the causal ordering.
- There is $n > 0$ such that for all $t \succ t_0$, we have $t \in$ TIME($[t_0]$, n). In fact, the shortest duration satisfying this condition is $n \coloneqq \Delta(t_0, t^*)$.
- Generally, for any moment t in a factual SOA $\mathcal{N}$, we have: ($\mathcal{N}$, t) will obtain iff there is s with $t \prec s$ such that ($\mathcal{N}$, s) obtains.

(I.iii) **'Is a process of coming to obtain'**: The set $\{x : t_0 \prec x\}$ is non-empty and has no maximum. In this case, it is said that ($\mathcal{M}$, t_0) *is a process of coming to obtain*.

- The set T has neither a maximum nor even a maximal element (since here the causal ordering is linear).
- If ($\mathcal{M}$, t_0) is a process of coming to obtain, there is no $t \succ t_0$ such that ($\mathcal{M}$, t) obtains.

We may note that by the above definitions, indeed the features 'obtains', 'will obtain' and 'is a process of coming to obtain' are mutually exclusive and that every factual centered SOA has one of these features.

There are two alternative sources of modality, which are jointly exhaustive but *not* mutually exclusive:

(II.i) **'Historically'**: There are moments s and t such that s, t, and t_0 are pairwise distinct and $s = \inf_{\preccurlyeq}\{t, t_0\}$, where t and t_0 are incomparable in terms of the causal ordering $\prec$. That is, at s, the moment t could have become actualized, but did not. When this condition holds, it is said that $(\mathcal{M}, t_0)$ is *historically modal*.

(II.ii) **'Presently'**: There is at least one moment t with $t_0 \prec t$ such that for no $n > 0$ do we have that t is the endpoint of TIME($[t_0]$, n). That is, t is a possible moment later than t_0 that will never be actualized by the passage of time. In this case, $(\mathcal{M}, t_0)$ is said to be *presently modal*.

- If $(\mathcal{M}, t_0)$ is historically modal, the set T is not linearly ordered by the causal ordering relation: there is a moment incomparable with t_0, this moment being later than a certain predecessor of t_0.
- If $(\mathcal{M}, t_0)$ is presently but not historically modal, the set T may but need not be linearly ordered by the causal ordering. If the order happens to be linear, there is a moment t later than t_0 such that t is not the endpoint of TIME(t_0, n) for any $n > 0$, but the immediate predecessor t' of t indeed is the endpoint of TIME(t_0, n_0) for some $n_0 \geq 0$. (In this case, the set of successors of t_0 attainable along the passage of time is finite.) It can happen that $t' = t_0$.

We may observe that by the above definitions, indeed one and the same centered SOA can be both presently and historically modal.

The features of being factual and being modal are ascribable to centered SOAs only from a meta-theoretical perspective. At t_0, it is as yet undetermined how time will unfold, so from the temporal perspective of the moment t_0, there is no way of reasoning in terms of the factual/modal distinction regarding what happens after t_0. This, however, by no means prevents us from using these features in our semantic theorizing. We may divide arbitrary centered SOAs into those that obtain, those that will obtain, those that are processes of coming to obtain, and those that are modal. This division is sufficiently fine-grained for my purposes. The qualifiers *obtains*, *will obtain*, and *is a process of coming to obtain* apply only to factual centered SOAs. I do not say that a modal circumstance of evaluation obtains, or that it will obtain, or that it is a process of coming to obtain. As implied above, among modal circumstances we may, however, discern those that are presently but not historically modal, those that are historically but not presently modal, and those that are both presently and historically modal.

The above-described characteristics of centered SOAs are directly applicable to truth-makers. After all, a truth-maker of a proposition *is* a centered SOA of a special kind—namely, one induced by a truth-evaluation. I began the background paper by a discussion of the semantic analysis of propositions about future events that Boethius and Abelard seemed to have put forward. According to this analysis, proposition Fq *is* true at a given time t_0 iff a truth-maker of Fq *will* obtain, though for the moment none does. (Since q is atomic, any truth-maker of Fq is factual.) The semantics I have formulated provides a systematic reconstruction of this analysis. Whenever a proposition of the form Fq is true in centered SOA, a truth-maker of Fq *will* obtain but does not, according to the senses of 'will obtain' and 'obtains' that were clarified above. That is, the passage of time will yield, when a sufficiently long finite duration has passed, a later moment at which q is true.

6. Determinate and indeterminate truth-makers

It was noted at the beginning of the background paper that Boethius and Abelard maintained that an *eventus rei* or a state of affairs is determinate, if it obtains or has obtained (in this case determinacy is based on the past or current presence of something), or it will obtain out of a certain kind of metaphysical necessity (in which case determinacy is based on the natures of things). Now, Boethius took Aristotle to hold, on the one hand, that the notions of determinacy and necessity (inevitability) are related so that determinate truth of a proposition entails its necessity. On the other hand, according to Boethius, Aristotle took this link to require argumentation, instead of being a direct consequence of a definition. Supposing that $\Box$ represents necessity in the relevant sense, can we define the notion of determinacy of a truth-maker in such a way that the following always holds: if there is a determinate truth-maker for ϕ at t_0, then $\Box\phi$ is true at t_0? We can, and it turns out that under this definition, phrased in terms of the semantic framework formulated above, there are indeed two essentially different grounds of determinacy, as in Boethius.

In order to lay bare the conceptual details, some auxiliary notions are needed.

Definition 6.1 (Historical *t*-equivalence) Let $\mathcal{M}_1 = (\mathfrak{S}_1, \text{TIME}_1)$ and $\mathcal{M}_2 = (\mathfrak{S}_2, \text{TIME}_2)$ be SOAs, with $\mathfrak{S}_1 = (T_1, \prec_1, V_1)$ and $\mathfrak{S}_2 = (T_2, \prec_2, V_2)$ being event structures. If $t \in T_1 \cap T_2$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are *historically t-equivalent*, in symbols $\mathcal{M}_1 \cong[t]^{\text{hist}} \mathcal{M}_2$, iff the following three conditions hold:

1. *Shared predecessors of t:* there is a set X_t such that $\{x : x \preccurlyeq_1 t\} = X_t = \{x : x \preccurlyeq_2 t\}$.
2. *Shared order in the past of t:* $\prec_1 \cap (X_t \times X_t) = \prec_2 \cap (X_t \times X_t)$.
3. *Qualitative indistinguishability of the past up to t:* $x \in V_1(p)$ iff $x \in V_2(p)$, for all moments $x \in X_t$ and all atoms p. ∎

Note that whether or not $\mathcal{M}_1$ and $\mathcal{M}_2$ are historically *t*-equivalent, this does not depend at all on their time-resources TIME_1 and TIME_2, but depends exclusively on the event structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ on which they are based. For what follows, recall the definition of 'course of events' from Subsection 1.1.

Definition 6.2 (*t*-completeness) If $\mathcal{M} = (\mathfrak{S}, \text{TIME})$ is a SOA with $\mathfrak{S} = (T, \prec, V)$, and S is a course of events in $\mathfrak{S}$ with $t \in S$, then S is *t-complete* in $\mathcal{M}$ iff for every $s \in S$ with $t \prec s$, there is n such that s is the endpoint of $\text{TIME}([t], n)$. ∎

In a *t*-complete course of events S, all successors of t belonging to S are actualized by the passage of time. Note that if S is *t*-complete and has a minimum, it may happen that there is a moment s in S with $t \prec s$ and a corresponding duration n_s such that the set $[s] = \text{TIME}([t], n_s)$ contains moments earlier than $\min(S)$. The course of events S may have a beginning, while in the temporal frame there are moments earlier than $\min(S)$. Those moments belong to the partial history $[s]$ leading to s, even if they do not belong to S. Observe likewise that the *t*-completeness of S does not entail that S is t'-complete for a given moment $t' \prec t$ (whether or not t' belongs to S). In particular, it can happen that there is no n such that t belongs to $\text{TIME}([t'], n)$; the endpoint of $\text{TIME}([t'], n)$ can be a moment incomparable with t. Likewise, the *t*-completeness of S does not entail that S is t'-complete for all moments t' with $t \prec t'$; it only entails that S is t'-complete for all *those* moments t' with $t \prec t'$ that moreover satisfy $t' \in S$. Finally, observe that if $\mathcal{E}$ is an evaluation of a formula ϕ in a centered SOA $(\mathcal{M}, t)$, the SOA $\mathcal{M}[t, \mathcal{E}]$ induced by $\mathcal{E}$ is automatically *t*-complete. Consequently, in particular every truth-maker $(\mathcal{M}, t)$ of ϕ is *t*-complete.

A certain event structure $\mathfrak{S}_w$ represents 'our indeterminist world' *from a static viewpoint*, indicating for each partial history the courses of events that are its causally possible continuations. A certain state of affairs $\mathcal{M}_w$ based on this event structure $\mathfrak{S}_w$ represents 'our indeterminist world' *from a dynamic viewpoint*. For every partial history g of $\mathfrak{S}_w$ and for every duration n, the time-resource of the state of affairs $\mathcal{M}_w$ yields exactly one possible continuation of g as the very course of events that indeed gets actualized in duration n. This course of events is by definition t-complete for $t = \max(g)$. However, the correlation between the pair (g, n) and the course of events is established only *ex post facto*—the state of affairs $\mathcal{M}_w$ itself does not contain information as to which course of events gets correlated with which partial history / duration pair. The fact remains that once n time units have passed since g, exactly one out of a multitude of possible continuations of g has got singled out; the others could have got actualized but did not. If we wish to reason counterfactually about those other possible continuations from the dynamic viewpoint—i.e., reason about what would have happened had the passage of time actualized one of those courses of events that it did not in fact actualize—we must consider, instead of 'our indeterminist world' in its dynamic aspect, *another* state of affairs $\mathcal{M}'_w$ which is based on the same event structure $\mathfrak{S}_w$ as $\mathcal{M}_w$, but employs a divergent time-resource, one that leads from g in n time units to *another* course of events, a course of events that is t-complete in $\mathcal{M}'_w$ (but not in $\mathcal{M}_w$). For strictly semantic purposes, we never need to consider variants of $\mathcal{M}_w$ that result from modifying its time-resource. The semantics of modal formulas utilizes simply the state of affairs $\mathcal{M}_w$ itself, the effect of a modal operator being that it changes the value of the history parameter of evaluation while keeping the time-resource intact. Certain meta-theoretic considerations, however, make use of the idea of varying the time-resource of $\mathcal{M}_w$ while keeping the event structure $\mathfrak{S}_w$ intact. This is why the following definition proves useful.

Definition 6.3 (Dynamic *t*-equivalence) Let $\mathfrak{S} = (T, \prec, V)$ be an event structure. Let S and S' be courses of events in $\mathfrak{S}$ such that $t \in S \cap S'$. Let $\mathfrak{S}_S$ and $\mathfrak{S}_{S'}$ be the static substructures of $\mathfrak{S}$ whose domains are respectively S and S'. Finally, let $\mathcal{M}_S = (\mathfrak{S}_S, \text{TIME}_S)$ and $\mathcal{M}_{S'} = (\mathfrak{S}_{S'}, \text{TIME}_{S'})$ be SOAs based respectively on $\mathfrak{S}_S$ and $\mathfrak{S}_{S'}$. I say that $\mathcal{M}_S$ and $\mathcal{M}_{S'}$ are *dynamically t-equivalent*, in symbols $\mathcal{M}_S \cong[t]^{\text{dyn}} \mathcal{M}_{S'}$, iff the following two conditions hold:

1. *Shared predecessors of t:* there is a set X such that $\{x \in S : x \preccurlyeq t\} = X = \{x \in S' : x \preccurlyeq t\}$.
2. *The successors of t are actualized by the passage of time in both S and S′ or in neither:* S is t-complete in $\mathcal{M}_S$ iff S' is t-complete in $\mathcal{M}_{S'}$. ∎

Note that if $\mathcal{M}_S$ and $\mathcal{M}_{S'}$ are t-equivalent *dynamically*, then directly by definition, the causal orderings of $\mathcal{M}_S$ and $\mathcal{M}_{S'}$ are linear, and by definition these SOAs are t-equivalent *historically*. Indeed, since $\mathfrak{S}_S$ and $\mathfrak{S}_{S'}$ are static substructures of the same event structure, the defining condition (1) of dynamic t-equivalence entails all the three defining conditions (1), (2), and (3) of historical t-equivalence. Actually, condition (1) could be replaced by the condition $\mathcal{M}_S \cong[t]^{\text{hist}} \mathcal{M}_{S'}$. Note also that if there are $s \in S$ and $s' \in S'$ that are incomparable in terms of the causal ordering of $\mathfrak{S}$, and one of the courses of events S and S' is t-complete, then both SOAs $\mathcal{M}_S$ and $\mathcal{M}_{S'}$ cannot be *dynamic* substructures of any one SOA based on the event structure $\mathfrak{S}$.

I take the notion of determinacy of a truth-maker to be definable in terms of the following notions of stability and fact-basedness, applicable to *factual truth-makers*. The definition is not intended as a nominal definition. Recall that the causal order of a factual truth-maker is always linear.

Definition 6.4 (Stable and instable truth-makers) Suppose ϕ is a formula and $(\mathcal{M}, t)$ is a factual truth-maker of ϕ in 'our indeterminist world' $\mathcal{M}_w$. Let $\mathcal{E}$ be an evaluation such that $\mathcal{M} = \mathcal{M}_w[t, \mathcal{E}]$. The truth-maker $(\mathcal{M}, t)$ of ϕ is *stable* iff the set of moments of $\mathcal{M}$ has a maximum, t^*, and for all linear SOAs $\mathcal{N}$, we have: if $\mathcal{M} \cong[t^*]^{\text{hist}} \mathcal{N}$, then the same evaluation $\mathcal{E}$ is a truth-evaluation of ϕ even in $(\mathcal{N}, t)$. Otherwise $(\mathcal{M}, t)$ is *instable*. ∎

In the above definition, it is *not* required that $\mathcal{N} \sqsubseteq \mathcal{M}_w$. There may well be in $\mathcal{N}$ 'hypothetical' moments that are later than t^* but *not* contained in $\mathcal{M}_w$. Stability requires that $\mathcal{E}$'s being a truth-evaluation of ϕ be totally insensitive to the way in which time evolves after t^*—the result of replacing the successors of t^* in $\mathcal{M}$ by *any* linearly ordered succession of objects, even by objects from outside $\mathcal{M}_w$, still leaves $\mathcal{E}$ being a truth-evaluation of ϕ in this modified SOA (as long as the obtained SOA $\mathcal{N}$ itself is linearly ordered). Note that the condition $\mathcal{M} \cong[t^*]^{\text{hist}} \mathcal{N}$ entails that the event structure $(T_{\mathcal{N}}, \prec_{\mathcal{N}}, V_{\mathcal{N}})$ of $\mathcal{N}$ is the result of taking the 'ordered sum' of event structures $(T_X, \prec_X, V_X)$ and $(T_Y, \prec_Y, V_Y)$, where T_X contains the moments of s of $\mathcal{M}$ with $s \preccurlyeq t^*$ and $(T_Y, \prec_Y, V_Y)$ is an arbitrary linear event structure whose domain T_Y is disjoint from T_X.[25] (By contrast, the time-resources of $\mathcal{M}$ and $\mathcal{N}$ need not be related in any way.) Some or all of the elements of T_Y may be objects not belonging to $\mathcal{M}_w$, and even if they indeed belonged all to $\mathcal{M}_w$ (without belonging to T_X), they might not appear in the same order as they do in $\mathcal{M}_w$, and the propositional atoms true at any one of these elements might not be the same as those that are true at that element according to the valuation of $\mathcal{M}_w$. Observe, finally, that for stability it is *not* enough simply that the *truth* of ϕ (as witnessed by this or that evaluation) is preserved when moving from ($\mathcal{M}$, t) to a centered SOA ($\mathcal{N}$, t) with $\mathcal{M} \cong[t^*]^{\text{hist}} \mathcal{N}$. Instead, the *same* evaluation that witnesses the truth of ϕ in ($\mathcal{M}$, t) must witness its truth in ($\mathcal{N}$, t).

Definition 6.5 (Fact-based truth-maker) Suppose ϕ is a formula and ($\mathcal{M}$, t) is a factual truth-maker of ϕ in $\mathcal{M}_w = (\mathfrak{S}_w, \text{TIME}_w)$. The truth-maker ($\mathcal{M}$, t) of ϕ is *fact-based* iff for all SOAs $\mathcal{N} = (\mathfrak{S}_{\mathcal{N}}, \text{TIME}_{\mathcal{N}})$, we have: if $\mathcal{N}$ is a maximal linear SOA such that $\mathfrak{S}_{\mathcal{N}} \Subset \mathfrak{S}_w$ and $\mathcal{M} \cong[t]^{\text{dyn}} \mathcal{N}$, then $\mathcal{N}, t \vDash \phi$. Here it is *not* required that one and the same evaluation witness the truth of ϕ in both ($\mathcal{M}$, t) and ($\mathcal{N}$, t). ▮

According to the above definition, in order for ($\mathcal{M}$, t) to be not merely a (factual linear) truth-maker of ϕ, but its fact-based truth-maker, the truth of ϕ must be preserved in the transition from ($\mathcal{M}$, t) to ($\mathcal{N}$, t), where $\mathcal{N} = (\mathfrak{S}_{\mathcal{N}}$, $\text{TIME}_{\mathcal{N}})$ is any linear SOA whose event structure $\mathfrak{S}_{\mathcal{N}}$ can be formed as an ordered sum of the event structures (T_X, $\prec_X$, V_X) and (T_Y, $\prec_Y$, V_Y) such that T_X and T_Y satisfy the following: T_X contains the moments s of $\mathcal{M}$ with $s \preccurlyeq t$, and T_Y is one of the maximal courses of events in 'our indeterminist world' $\mathcal{M}_w$ that consists of moments s with $t \prec s$ (that is, T_Y is the final segment of a history of $\mathcal{M}_w$ consisting of moments strictly later than t), and furthermore, T_Y is t-complete in $\mathcal{N}$ (whence generally $\mathcal{N}$ is not a dynamic substructure of $\mathcal{M}_w$). Generally, ($\mathcal{N}$, t) differs from ($\mathcal{M}$, t) in two ways: in $\mathcal{N}$, the passage of time actualizes after t moments that the passage of time does not actualize after t in $\mathcal{M}$ (unless $\mathcal{M}$ happens to be a dynamic substructure of $\mathcal{N}$), and $\mathcal{N}$ continues from t by a *maximal* causally possible course of events, while $\mathcal{M}$ may continue by a course of events that could be further extended within $\mathcal{M}_w$. In short, $\mathcal{N}$ represents in general a counterfactual maximal passage of

[25] If $\mathfrak{S}_1 = (T_1, \prec_1, V_1)$ and $\mathfrak{S}_2 = (T_2, \prec_2, V_2)$, with T_1 and T_2 being disjoint, the *ordered sum* of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ (in this order) is the event structure $(T, \prec, V)$, where: $T = T_1 \cup T_2$ and $\prec = (\prec_1 \cup \prec_2 \cup [T_1 \times T_2])$ and $V(p) = V_1(p) \cup V_2(p)$ for all $p \in \textbf{prop}$.

time from t, compatible with the event structure $\mathfrak{S}_w$. The restriction to *maximal* counterfactual passages of time is motivated, because my notion of fact-basedness is meant to capture the idea of 'settled truth' when the past is fixed but future can evolve in the sense of temporal becoming in any way allowed by the causal ordering. The notion of dynamic t-equivalence was defined for SOAs whose domains S and S' are linearly ordered subsets in one and the same event structure $\mathfrak{S}$. For the definition of a fact-based truth-maker, take S to be the domain of $\mathcal{M}$, take S' to be the domain of $\mathcal{N}$, and take $\mathfrak{S}$ to be the event structure $\mathfrak{S}_w$ of 'our indeterminist world' $\mathcal{M}_w$. Writing $\mathcal{M} = (\mathfrak{S}_{\mathcal{M}}, \text{TIME}_{\mathcal{M}})$ and $\mathcal{N} = (\mathfrak{S}_{\mathcal{N}}, \text{TIME}_{\mathcal{N}})$, the event structures $\mathfrak{S}_{\mathcal{M}}$ and $\mathfrak{S}_{\mathcal{N}}$ are static substructures of $\mathfrak{S}_w$. That $\mathfrak{S}_{\mathcal{M}}$ is a static substructure of $\mathfrak{S}_w$ follows from the fact that $(\mathcal{M}, t)$ is a truth-maker of ϕ in $\mathcal{M}_w$.

Observe that in order for $(\mathcal{M}, t)$ to be a *stable* truth-maker of ϕ (as opposed to its *fact-based* truth-maker), this formula ϕ must be true at t in *all* those dynamic linear extensions $\mathcal{N}$ of $\mathcal{M}$ that are not merely historically t-equivalent, but even historically t^*-equivalent with $\mathcal{M}$, where t^* is the maximum of the causal ordering of $\mathcal{M}$. Here t^* is later than t and $(\mathcal{M}, t)$ *will obtain*, unless t itself is the maximum of the causal ordering of $\mathcal{M}$—i.e., unless $(\mathcal{M}, t)$ *obtains*. Furthermore, it is not enough that ϕ is true at t in $\mathcal{N}$, but its truth therein must be witnessed by exactly the same truth-evaluation as in $(\mathcal{M}, t)$. Indeed, compare the requirement of stability with being fact-based. The latter condition concerns only certain courses of events within 'our indeterminist world' and the preservation of *truth* (as opposed to the preservation of a truth-evaluation) from a truth-maker to its suitable extension, whereas the former concerns arbitrary dynamic extensions of $\mathcal{M}$ that are t^*-equivalent with it and the transportability of a *truth-evaluation* from the truth-maker to its relevant dynamic extensions. In order for $(\mathcal{M}, t)$ to be a fact-based truth-maker of ϕ, this formula ϕ must be true at t in all those maximal linear SOAs $\mathcal{N}$ whose event structure is a static substructure of $\mathfrak{S}_w$ and that are not only historically but also dynamically t-equivalent with $\mathcal{M}$.

Generally, the properties of stability and being fact-based are independent of each other. Here are some examples. Suppose that $\mathcal{M}_w = (\mathfrak{S}_w, \text{TIME}_w)$ consists of five moments, with the causal ordering $\prec$ satisfying the following: $t_{-1} \prec t_0 \prec t_1 \prec t_2$ and $t_0 \prec t'_1$. The moment t_0 has, then, two immediate successors, t_1 and t'_1. Suppose $V(q) = \{t_{-1}, t_1, t_2\}$. And suppose $\text{TIME}_w([t_0], 1) = \{t_{-1}, t_0, t_1\}$ and $\text{TIME}_w([t_0], 2) = \{t_{-1}, t_0, t_1, t_2\}$, with $[t_0] = \{t_{-1}, t_0\}$. Thus, the course of events $\{t_{-1}, t_0, t_1, t_2\}$ is t_0-complete, but the course of events $\{t_{-1}, t_0, t'_1\}$ is not. Note that each of the following four formulas is true in $\mathcal{M}_w$ at t_0: $\phi_1 \coloneqq Pq$, $\phi_2 \coloneqq Fq$, $\phi_3 \coloneqq GPq$, and $\phi_4 \coloneqq Gq$. Now, let $\mathcal{M}_1$ be the dynamic substructure of $\mathcal{M}_w$ with the domain $\{t_{-1}, t_0\}$; let $\mathcal{M}_2$ be the substructure of $\mathcal{M}_w$ whose domain is $\{t_0, t_1\}$; let $\mathcal{M}_3$ be the substructure of $\mathcal{M}_w$ whose domain is $\{t_{-1}, t_0, t_1, t_2\}$; and let $\mathcal{M}_4$ be the substructure of $\mathcal{M}_w$ with the domain $\{t_0, t_1, t_2\}$. Then, evidently $(\mathcal{M}_i, t_0)$ is a truth-maker of ϕ_i in $\mathcal{M}_w$, for all $1 \leq i \leq 4$. In each case, let us take up the question of whether the truth-maker $(\mathcal{M}_i, t_0)$ is stable and/or fact-based.

First, we may note that $(\mathcal{M}_1, t_0)$ is a stable and fact-based truth-maker of ϕ_1. Actually, if $\mathcal{N}$ is *any static extension* of $\mathcal{M}_1$ (independently of whether $\mathcal{N}$ is a *dynamic* extension of $\mathcal{M}_1$ or not), the unique truth-evaluation of ϕ_1 in $(\mathcal{M}_1, t_0)$, which consists of selecting t_{-1} as a witness of P, is itself a truth-evaluation of ϕ_1 in $(\mathcal{N}, t_0)$. Thus, $\mathcal{N}, t_0 \vDash \phi_1$. It follows that the truth-maker $(\mathcal{M}_1, t_0)$ of ϕ_1 is fact-based. Since t_0 is the maximum of the causal order of $\mathcal{M}_1$, we may likewise conclude that $(\mathcal{M}_1, t_0)$ is a stable truth-maker $(\mathcal{M}_1, t_0)$ of ϕ_1.

Second, actually $(\mathcal{M}_2, t_0)$ is a stable truth-maker of ϕ_2, but not fact-based. If $\mathcal{N}$ is *any static extension* of $\mathcal{M}_2$ (whence $\mathcal{N}$ might not be historically t_0-equivalent to $\mathcal{M}_2$, as $\mathcal{N}$ might contain moments earlier than t_0), the truth-evaluation of ϕ_2 in $(\mathcal{M}_2, t_0)$—which consists of selecting t_1 as a witness of F—is itself a truth-evaluation of ϕ_2 in $(\mathcal{N}, t_0)$. Therefore, $(\mathcal{M}_2, t_0)$ is a stable truth-maker of ϕ_2. However, the centered SOA $(\mathcal{M}_2, t_0)$ is *not* a fact-based truth-maker of ϕ_2. Namely, let $\mathcal{N}_0$ be the static substructure of $\mathcal{M}_w$ whose domain consists of the moments t_0 and t'_1 and whose time-resource maps the pair $(\{t_0\}, 1)$ to the partial history $\{t_0, t'_1\}$. Then $\mathcal{N}_0$ is dynamically t_0-equivalent to $\mathcal{M}_2$, but $\mathcal{N}_0, t_0 \nvDash Fq$ (since q is false at t'_1). Thus, $(\mathcal{M}_2, t_0)$ is not a fact-based truth-maker of Fq.

Third, we may check that $(\mathcal{M}_3, t_0)$ is a fact-based truth-maker of ϕ_3, but not its stable truth-maker. To see that it is fact-based, note that there are only two static substructures of $\mathcal{M}_w$ that are (maximal and) dynamically t_0-equivalent to $\mathcal{M}_3$—namely, $\mathcal{M}_3$ itself and the static substructure $\mathcal{M}'_3$ of $\mathcal{M}_w$ whose domain is $\{t_{-1}, t_0, t'_1\}$ and whose time-resource maps the pair $(\{t_0\}, 1)$ to the partial history $\{t_0, t'_1\}$. Now, since both in $\mathcal{M}_3$ and in $\mathcal{M}'_3$ the formula Pq is true at t_0, it follows that even the formula GPq is true in them both at t_0. We may conclude that $(\mathcal{M}_3, t_0)$ is a fact-based truth-maker of ϕ_3. By contrast, let $\mathcal{N}^*$ be a dynamic extension of $\mathcal{M}_3$ such that first, $\mathcal{N}^*$ is historically t_2-equivalent to $\mathcal{M}_3$; second, $\mathcal{N}^*$ contains exactly one successor, s^*, of t_2; and third, $\mathcal{N}^*$ is t_0-complete (whence at t_0 the operator G ranges over t_1, t_2, and s^*). Although the truth of GPq at t_0 is indeed preserved when moving $\mathcal{M}_3$ to $\mathcal{N}^*$, still no truth-evaluation of GPq in $(\mathcal{M}_3, t_0)$ is itself a truth-evaluation of this same formula in $(\mathcal{N}^*, t_0)$. This is because a truth-evaluation of ϕ_3 in $(\mathcal{N}^*, t_0)$ contains inevitably an evaluation sequence with the position $(Pq, s^*, [s^*])$—so that s^* appears in the moment set of that truth-evaluation—whereas no truth-evaluation of ϕ_3 in $(\mathcal{M}_3, t_0)$ may contain an evaluation sequence with such a position, s^* not being a moment in $\mathcal{M}_3$. It follows that $(\mathcal{M}_3, t_0)$ is not a stable truth-maker of ϕ_3.

Fourth, actually the truth-maker $(\mathcal{M}_4, t_0)$ of ϕ_4 is neither fact-based nor stable. To see that it is not fact-based, it suffices to consider the SOA $\mathcal{N}_0$ that was used to show that $(\mathcal{M}_2, t_0)$ is not a fact-based truth-maker of ϕ_2. Here, $\mathcal{N}_0, t_0 \vDash F\neg q$, whence $\mathcal{N}_0, t_0 \nvDash Gq$. Since $\mathcal{N}_0$ is dynamically t_0-equivalent to $\mathcal{M}_4$ and a static substructure of $\mathcal{M}_w$, it follows that $(\mathcal{M}_4, t_0)$ is not a fact-based truth-maker of Gq. (A counterfactual passage of time renders Gq false at t_0.) Further, $(\mathcal{M}_4, t_0)$ is not even stable: if $\mathcal{N}^*$ is the SOA that was employed to prove that $(\mathcal{M}_3, t_0)$ is not a stable truth-maker of ϕ_3, we note that $\mathcal{N}^*$ is a dynamic extension of $\mathcal{M}_4$ historically t_2-equivalent to $\mathcal{M}_4$, but the moment set of any evaluation of Gq in $(\mathcal{N}^*, t_0)$ comprises the moment s^* that cannot appear in the moment set of a truth-evaluation of Gq in $(\mathcal{M}_4, t_0)$.

Definition 6.6 (Determinacy, indeterminacy) Suppose ϕ is a formula and $(\mathcal{M}, t)$ is a truth-maker of ϕ in $\mathcal{M}_w$. The truth-maker of ϕ is *determinate* iff it is factual, stable, and fact-based. It is *indeterminate* iff it is not determinate—that is, iff it is either modal, or else factual but either not stable or not fact-based. ∎

If ϕ is a formula containing no occurrences of modal operators or future-tense operators, then any truth-maker of ϕ obtains. Moreover, such a truth-maker is both stable and fact-based, and consequently determinate. If a formula such as Fq is true, it has a stable truth-maker that will obtain, but generally it fails to have a fact-based truth-maker—the formula might be false if the passage of time was different from what it in fact will be. Such a truth-maker is indeterminate. Any truth-maker $(\mathcal{M}, t)$ of the formula FPq will obtain and is stable. Such a truth-maker may but need not be fact-based, as well. It is guaranteed to be fact-based and therefore determinate, if for example q is true at t, or if $\mathcal{M}$ contains at least two successors of t and q is true at the immediate successor of t.

If the passage of time actualizes at least one moment after t_0, then the formula $F(q \vee \neg q)$ has a stable and fact-based truth-maker $(\mathcal{M}, t_0)$ that will obtain. Consequently, the truth-maker is determinate. The formula $q \vee \neg q$ is true at some moment—indeed, all moments—of *any* causally possible course of events following t_0, so whichever of these courses of events is by hypothesis actualized by the passage of time, the course of events in question comprises a moment making this formula true. Similarly, if p is a formula that happens to be true in 'our indeterminist world' somewhere on every causally possible course of events following t_0, then the formula Fp has a determinate truth-maker $(\mathcal{M}, t_0)$. The formula Gq has no stable truth-maker, and therefore no determinate truth-maker. For, any of its truth-makers is either infinite (and so lacks a maximum, whence the truth-maker cannot be stable) or else has a maximum s^* and admits of a suitable extension in which a successor of s^* renders q false. Further, even $G(q \vee \neg q)$ has no stable truth-maker. Namely, if $(\mathcal{M}, t_0)$ is its truth-maker that has a maximum, the truth-evaluation of $G(q \vee \neg q)$ in $(\mathcal{M}, t_0)$ is not itself a truth-evaluation of $G(q \vee \neg q)$ in any extension of $(\mathcal{M}, t_0)$.

7. Analyzing the conditions of determinacy

Let us study how the logical form of a formula and the status of its truth-maker with respect to obtaining affect the truth-maker's being or not being determinate. In order to be conveniently in a position to comment more systematically on this issue, I propose to analyze the types of conditions that can be expressed by a *prefix formula* of the form $O_1 \ldots O_n\, \beta$, where $n \geq 0$ and $O_i \in \{P, H, F, G\}$ for all $1 \leq i \leq n$, and β is a Boolean combination of literals in terms of conjunction and/or disjunction. (Prefix formulas are by definition non-modal.) The generalization of the analysis to arbitrary L-formulas is left to another occasion. A prefix formula $O_1 \ldots O_n\, \beta$ is *pure past*, if $O_i \in \{P, H\}$ for all $1 \leq i \leq n$ with $n \geq 1$. It is *non-future*, if it is either pure past or else equals β (the case $n \coloneqq 0$).

All truth-makers of all true (non-modal) non-future formulas trivially *obtain*. However, a prefix formula may have an obtaining truth-maker even if its prefix contains an occurrence of F. For example, if t_0 has at least two predecessors and q is true at the immediate predecessor of t_0, then PFq is true at t_0 and this formula has a truth-maker that obtains. The domain of this truth-maker consists of t_0, its immediate predecessor, and the immediate predecessor of this latter moment. Of course not all truth-makers of PFq obtain. For instance, consider a SOA consisting of t_0, the immediate successor of t_0, and the immediate predecessor of t_0, where q is true at the immediate successor of t_0. Then centering this SOA on t_0 results in a truth-maker of *PFq* that does not obtain but *will obtain*. Even a formula whose prefix contains an occurrence of *G* may have a truth-maker ($\mathcal{M}$, t_0) that *obtains*, but only if the passage of time fails to lead from t_0 to a causally possible moment later than t_0. Some formulas admit of truth-makers that are *processes of coming to obtain*. E.g., a certain truth-maker of the formula *HFp* is a process of coming to obtain, although the formula also has truth-makers that *will obtain* and even ones that *obtain*. An example of the first kind is a truth-maker that consists of a countable infinity of successors of t_0 and a countable infinity of its predecessors, with *p* true at all successors of t_0, this truth-maker being induced by the truth-evaluation that assigns to the *n*-th predecessor of t_0 (corresponding to *H*) the *n*-th successor of t_0 (corresponding to *F*). An example of the second kind, again, is a truth-maker that consists of a single successor of t_0 and a countable infinity of predecessors of t_0, with *p* true at the unique successor of t_0, the truth-maker being triggered by the truth-evaluation that assigns uniformly to all predecessors of t_0 the unique successor of t_0. Finally, an example of the third kind is a truth-maker in which t_0 has no successors at all but has a countable infinity of predecessors, with *p* true at t_0, the truth-maker being triggered by the truth-evaluation that assigns uniformly to all predecessors of t_0 the moment of evaluation t_0 itself. The formula *GFp* has no truth-maker that will obtain. It has only truth-makers that are processes of coming to obtain, except if the passage of time through the moment of evaluation leads to no further moments, in which case this formula has an obtaining truth-maker. Trivially, a truth-maker that is a process of coming to obtain cannot be determinate—as it lacks a maximum and therefore cannot even be stable.

Fact 7.1 Let ϕ be a formula. If a factual truth-maker of ϕ in $\mathcal{M}_{\mathrm{w}}$ is a process of coming to obtain, it is not determinate.

Proof. A truth-maker that is a process of coming to obtain is not stable (as it has no maximum), and therefore it is *a fortiori* not determinate. ∎

The fact that a truth-maker of a non-modal formula ϕ obtains or will obtain does not by itself guarantee that the truth-maker is determinate. Whether it is or not, that depends on the *form* of the formula. Let us, now, look more systematically into the form of non-modal prefix formulas and see how their form affects their possibility of having determinate truth-makers. The considerations are relativized to ‘our indeterminist world’ $\mathcal{M}_w$. Recall that by definition, a 'real truth-maker' of a formula is a truth-maker of this formula *in* the specific state of affairs $\mathcal{M}_w$ (see Definition 4.8).

7.1 Formulas with an occurrence of *G*

The presence of G in a prefix formula renders it impossible for the formula to have a determinate truth-maker.

Lemma 7.2 Suppose $\phi \coloneqq \mathrm{O}_1 \ldots \mathrm{O}_n\, \beta$ is a prefix formula with $n \geq 1$ such that G occurs at least once in the prefix. Then all truth-makers of ϕ are instable. Consequently, no truth-maker of ϕ is determinate.

Proof. Let ϕ be as in the statement of the lemma. Suppose $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{M}_w$, with $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$. Since ϕ contains G, either (1) the moment set $\mathrm{M}(\mathcal{E})$ has no maximum, or else (2) the passage of time comes to an end in a finite duration—i.e., there is a minimal duration, n^*, such that $\mathrm{TIME}_w([t_0], n^*) = \mathrm{TIME}_w([t_0], n^* + 1)$, whence the passage of time does not lead to later moments from the endpoint t^* of the partial history $\mathrm{TIME}_w([t_0], n^*)$. In case (1), the causal ordering of the truth-maker $(\mathcal{M}, t_0)$ has no maximum, whence $(\mathcal{M}, t_0)$ is instable. In case (2), the passage of time through t_0 has a maximum, t^*. Let s^* be any object from outside the domain of $\mathcal{M}_w$ and let $\mathcal{N}$ be the linear SOA whose domain consists of the domain of $\mathcal{M}$ together with the object s^*; the causal order of $\mathcal{N}$ is the transitive closure of the result of adding the pair (t^*, s^*) to the causal order of $\mathcal{M}$; the valuation of $\mathcal{N}$ is simply that of $\mathcal{M}$ (whence all propositional atoms are false at s^* in $\mathcal{N}$); and the time-resource $\mathrm{TIME}_{\mathcal{N}}$ of $\mathcal{N}$ is otherwise like the time-resource $\mathrm{TIME}_{\mathcal{M}}$ of $\mathcal{M}$ except that $\mathrm{TIME}_{\mathcal{N}}([t^*], 1) = [s^*]$, whereas—we note—$\mathrm{TIME}_{\mathcal{M}}([t^*], 1) = \mathrm{TIME}_w([t^*], 1) = \mathrm{TIME}_w([t^*], 0) = [t^*]$. Thus, the maximal t_0-complete course of events beginning at t_0 in $\mathcal{N}$ is $[t^*] \cup \{s^*\}$, whereas the maximal t_0-complete course of events beginning at t_0 in $\mathcal{M}$ is $[t^*] \neq [t^*] \cup \{s^*\}$. By construction, $\mathcal{N}$ is a linear SOA historically t^*-equivalent to $\mathcal{M}$, where t^* is the maximum of the causal ordering of $\mathcal{M}$. Now, because the prefix of ϕ contains G, if $\mathcal{E}'$ is any evaluation of ϕ in $(\mathcal{N}, t_0)$, its moment set contains s^*. Yet the moment set of $\mathcal{E}$ is included in the domain of $\mathcal{M}_w$ (indeed, included in the domain of $\mathcal{M}$) and therefore cannot contain s^*. It follows that $\mathcal{E}$ is not even an evaluation in $(\mathcal{N}, t_0)$ and therefore *a fortiori* not a truth-evaluation. Thus, we may conclude that $(\mathcal{M}, t_0)$ is instable. We have seen, then, that the stability of $(\mathcal{M}, t_0)$ fails in both cases (1) and (2). It follows that $(\mathcal{M}, t_0)$ is not determinate. ∎

7.2 Generally on *G*-free formulas

From now on, I restrict my attention to *G-free* prefix formulas—i.e., formulas of the form $O_1 \dots O_n\ \beta$, where β is a Boolean combination of literals and $O_i \in \{F, P, H\}$ for all $1 \leq i \leq n$.

Lemma 7.3 (Transportability of truth-evaluations) Let ϕ be a *G*-free prefix formula. Suppose $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{M}_w, t_0)$. Let $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$. If the passage of time from t_0 has a maximum, t^*, and $\mathcal{N}$ is any linear SOA that is historically t^*-equivalent to $\mathcal{M}$, then $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{N}, t_0)$ and $\mathcal{M} = \mathcal{N}[t_0, \mathcal{E}]$.

Proof. Suppose that $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{M}_w, t_0)$ and let $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$, ϕ being *G*-free. Suppose the passage of time from t_0 has a maximum t^* (as the passage of time is determined by the time-resource of $\mathcal{M}_w$), and let $\mathcal{N}$ be any linear SOA historically t^*-equivalent to $\mathcal{M}$. Thus, $\mathcal{N}$ contains all the moments of $\mathcal{M}$, but—except in the special case $\mathcal{N} = \mathcal{M}$—it contains even further moments (which need not even be moments of $\mathcal{M}_w$). Any such moments in $\mathcal{N}$ are later than t^* in the causal order of $\mathcal{N}$. We claim that since ϕ contains no occurrence of *G*, the set $\mathcal{E}$ itself is an evaluation of ϕ in $(\mathcal{N}, t_0)$, and not merely in $(\mathcal{M}, t_0)$. If ϕ is non-future, this is trivial, since in that case the moments available for constructing an evaluation in $(\mathcal{N}, t_0)$ are the same in the two cases (namely, the moment t_0 and the moments that are earlier than t_0 according to the causal ordering of $\mathcal{M}$). If, again, ϕ contains occurrences of *F*, there may be in $(\mathcal{N}, t_0)$ more moments to be used in an evaluation than in $(\mathcal{M}, t_0)$. (This is so if not only $\mathcal{N} \neq \mathcal{M}$, but also at least the immediate causally possible successor of t^* in $\mathcal{N}$ is attainable from t^* via the time-resource of $\mathcal{N}$.) However, the construction of an evaluation just requires choosing for each occurrence of *F* in every evaluation sequence one witness, and the possible availability in $\mathcal{N}$ of extra moments to be used as witnesses of *F*—namely, moments later than t^* and furthermore attained via the time-resource of $\mathcal{N}$—still leaves us the possibility of constructing an evaluation by effectively using as witnesses only the moments used when evaluating ϕ in $(\mathcal{M}, t_0)$. We may conclude that whether *F* occurs in ϕ or not, $\mathcal{E}$ is an evaluation of ϕ in $(\mathcal{N}, t_0)$. Since $\mathcal{E}$ is a truth-evaluation of ϕ in $(\mathcal{M}, t_0)$, it is likewise a truth-evaluation of ϕ in $(\mathcal{N}, t_0)$. Further, the dynamic substructure of $\mathcal{M}$ generated by the moment set $\mathrm{M}(\mathcal{E})$ is the same as the dynamic substructure of $\mathcal{N}$ generated by the very same moment set $\mathrm{M}(\mathcal{E})$. That is, $\mathcal{N}[t_0, \mathcal{E}] = \mathcal{M}[t_0, \mathcal{E}] = \mathcal{M}$. ∎

We move on to study determinacy in connection with G-free prefix formulas, and relate the question of determinacy to the question of whether they have a truth-maker that obtains or will obtain. (It was already noted at the beginning of the present section that such a formula *may* have a truth-maker that is a process of coming to obtain, cf. the case of *HFp*.) We begin by observing that if a truth-maker of a G-free formula *obtains*, it is determinate. By contrast, as I noted above, a formula in which G occurs might have an obtaining truth-condition which nevertheless would not be determinate. For example, any truth-maker whose evaluation moment has no successors would be an obtaining but indeterminate truth-maker of any formula of the form $G\phi$—actually, such a truth-maker would be neither stable nor fact-based.

Lemma 7.4 Let ϕ be a G-free prefix formula.

(a) If ϕ has a real truth-maker that *obtains*, the truth-maker is *determinate*.
(b) If ϕ has a real truth-maker that *will obtain*, the truth-maker is *stable*.

Proof. Suppose ϕ has a truth-maker $(\mathcal{M}, t)$ in $\mathcal{M}_w$, with $\mathcal{E}$ being an evaluation such that $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$. Since ϕ is non-modal, the truth-maker is automatically factual. I show first that whether $(\mathcal{M}, t)$ obtains or will obtain, in both cases it is stable. So, suppose that $(\mathcal{M}, t)$ obtains or will obtain. In any event, then, the causal ordering $\prec$ of $\mathcal{M}$ has a maximum, t^*, satisfying $t \preccurlyeq t^*$. Now, by Lemma 7.3, $\mathcal{E}$ is a truth-evaluation of ϕ at t in any linear SOA historically t^*-equivalent to $\mathcal{M}$. Consequently, $(\mathcal{M}, t)$ is stable. This constitutes the proof of (b) and half of the proof of (a). For (a), it remains to show that if in particular $(\mathcal{M}, t)$ obtains, it is fact-based, as well.

Now, suppose that $(\mathcal{M}, t)$ indeed obtains, whence $t^* = t$. Again by Lemma 7.3, $\mathcal{E}$ is a truth-evaluation of ϕ at t in absolutely any linear SOA historically t-equivalent to $\mathcal{M}$. Consequently, it is a truth-evaluation of ϕ in any *such* linear SOA $\mathcal{N}$ that not only is historically t-equivalent to $\mathcal{M}$, but is in particular a static substructure of $\mathcal{M}_w$ and *dynamically* t-equivalent to $\mathcal{M}$. It follows that $\mathcal{N}, t \vDash \phi$. We may conclude that $(\mathcal{M}, t)$ is fact-based. Since, then, $(\mathcal{M}, t)$ is factual, stable, and fact-based, we may conclude that it is determinate. This concludes the proof of (a). ∎

When a truth-maker of a non-modal formula will obtain but fails to be determinate, this failure is, then, due to the truth-maker not being fact-based.

Indeed, a truth-maker that will obtain *need not* be fact-based. This is precisely the case of truth-conditions of future contingent propositions. Just suppose Fp and $\Diamond{\sim}Fp$ (i.e., $\Diamond G\neg p$) are both true at t_0 in $\mathcal{M}_w = (\mathfrak{S}_w, \mathrm{TIME}_w)$. Then Fp has a truth-maker $(\mathcal{M}, t_0)$ with a finite t_0-complete domain $\{t_0,t_1,\ldots,t_n\}$ for some $n \geq 1$, a linear causal ordering, and a valuation according to which p is true at t_n. Further, the fact that $\Diamond G\neg p$ is true at t_0 means that there is a finite or infinite course of events S in $\mathcal{M}_w$ such that: (*i*) all moments in S are later than t_0, (*ii*) all moments in S render q false, and *(iii)* $[t_0] \cup S$ is a history of $\mathcal{M}_w$ (not merely a partial history). Now, let $\mathfrak{S}_{\mathcal{N}}$ be the static substructure of $\mathfrak{S}_w$ determined by the set $[t_0] \cup S$, and let $\mathrm{TIME}_{\mathcal{N}}$ be a (counterfactual) time-resource such that $S = \cup_{n>0} \mathrm{TIME}_{\mathcal{N}}([t_0], n)$. Setting $\mathcal{N} \coloneqq (\mathfrak{S}_{\mathcal{N}}, \mathrm{TIME}_{\mathcal{N}})$, we have that S is t_0-complete in $\mathcal{N}$, and that $\mathcal{N}$ is a *maximal* linear static substructure of $\mathcal{M}_w$. Further, we have $\mathcal{N}, t_0 \nvDash Fp$. Here, $\mathcal{M}$ and $\mathcal{N}$ are both static substructures of $\mathcal{M}$, and $\mathcal{N}$ is dynamically t_0-equivalent to $\mathcal{M}$. So, if $(\mathcal{M}, t_0)$ was a fact-based truth-maker of Fp, then Fp would be true at t_0 in $\mathcal{N}$. However, this is not the case. It follows that $(\mathcal{M}, t_0)$ is not fact-based.

7.3 *G*-free formulas and truth-makers that obtain or will obtain

To complete my study of determinacy in connection with prefix formulas, it remains to investigate the conditions under which a *G*-free formula can have a truth-maker that will obtain and is furthermore fact-based. To this end, we may begin by studying how the prefix of a *G*-free formula affects its having a truth-condition that *will obtain*. Thereafter we may pose the further question of what more it takes for such a truth-maker to be fact-based. Now, no non-future formula has a truth-maker that will obtain; so let us turn attention to prefixes with at least one occurrence of *F*. Obviously all truth-makers of non-modal formulas of the form $F\chi$ will obtain. Some, but not all, truth-makers of PFq will obtain, and the same holds true of $HHFq$. But what can be said generally of *G*-free formulas in which *F* appears? In order to study this question, I will take a look at a specific class of SOAs—namely, the class $\mathcal{K}$ of all SOAs whose frame has no minimum, and whose time-resource yields from every moment an infinite passage of time forward. In this paper, I content myself in this connection with the class $\mathcal{K}$ and ignore SOAs whose frames have a minimum (a root) and SOAs whose frames have a finitely terminating passage of time through at least one moment.

In passing, we note that a pure-past prefix formula is $\mathcal{K}$-equivalent to a formula of the following four forms: $H^n\beta$, $P^n\beta$, $P^mH^n\beta$, and $H^mP^n\beta$, where β is a Boolean combination of literals and n, $m \geq 1$.[26] This ensues from the fact that we have the following $\mathcal{K}$-equivalences: $H^kP^mH^n\beta \equiv_{\mathcal{K}} P^{k+m}H^n\beta$ and $P^kH^mP^n\beta \equiv_{\mathcal{K}} H^{k+m}P^n\beta$.

I formulate an analysis of truth-conditions of G-free formulas by utilizing an augmented language in which an additional tense operator $\ominus$ is available that shifts the value of the moment parameter to the immediate predecessor of the moment of evaluation. (Whenever $\mathcal{M} \in \mathcal{K}$ and t is a moment of $\mathcal{M}$, a unique immediate predecessor of t exists.) That is, $\ominus\chi$ is true at t iff χ is true at the immediate predecessor of t. Once the analysis is effected, we can return to the question of the status of obtaining of truth-conditions of G-free formulas (in which, by definition, $\ominus$ does not occur).

It turns out that over $\mathcal{K}$, any G-free prefix formula $O_1 \ldots O_n\ \beta$ falls into one of the following four categories: either such a formula is non-future, or it is of the form $F^m\chi$ with χ being non-future, or else it is $\mathcal{K}$-equivalent to a formula of one of the following two forms: $(P\chi \vee \chi \vee F\chi)$ or $\ominus^nF^m\chi$, where χ is non-future. In order to prove this, I first establish the following lemma.

Lemma 7.5 Consider G-free prefix formulas in which F appears at least once preceded either by P or by H, and θ is an arbitrary prefix formula.

(a) Let $\phi \coloneqq O_1 \ldots O_r\ PF^m\theta$, where $m \geq 1$ and $O_i \in \{P, H\}$ for all $0 \leq i \leq r$. Then ϕ is $\mathcal{K}$-equivalent to $(P\theta \vee \theta \vee F\theta)$.
(b) Let $\phi \coloneqq H^nF^m\theta$, where $n, m \geq 1$. Then ϕ is $\mathcal{K}$-equivalent to $\ominus^nF^m\theta$.
(c) Let $\phi \coloneqq O_1 \ldots O_r\ PH^nF^m\theta$, where n, $m \geq 1$ and $O_i \in \{P, H\}$ for all $0 \leq i \leq r$. Then ϕ is $\mathcal{K}$-equivalent to $(P\theta \vee \theta \vee F\theta)$.

Proof. For (a), we note first that

(1) $P^nF^m\theta \equiv_{\mathcal{K}} (P\theta \vee \theta \vee F\theta)$.

Here, $(P\theta \vee \theta \vee F\theta)$ says that θ was true, is true or will be true. Hence, if $(P\theta \vee \theta \vee F\theta)$ is true at t, the formula $(P\theta \vee \theta \vee F\theta)$ is likewise true at all moments in the past of t and at all moments yielded by the passage of time from t. Consequently,

[26] Formulas ϕ and ψ are by definition *$\mathcal{K}$-equivalent* (in symbols $\phi \equiv_{\mathcal{K}} \psi$) iff for all $\mathcal{M} \in \mathcal{K}$ and all t in $\mathcal{M}$, we have: $\mathcal{M}, t \vDash \phi$ iff $\mathcal{M}, t \vDash \psi$. If O is an operator and χ is a formula, then $O^n\chi$ stands for a string of n consecutive occurrences of O followed by χ. In the special case $n \coloneqq 0$, $O^n\chi$ stands for χ (i.e., O^n stands for the empty string).

(2) $H^kP^nF^m\theta \equiv_{\mathcal{K}} H^k(P\theta \vee \theta \vee F\theta) \equiv_{\mathcal{K}} P^nF^m\theta \equiv_{\mathcal{K}} (P\theta \vee \theta \vee F\theta)$.

Together (1) and (2) entail that $O_1\ldots O_r\, PF^m\chi \equiv_{\mathcal{K}} (P\theta \vee \theta \vee F\theta)$ for any string $O_1\ldots O_r$ of occurrences of P and H. This completes the proof of (a). Regarding (b), we note that generally, we have:

(3) $H^nF^m\theta \equiv_{\mathcal{K}} \ominus^nF^m\theta$,

since n occurrences of the universal past-tense operator shift the value of the moment parameter to the n-th predecessor of the moment of evaluation—or further in the past. If $n = m$, a $\mathcal{K}$-equivalent form of $H^nF^m\theta$ can be expressed in which H does not appear (if not in θ): $\theta \vee F\theta$. And if $m > n$, the formula has the $\mathcal{K}$-equivalent $F^{m-1}\theta$. However, generally, in order to eliminate H in the case that $n < m$, the operator $\ominus$ is needed. Hereby the proof of (b) is completed.

It remains to prove (c). We note that by (1) and (3), we have for all $k \geq 1$:

(4) $P^kH^nF^m\theta \equiv_{\mathcal{K}} P^k\ominus^nF^m\theta \equiv_{\mathcal{K}} P^{k+n}F^m\theta \equiv_{\mathcal{K}} (P\theta \vee \theta \vee F\theta)$.

Further, by (4), we have for all $x \geq 1$:

(5) $H^xP^kH^nF^m\theta \equiv_{\mathcal{K}} H^x(P\theta \vee \theta \vee F\theta) \equiv_{\mathcal{K}} (P\theta \vee \theta \vee F\theta)$.

Together (4) and (5) entail that $O_1\ldots O_r\, PH^nF^m\chi \equiv_{\mathcal{K}} (P\theta \vee \theta \vee F\theta)$ for any string $O_1\ldots O_r$ of occurrences of P and H. This brings the proof of (c) to an end. ∎

Directly by Lemma 7.5, we have:

Corollary 7.6 Let $\phi \coloneqq O_1\ldots O_n\, F^m\chi$ be a prefix formula, where $n, m \geq 1$ and $O_i \in \{P, H\}$ for all $1 \leq i \leq n$, and χ is an arbitrary prefix formula. Then, ϕ is $\mathcal{K}$-equivalent to one of the following two formulas: $(P\chi \vee \chi \vee F\chi)$ or $\ominus^nF^m\chi$. ∎

Let us, then, say that a string of operators $O_1\ldots O_n\, F^m$ is a *Past-Future prefix*, if $n, m \geq 1$ and $O_i \in \{P, H\}$ for all $1 \leq i \leq n$. Note that if a formula has a Past-Future prefix and is therefore of the form $O_1\ldots O_n\, F^m\chi$, it may well happen that the subformula χ, too, has a Past-Future prefix. Now, if a prefix formula

ϕ has a Past-Future prefix, it has automatically a (proper or improper) subformula which consists of a Past-Future prefix applied to a non-future formula. Of course, it cannot happen that its all subformulas having a Past-Future prefix are applied to a formula that itself has a Past-Future prefix, already because each formula consists of a finite number of symbols. I move on to prove the theorem announced above.

Theorem 7.7 Let $\phi \coloneqq O_1 \ldots O_N\ \beta$ be a prefix formula, where $N \geq 0$ and $O_i \in \{P, H, F\}$ for all $1 \leq i \leq N$ and β is a Boolean combination of literals. Then, there is a non-future formula χ such that either ϕ equals χ, or ϕ is of the form $F^m\chi$ for some $m \geq 1$, or else ϕ is $\mathcal{K}$-equivalent to a formula of the form $(P\chi \vee \chi \vee F\chi)$ or the form $\ominus^x F^y \chi$, where $x, y \geq 1$. More specifically, either χ equals β or is of the form $H^k\beta$, $P^k\beta$, $P^rH^k\beta$, and $H^rP^k\beta$ for some $k, r \geq 1$.

Proof. Let ϕ be an arbitrary formula of the relevant form. If its prefix is empty, or contains no occurrence of F, or contains exclusively occurrences of F, there is nothing to prove. Similarly, if ϕ is of the form $F^m\chi$ with χ being non-future, there is nothing to prove. So suppose ϕ is of the form

$$O_{1.1} \ldots O_{1.k(1)}\ F^{m(1)}\ O_{2.1} \ldots O_{2.k(2)}\ F^{m(2)}\ \ldots\ O_{n.1} \ldots O_{n.k(n)}\ F^{m(n)}\ \chi$$

or of the form

$$F^{m(0)}\ O_{1.1} \ldots O_{1.k(1)}\ F^{m(1)}\ O_{2.1} \ldots O_{2.k(2)}\ F^{m(2)}\ \ldots\ O_{n.1} \ldots O_{n.k(n)}\ F^{m(n)}\ \chi,$$

where χ is non-future and $n \geq 1$ and $k(i) \geq 1$ and $m(i) \geq 1$ and $O_i \in \{P, H\}$ for all $1 \leq i \leq n$. In the latter case, it is further supposed that $m(0) \geq 1$.

Let us think of the first alternative first. The formula comprises, then, n Past-Future prefixes followed by a non-future formula. Let us consider these Past-Future prefixes one by one, starting with the outermost and moving toward the innermost. We already know by Corollary 7.6 that a formula with a Past-Future prefix followed by an arbitrary formula ψ is $\mathcal{K}$-equivalent to one of the two formulas $(P\psi \vee \psi \vee F\psi)$ or $\ominus^x F^y \psi$ for some $x, y \geq 1$. In order to find out whether nested Past-Future prefixes likewise give rise to $\mathcal{K}$-equivalents having these forms, we must study what happens when the formula ψ in $(P\psi \vee \psi \vee F\psi)$ or $\ominus^x F^y \psi$ is itself of one these two forms. So, suppose $\psi = O_{2.1} \ldots O_{2.k(2)}\ F^{m(2)}\theta$ with $\theta = O_{3.1} \ldots O_{3.k(3)}\ F^{m(3)} \ldots O_{n.1} \ldots O_{n.k(n)}\ F^{m(n)}\chi$. Recall that $\phi = O_{1.1} \ldots O_{1.k(1)}\ F^{m(1)}\psi$ is $\mathcal{K}$-equivalent to $(P\psi \vee \psi \vee F\psi)$

or to $\ominus^{k(1)}F^{m(1)}\psi$, and $\psi = \mathrm{O}_{2.1}\ldots\mathrm{O}_{2.k(2)}\ F^{m(2)}\theta$ is $\mathcal{K}$-equivalent to $(P\theta \vee \theta \vee F\theta)$ or to $\ominus^{k(2)}F^{m(2)}\theta$. It follows that ϕ is $\mathcal{K}$-equivalent to one of the four formulas:

1. $(P[(P\theta \vee \theta \vee F\theta)] \vee [(P\theta \vee \theta \vee F\theta)] \vee F[(P\theta \vee \theta \vee F\theta)])$. This formula is (logically) equivalent to $(PP\theta \vee P\theta \vee PF\theta \vee P\theta \vee \theta \vee F\theta \vee FP\theta \vee F\theta \vee FF\theta)$, which is in fact $\mathcal{K}$-equivalent to $(P\theta \vee \theta \vee F\theta)$.
2. $(P[\ominus^{k(2)}F^{m(2)}\theta] \vee [\ominus^{k(2)}F^{m(2)}\theta] \vee F[\ominus^{k(2)}F^{m(2)}\theta])$. This formula is $\mathcal{K}$-equivalent to $(P^{k(2)+1}F^{m(2)}\theta \vee \ominus^{k(2)}F^{m(2)}\theta \vee \ominus^{k(2)}F^{m(2)+1}\theta)$, which is $\mathcal{K}$-equivalent to $(P\theta \vee \theta \vee F\theta \vee \ominus^{k(2)}F^{m(2)}\theta \vee \ominus^{k(2)}F^{m(2)+1}\theta)$ and therefore $\mathcal{K}$-equivalent simply to $(P\theta \vee \theta \vee F\theta)$.
3. $\ominus^{k(1)}F^{m(1)}(P\theta \vee \theta \vee F\theta)$. This formula is (logically) equivalent to $\ominus^{k(1)}(F^{m(1)}P\theta \vee F^{m(1)}\theta \vee F^{m(1)}F\theta)$, which is $\mathcal{K}$-equivalent to $\ominus^{k(1)}(P\theta \vee \theta \vee F\theta \vee F^{m(1)}\theta \vee F^{m(1)+1}\theta)$, this latter formula being $\mathcal{K}$-equivalent to $(P\theta \vee \theta \vee F\theta)$.
4. $\ominus^{k(1)}F^{m(1)}[\ominus^{k(2)}F^{m(2)}\theta]$. Now, this formula is $\mathcal{K}$-equivalent to $\ominus^{k(1)+k(2)}F^{m(1)+m(2)}\theta$.

All these formulas have one of the two requisite forms. By repeated applications of the above $\mathcal{K}$-equivalences, any formula of the form $\mathrm{O}_{1.1}\ldots\mathrm{O}_{1.k(1)}\ F^{m(1)}\ \mathrm{O}_{2.1}\ldots\mathrm{O}_{2.k(2)}\ F^{m(2)}\ \ldots\ \mathrm{O}_{n.1}\ldots\mathrm{O}_{n.k(n)}\ F^{m(n)}\ \chi$ (with χ being non-future) can be seen to have a $\mathcal{K}$-equivalent that is of the form $(P\chi \vee \chi \vee F\chi)$ or $\ominus^{x}F^{y}\chi$ for some positive integers x, y.

Coming back to the second alternative discerned above, according to which ϕ is of the form $F^{m(0)}\ \mathrm{O}_{1.1}\ldots\mathrm{O}_{1.k(1)}\ F^{m(1)}\ \mathrm{O}_{2.1}\ldots\mathrm{O}_{2.k(2)}\ F^{m(2)}\ \ldots\ \mathrm{O}_{n.1}\ldots\mathrm{O}_{n.k(n)}\ F^{m(n)}\ \chi$, we conclude that by the above observations, this formula has either a $\mathcal{K}$-equivalent of the form $F^{m(0)}(P\chi \vee \chi \vee F\chi)$ or a $\mathcal{K}$-equivalent of the form $F^{m(0)}\ominus^{x}F^{y}\chi$. Now, the former formula is $\mathcal{K}$-equivalent to $(P\chi \vee \chi \vee F\chi)$. And the latter formula is $\mathcal{K}$-equivalent to $\ominus^{x}F^{m(0)+y}\chi$. So, even in this case the formula has a $\mathcal{K}$-equivalent of one of the two requisite forms. The specific forms that the formula χ can take, mentioned in the statement of the theorem, follow from the observation made at the beginning of the present subsection. ∎

The fact that the above transformations do not remain within L but lead to its extension in which the operator $\ominus$ is available does not matter for our purposes. (It would, of course, matter had we wished to formulate a normal form for G-free prefix formulas of L.) Namely, we need the above theorem merely in order to be able to reason conveniently about the semantics of the relevant prefix formulas, and to this end it does not matter how the relevant equivalent forms are expressed, as long as the equivalent forms allow us to have a synoptic view on the totality of the prefix formulas in question.

We cannot show that all truth-makers of all *G*-free prefix formulas obtain or will obtain. Certain formulas in whose prefix *F* occurs admit a truth-maker that is a process of coming to obtain. A case in point is *HFp*, as noted at the beginning of the present section. However, we *can* show that there is no *G*-free prefix formula that is true at a time in a SOA belonging to $\mathcal{K}$ and has *exclusively* truth-makers that are processes of coming to obtain. Any such formula *has* a truth-maker that obtains or will obtain. That is, a *G*-free prefix formula cannot be true without having a truth-maker that obtains or will obtain.

Corollary 7.8 Let $\phi \coloneqq \mathrm{O}_1 \ldots \mathrm{O}_N\, \beta$ be a *G*-free prefix formula. If $\mathcal{M}^* \in \mathcal{K}$ and $\mathcal{M}^*, t \vDash \phi$, then ϕ has a truth-maker $(\mathcal{M}, t)$ in $\mathcal{M}^*$ such that $(\mathcal{M}, t)$ obtains or $(\mathcal{M}, t)$ will obtain.

Proof. Let $\mathcal{M}^* \in \mathcal{K}$ be arbitrary. Suppose $\mathcal{M}^*, t \vDash \phi$, where ϕ is as in the statement of the corollary. By Theorem 7.7, either ϕ is non-future or of the form $F^m\chi$, where χ is non-future and $m \geq 1$, or else there is a non-future formula χ and integers $n, m \geq 1$ such that ϕ is $\mathcal{K}$-equivalent to $(P\chi \vee \chi \vee F\chi)$ or $\ominus^n F^m \chi$. If ϕ is non-future, ϕ has a truth-maker $(\mathcal{M}, t)$ that obtains. If ϕ is of the form $F^m\chi$, it has a truth-maker $(\mathcal{M}, t)$ that will obtain. If, again, ϕ is $\mathcal{K}$-equivalent to $(P\chi \vee \chi \vee F\chi)$, it has either a truth-maker $(\mathcal{M}, t)$ that obtains or a truth-maker $(\mathcal{M}, t)$ that will obtain. Suppose, then, that ϕ is $\mathcal{K}$-equivalent to $\ominus^n F^m \chi$. If $n < m$, then ϕ has a truth-maker $(\mathcal{M}, t)$ that will obtain, whereas if $n \geq m$, then ϕ has a truth-maker $(\mathcal{M}, t)$ that obtains or will obtain. ∎

7.4 *G*-free formulas and determinate truth-makers

Let us now pose the question of determinacy regarding *G*-free prefix formulas, under the hypothesis that $\mathcal{M}_w \in \mathcal{K}$. By Lemma 7.4(a), if a real truth-maker of a *G*-free prefix formula *obtains*, it is determinate. By Corollary 7.8, any remaining *G*-free prefix formulas that have a real truth-maker at all, have a real truth-maker that *will obtain*. Now, if a truth-maker of a *G*-free formula will obtain, under what conditions is the truth-maker determinate?

By Lemma 7.4(b), the information that a factual truth-maker *will obtain* allows indeed us to infer that it is stable, but it does not allow deducing that it is fact-based. So, it remains to consider real truth-makers $(\mathcal{M}, t)$ of G-free prefix formulas ϕ that will obtain, and ask whether some, though not all, such truth-makers are, after all, fact-based and not merely stable. Since the truth-maker $(\mathcal{M}, t)$ of ϕ will obtain, in any event ϕ is not non-future. By Theorem 7.7, there is, then, a non-future prefix formula χ such that either ϕ is directly of the form $F^m\chi$, or else ϕ is $\boldsymbol{\mathcal{K}}$-equivalent to a formula of the form $(P\chi \vee \chi \vee F\chi)$ or $\ominus^x F^y\chi$. Now, since $(\mathcal{M}, t)$ *will obtain*, in each of these three cases there is a positive integer k such that $(\mathcal{M}, t)$ is, in particular, a truth-maker of the formula $F^k\chi$, in addition to being a truth-maker of ϕ. That is, in each case, it cannot happen that $(\mathcal{M}, t)$ makes ϕ true without making $F^k\chi$ true, as well. (In the first case $k \coloneqq m$, in the second case $k \coloneqq 1$, and in the third case $k \coloneqq \max\{1, y - x\}$.) Further, in each case, any truth-maker of $F^k\chi$ makes likewise ϕ true. Whenever a G-free prefix formula has a truth-maker that will obtain, this truth-maker is, then, a truth-maker of a formula of the form $F^k\chi$ with χ being non-future. In our investigation of centered SOAs that are truth-makers of G-free prefix formulas and that furthermore will obtain, we may, then, without loss of generality concentrate on formulas of the form $F^k\chi$.

Such formulas $F^k\chi$ may be merely future-tense reports about what has already happened; so is the case with FPq, if its truth at t_0 is based on the truth of q at t_0 or on that of Pq at t_0. Or such formulas $F^k\chi$ may concern both the future and the past. This is what happens with $FFHq$, whose truth at t_0 entails that q is true at the immediate successor of t_0, in addition to being true at t_0 and always before. Or such formulas $F^k\chi$ may concern exclusively the future, as is the case with Fp. Another such example is the truth of FPq at t_0, supposing that q is false at t_0 and always earlier. Finally, such formulas $F^k\chi$ can be true for logical, physical or metaphysical reasons, in which case they fail to express any ‘substantial’ condition on their moment of evaluation. Cases in point are $F(p \vee \neg p)$ and $FH(p \vee \neg p)$, as well as Fq, supposing that q happens to occur after the moment of evaluation at some moment of each branch of the temporal frame. If Socrates exists at t_0, then q might be the proposition *that Socrates dies*. The different grounds for the truth of a formula of the form $F^k\chi$ motivate a division of fact-based truth-makers into strong and weak. The division can be formulated using the following notion of factive evaluation.

Definition 7.9 (Factive evaluation) Let ϕ be a formula. Let $\mathcal{E}$ be an evaluation of ϕ on $(\mathcal{M}_w, t_0)$, with $\prec$ being the causal ordering of $\mathcal{M}_w$. The evaluation $\mathcal{E}$ is *factive* iff for all evaluation sequences Σ in $\mathcal{E}$, we have that the moment component $m(\Sigma)$ of the last position of Σ satisfies $m(\Sigma) \preccurlyeq t_0$. ▮

Suppose $(\mathcal{M}, t_0)$ is a truth-maker of a formula ϕ in $\mathcal{M}_w$, and $\mathcal{E}$ is an evaluation such that $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$. Then, trivially, if $(\mathcal{M}, t_0)$ *obtains*, $\mathcal{E}$ is factive. In this case not only is there no evaluation sequence in $\mathcal{E}$ whose last position has a moment component later than t_0, but no moment of the *entire moment set* of $\mathcal{E}$ is later than t_0. Now, it can even happen that $(\mathcal{M}, t_0)$ *will obtain* but still $\mathcal{E}$ is factive—this is the case of future-tense reports of present or past facts. In such a case the possibility of constructing the evaluation requires that the passage of time extend beyond t_0, but apart from this 'structural' requirement, no further condition is imposed on the future. The substantial requirement about the truth of propositional atoms concerns exclusively t_0 and/or moments earlier than t_0.

When attention is confined to the class $\boldsymbol{\mathcal{K}}$ of SOAs, it can be shown that if ϕ has a truth-maker that will obtain, then this truth-maker being induced by a *factive* truth-evaluation is a sufficient condition for the truth-maker to be determinate.

Fact 7.11 (Determinacy via factiveness) Let ϕ be a G-free prefix formula. Suppose $(\mathcal{M}, t_0)$ is a truth-maker of ϕ in $\mathcal{M}_w$, with $\mathcal{E}$ being a truth-evaluation such that $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$. Suppose that $(\mathcal{M}, t_0)$ will obtain. If $\mathcal{M}_w$ belongs to $\boldsymbol{\mathcal{K}}$ and $\mathcal{E}$ is factive, then $(\mathcal{M}, t_0)$ is determinate.

Proof. Let ϕ, $(\mathcal{M}, t_0)$, and $\mathcal{E}$ be as in the statement of the Fact, with $\mathcal{M}_w \in \boldsymbol{\mathcal{K}}$. By Lemma 7.4(b), if a truth-maker of a G-free prefix formula has a truth-maker that will obtain, this truth-maker is stable. To show that $(\mathcal{M}, t_0)$ is determinate, it remains, therefore, to prove that it is fact-based. To this end, let $\mathcal{N} = (\mathfrak{S}_{\mathcal{N}}, \mathrm{TIME}_{\mathcal{N}})$ be a maximal linear SOA satisfying $\mathfrak{S}_{\mathcal{N}} \Subset \mathfrak{S}_w$ and $\mathcal{M} \cong[t_0]^{\mathrm{dyn}}\, \mathcal{N}$. I claim that there is a truth-evaluation $\mathcal{E}'$ of ϕ in $(\mathcal{N}, t_0)$. Note that because $\mathcal{M}_w \in \boldsymbol{\mathcal{K}}$, the passage of time from t_0 onward in $\mathcal{N}$ is infinite. Consequently, there are in particular enough moments after t_0 so that $\mathcal{E}'$ can be constructed by copying $\mathcal{E}$: whenever in an evaluation sequence of $\mathcal{E}$ the n-th successor of t_0 from $\mathcal{M}$ is chosen, choose here the n-th successor of t_0 from $\mathcal{N}$. Further, whenever t_0 or one of its predecessors is chosen in $\mathcal{M}$, choose the very same moment in $\mathcal{N}$. Thus constructed, $\mathcal{E}'$ ends up being factive because $\mathcal{E}$ is factive. Now, $\mathcal{E}$ is a truth-evaluation. Because $\mathcal{M}_w \in \boldsymbol{\mathcal{K}}$, no evaluation sequence in $\mathcal{E}$ can have a last position of the form $(\mathrm{O}\chi, s, g)$ where $\mathrm{O} \in \{P, H\}$ and s has no predecessor, or of the form $(F\chi, s, g)$ where s has no successor attainable via the time-resource. Therefore its all evaluation sequences end with a position of the form (θ, s, g), where $\theta \in \mathbf{lit}$ and θ is true at s.[27] Since $\mathcal{E}$ is factive, in each such case $s \preccurlyeq t_0$. Given the way in which $\mathcal{E}'$ is constructed, there is a one-one correspondence between the last positions of its evaluation sequences and the last positions of evaluation sequences of $\mathcal{E}$ (the corresponding positions differ only in the values of their history parameter, sharing their literal and sharing their value of the moment parameter). Since $\mathcal{E}$ is a truth-evaluation, it follows that so is $\mathcal{E}'$. ∎

Recall that a truth-maker of a prefix formula obtains or will obtain or is a process of coming to obtain. Truth-makers of the first kind are trivially both factive and fact-based. Those of the second kind were just shown to be fact-based if factive. And those of the third kind are never factive. So, whenever a truth-maker of a prefix formula *is* factive, it is thereby fact-based. Now, fact-based truth-makers can be divided into strong and weak according to whether or not they are induced by a factive truth-evaluation.

Definition 7.10 (Strongly vs. weakly fact-based truth-makers) Suppose $(\mathcal{M}, t_0)$ is a truth-maker of a prefix formula ϕ in $\mathcal{M}_w$ and $\mathcal{M} = \mathcal{M}_w[t_0, \mathcal{E}]$. The truth-maker $(\mathcal{M}, t_0)$ is *strongly fact-based* iff the evaluation $\mathcal{E}$ is factive (and thereby fact-based). It is *weakly fact-based* iff it is fact-based but not factive—i.e., there is in $\mathcal{E}$ at least one evaluation sequence whose last position contains a moment component later than t_0. ∎

Let us, now, come back to truth-makers of G-free formulas ϕ that *will obtain*. If $(\mathcal{M}, t_0)$ is such a truth-maker of ϕ, it may but need not be fact-based. By Lemma 7.4(b), it is *determinate* if it is fact-based, and there are two ways in which it can be fact-based. It can be *strongly* fact-based—in which case ϕ is a report about the present or the past. Or it can be *weakly* fact-based—and in this case ϕ involves a prediction regarding future necessities, whether logical, physical or metaphysical. Indeed, if $(\mathcal{M}, t_0)$ is *weakly* fact-based, ϕ ends up expressing a (non-future) condition χ regarding a moment later than t_0. However, given that $(\mathcal{M}, t_0)$ is *fact-based*, this same condition χ must be satisfied by some moment later than t_0 on *every* causally

[27] Recall the definition of the set **lit** of literals from the beginning of Section 2.

possible course of events proceeding from t_0. The truth of χ at some moment of every causally possible continuation of t_0 is, then, *not* a contingent matter, but at least physically or metaphysically necessary in the sense that 'our indeterminist world' happens to be such that χ gets realized somewhere on all causally possible histories after t_0.

The two variants of fact-based truth-makers—strong and weak—correspond to the sources of determinacy of states of affairs discerned by Boethius and Abelard: determinacy due to the past or current presence of something amounts to strongly fact-based truth-makers that obtain (or indeed will obtain), while determinacy of states of affairs that will obtain due to the natures of things amounts to weakly fact-based truth-makers.

Finally, those truth-makers ($\mathcal{M}$, t_0) that will obtain but are *not* fact-based render true *contingent predictions* ϕ. This means two things. First, the truth of ϕ genuinely depends on what happens after t_0—the state of affairs $\mathcal{M}$ is induced by a truth-evaluation that is *not* factive. Second, ϕ is contingent in the sense that *not* all counterfactual causally possible maximal courses of events extending t_0 render true the prediction that ϕ makes about the future, even though the actual maximal course of events extending t_0 indeed does. By Theorem 7.7, the prediction is actually of the form $F^k\chi$ with χ being non-future. Thus, there is at least one counterfactual causally possible maximal course of events that extends t_0 and contains no moment at which χ is true. Indeed, both $F^k\chi$ and $\Diamond{\sim}F^k\chi$ are true at t_0. Truth-makers of this latter variety, being *not* fact-based, are by definition indeterminate. They are indeterminate truth-makers that will obtain but currently do not obtain.

In order to complete my discussion of determinacy, it remains to show that a prefix formula ϕ with a determinate real truth-maker ($\mathcal{M}$, t_0) behaves in the desired way with respect to historical necessity: the truth of such a formula is settled—i.e., the formula $\Box\phi$ is true at t_0.

Theorem 7.12 (Having a determinate truth-maker entails being settled) Let ϕ be a prefix formula. If $\mathcal{M}^* \in \boldsymbol{\mathcal{K}}$ and ($\mathcal{M}$, t_0) is a *determinate* truth-maker of ϕ in $\mathcal{M}^*$, then ϕ is (G-free and) *settled*—i.e., satisfies $\mathcal{M}^*, t_0 \vDash \Box\phi$.

Proof. Let ϕ be a prefix formula. Suppose ($\mathcal{M}$, t_0) is a determinate truth-maker of ϕ with $\mathcal{M} = \mathcal{M}^*[t_0, \mathcal{E}]$, where $\mathcal{M}^* = (\mathfrak{S}^*, \text{TIME}^*) \in \boldsymbol{\mathcal{K}}$. By Lemma 7.2, ϕ is G-free. In order to show that $\mathcal{M}^*, t_0 \vDash \Box\phi$, it must be established that $\mathcal{M}^*, t_0, [x] \vDash \phi$, for all $t_0 \preccurlyeq x$. So, let s with $t_0 \preccurlyeq s$ be arbitrary. I wish to show that $\mathcal{M}^*, t_0, [s] \vDash \phi$. Note that $[s]$ is a partial history leading to s. Because $\mathcal{M}^* \in \boldsymbol{\mathcal{K}}$, the time-resource TIME* yields to $[s]$ an infinite future continuation Z such that $h_s = [s] \cup Z$ is a history in $\mathcal{M}^*$.

Since $(\mathcal{M}, t_0)$ is a determinate truth-maker of ϕ, it is in particular fact-based, whence we have $\mathcal{N}, t_0 \vDash \phi$, for all maximal linear SOAs $\mathcal{N} = (\mathfrak{S}_{\mathcal{N}},\mathrm{TIME}_{\mathcal{N}})$ satisfying $\mathfrak{S}_{\mathcal{N}} \Subset \mathfrak{S}^*$ and $\mathcal{M} \cong[t_0]^{\mathrm{dyn}}\ \mathcal{N}$. One such maximal linear SOA is the static substructure of $\mathfrak{S}^*$ determined by the set h_s, call it $\mathcal{N}^s$. Thus, we have $\mathcal{N}^s, t_0 \vDash \phi$. It follows from Theorem 7.7 that there is an evaluation $\mathcal{E}^s$ and a truth-maker $(\mathcal{X}^s, t_0)$ of ϕ in $\mathcal{N}^s$ such that $\mathcal{X}^s = \mathcal{N}^s[t_0, \mathcal{E}^s]$ and $(\mathcal{X}^s, t_0)$ obtains or will obtain. Consequently, the causal ordering of $\mathcal{X}^s$ has a maximum, r, satisfying $t_0 \preccurlyeq r$. The moment r is, then, comparable with s, and in particular we have $r \in h_s$. Here, $[r]$ is a partial history (not only of $\mathcal{N}^s$ but also) of $\mathcal{M}^*$, more specifically $[r]$ is an initial segment of h_s. Now, either $r \preccurlyeq s$ and $r \in [s]$. Or else $s \prec r$ and so r does not belong to $[s]$, but still is attained by TIME* from $[s]$. Consequently, in both cases, the truth-evaluation $\mathcal{E}^s$ of ϕ on $(\mathcal{N}^s, t_0, [t_0])$ induces a truth-evaluation $\mathcal{E}^*$ of ϕ on $(\mathcal{M}^*, t_0, [s])$. The positions in $\mathcal{E}^*$ differ from those in $\mathcal{E}^s$ only in their values of the history parameter; in $\mathcal{E}^*$, all moments up to s are directly given by the initial position $(\phi, t_0, [s])$, and only from s onward elements of the moment set of $\mathcal{E}^*$ are generated by the time-resource TIME* (if indeed $s \prec r$). It follows that $\mathcal{M}^*, t_0, [s] \vDash \phi$. We may conclude that $\mathcal{M}^*, t_0 \vDash \Box\phi$. ∎

8. Conclusion

An analysis was found that renders it meaningful to talk of indeterminate truth-makers that do not obtain but will. The discussion was based on the assumptions of 'objective indeterminism' (at any moment, there are normally several causally possible future courses of events, none of which has metaphysical priority over the others) and 'temporal becoming' (over a time span of any given duration, exactly one causally possible future course of events gets actualized).

Already in the background paper, I motivated my proposal conceptually, and I pointed out that Boethius and Abelard appear to have been early proponents of a semantic analysis of future contingent propositions in terms of states of affairs that do not obtain but will. In this follow-up paper, I wished to strengthen my proposal by presenting a formal semantics that utilizes the ideas put forward in the background paper. A precise model-theoretic definition of the notion of truth-maker was developed based on the notion of *truth-evaluation*, and it was defined what it means that a truth-maker of a non-modal formula obtains, or will obtain, or is a process of coming to obtain. Likewise, the notions of determinacy and indeterminacy of truth-makers of non-modal formulas were defined.

The determinacy of a truth-maker was taken to require that two conditions be fulfilled: *fact-basedness* and *stability*. A truth-maker ($\mathcal{M}$, t_0) of a non-modal formula ϕ is fact-based, if the truth of ϕ in $\mathcal{M}$ at t_0 does not depend on how the passage of time evolves from t_0 on, as long as the passage of time conforms to what is causally possible in 'our indeterminist world'. And a truth-maker ($\mathcal{M}$, t_0) of ϕ is stable, if it involves only finitely many moments after t_0 and if the very same truth-evaluation that witnesses the truth of ϕ in $\mathcal{M}$ at t_0, would witness its truth in any (linear) extension of $\mathcal{M}$, whether the extension employs causally possible or purely hypothetical moments. It was proven that a proposition having a determinate truth-maker entails, according to the semantics developed, that the truth of the proposition is settled (historically necessary, unpreventable). It was indeed an important *desideratum* for the semantic analysis I wished to formulate that this link between determinacy and historical necessity would be in force but would not be simply definitional. As Neil Lewis notes, also according to Boethius, the move from determinacy to necessity was *not* supposed to be obvious, but something calling for an argument.[28]

It was seen that true contingent predictions end up having an indeterminate truth-maker that will obtain but currently does not. Only one semantically relevant sense of 'true' was recognized in my analysis. My proposal was seen not to compromise the bivalence of future contingent propositions. If a proposition of the form Fq is true at t, the passage of time, as it in fact turns out to evolve, will actualize a truth-maker of this proposition by actualizing a moment later than t at which q is true. If Fq is not true at t, then from t on, the passage of time systematically produces moments at which q is false, and it continues to do so, unless the passage of time terminates.

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[28] Lewis (1987), p. 83.

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