

**A true mathematical statement  $\Psi$  with the predicate  $\mathcal{K}$  of the current mathematical knowledge, where  $\Psi$  may be false in the future and does not express the current knowledge about the provability of any mathematical statements  $\Psi_1, \dots, \Psi_n$  without  $\mathcal{K}$**

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ABSTRACT. The theorem of Royer and Case states that there exists a limit-computable function  $\beta_1 : \mathbb{N} \rightarrow \mathbb{N}$  which eventually dominates every computable function  $\delta_1 : \mathbb{N} \rightarrow \mathbb{N}$ . We present an alternative proof of this theorem.  $\mathcal{K}$  denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set  $\mathcal{K}$  is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to  $\mathcal{K}$ . We prove: (1) there exists a limit-computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  of unknown computability which eventually dominates every function  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  with a single-fold Diophantine representation. We present both constructive and non-constructive proof of (1). Statement (1) claims that there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $(f \text{ is computable in the limit}) \wedge (\neg \mathcal{K}(f \text{ is computable})) \wedge (\neg \mathcal{K}(f \text{ is uncomputable})) \wedge (f \text{ eventually dominates every function } \delta : \mathbb{N} \rightarrow \mathbb{N} \text{ with a single-fold Diophantine representation})$ . Since Martin Davis' conjecture on single-fold Diophantine representations disproves Statement (1), Statement (1) justifies the title of the article.

**Key words and phrases:** eventual domination, limit-computable function, predicate  $\mathcal{K}$  of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate  $\mathcal{K}$ .

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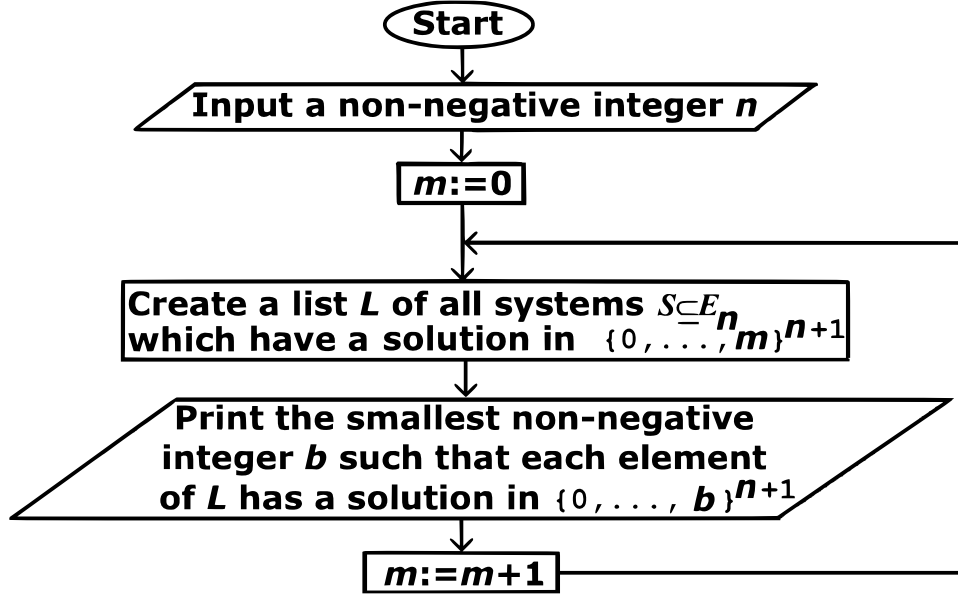
Let

$$E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \dots, n\}\}$$

### 1. CLASSICAL MATHEMATICS

**Theorem 1.** ([4, p. 118]). *There exists a limit-computable function  $\beta_1 : \mathbb{N} \rightarrow \mathbb{N}$  which eventually dominates every computable function  $\delta_1 : \mathbb{N} \rightarrow \mathbb{N}$ .*

We present an alternative proof of Theorem 1. For every  $n \in \mathbb{N}$ , we define  $\beta_1(n)$  as the smallest  $b \in \mathbb{N}$  such that if a system of equations  $S \subseteq E_n$  has a solution in  $\mathbb{N}^{n+1}$ , then this solution belongs to  $\{0, \dots, b\}^{n+1}$ . The function  $\beta_1 : \mathbb{N} \rightarrow \mathbb{N}$  is computable in the limit and eventually dominates every computable function  $\delta_1 : \mathbb{N} \rightarrow \mathbb{N}$ , see [5]. Flowchart 1 describes a semi-algorithm which computes  $\beta_1(n)$  in the limit.



Flowchart 1

A semi-algorithm which computes  $\beta_1(n)$  in the limit

**Conjecture 1.** ([1, pp. 341–342], [2, p. 42], [3]). *Every listable set  $\mathcal{X} \subseteq \mathbb{N}^k$  ( $k \in \mathbb{N} \setminus \{0\}$ ) has a single-fold Diophantine representation.*

Let  $\Phi$  denote the following statement: *the function  $\mathbb{N} \ni n \rightarrow 2^n \in \mathbb{N}$  eventually dominates every function  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  with a single-fold Diophantine representation.* For  $n \in \mathbb{N}$ , let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \beta_1(n), & \text{otherwise} \end{cases}$$

The function  $g : \mathbb{N} \rightarrow \mathbb{N}$  is computable if and only if  $\Phi$  holds. Currently,

$$(\neg \mathcal{K}(\Phi)) \wedge (\neg \mathcal{K}(\neg \Phi)) \wedge (\neg \mathcal{K}(g \text{ is computable})) \wedge (\neg \mathcal{K}(g \text{ is uncomputable}))$$

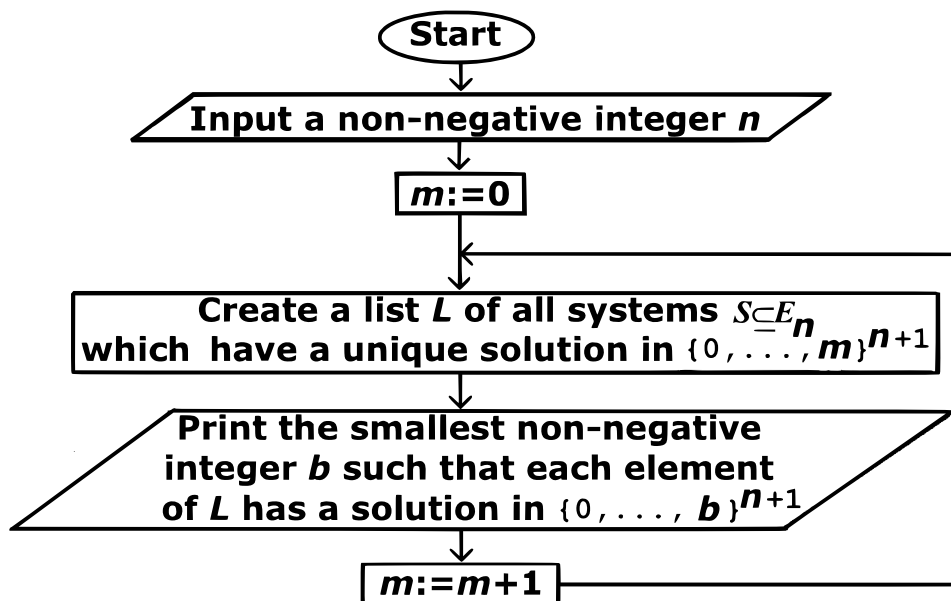
**Lemma 1.** *The function  $g$  is computable in the limit and eventually dominates every function  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  with a single-fold Diophantine representation.*

*Proof.* It follows from Theorem 1. □

For every  $n \in \mathbb{N}$ , we define  $\beta(n)$  as the smallest  $b \in \mathbb{N}$  such that if a system of equations  $S \subseteq E_n$  has a unique solution in  $\mathbb{N}^{n+1}$ , then this solution belongs to  $\{0, \dots, b\}^{n+1}$ .

**Theorem 2.** *The function  $\beta : \mathbb{N} \rightarrow \mathbb{N}$  is computable in the limit and eventually dominates every function  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  with a single-fold Diophantine representation.*

*Proof.* This is proved in [5]. The term "dominated" in the title of [5] means "eventually dominated". Flowchart 2 describes a semi-algorithm which computes  $\beta(n)$  in the limit.



Flowchart 2

A semi-algorithm which computes  $\beta(n)$  in the limit

□

## 2. MATHEMATICS WITH THE PREDICATE $\mathcal{K}$ OF THE CURRENT MATHEMATICAL KNOWLEDGE

$\mathcal{K}$  denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set  $\mathcal{K}$  is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to  $\mathcal{K}$ .

Let  $\mathcal{T}$  denote the set of twin primes. Let  $\Gamma$  denote the following true statement: *it is unknown whether or not there are infinitely many twin primes.*

**Proposition 1.** *The statement  $\Gamma$  does not justify the title of the article.*

*Proof.* The statement  $\Gamma$  expresses that

$$(\neg\mathcal{K}(\text{card}(\mathcal{T}) = \omega)) \wedge (\neg\mathcal{K}(\text{card}(\mathcal{T}) < \omega))$$

This conjunction expresses the current knowledge about the provability of the statements  $\text{card}(\mathcal{T}) = \omega$  and  $\text{card}(\mathcal{T}) < \omega$ . □

**Statement 1.** *There exists a limit-computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  of unknown computability which eventually dominates every function  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  with a single-fold Diophantine representation.*

*Proof.* Statement 1 follows constructively from Theorem 2 by taking  $f = \beta$  and the following conjunction:

$$(\neg\mathcal{K}(\beta \text{ is computable})) \wedge (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

Statement 1 follows non-constructively from Lemma 1 by taking  $f = g$  and the following conjunction:

$$(\neg\mathcal{K}(g \text{ is computable})) \wedge (\neg\mathcal{K}(g \text{ is uncomputable}))$$

□

**Proposition 2.** *Statement 1 justifies the title of the article.*

*Proof.* Statement 1 claims that there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$(f \text{ is computable in the limit}) \wedge (\neg\mathcal{K}(f \text{ is computable})) \wedge$$

$$(\neg\mathcal{K}(f \text{ is uncomputable})) \wedge (f \text{ eventually dominates every function } \delta : \mathbb{N} \rightarrow \mathbb{N} \text{ with a single-fold Diophantine representation})$$

Conjecture 1 disproves Statement 1. □

Since the function  $\beta_1$  in Theorem 1 is not computable, Statement 1 does not follow from Theorem 1. Ignoring the epistemic condition in Statement 1, Statement 1 follows from Theorem 1 by taking  $f = \beta_1$ .

In [7], the author showed that the predicate  $\mathcal{K}$  non-trivially extends constructive mathematics, see also [6], [8], [9].

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