NON-TRIVIALLY TRUE STATEMENTS IN COMPUTABILITY WITH THE PREDICATE $\mathcal{K}$ OF THE CURRENT MATHEMATICAL KNOWLEDGE, WHICH DO NOT EXPRESS THE CURRENT MATHEMATICAL KNOWLEDGE AND MAY BE FALSIFIED

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ABSTRACT. The theorem of Royer and Case states that there exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$. We present an alternative proof of this theorem. $\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. The statement $\neg\mathcal{K}(P \neq NP) \wedge \neg\mathcal{K}(P = NP)$ does not justify the title of the article. We prove: (1) there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation; (2) for every computable function $g : \mathbb{N} \to \mathbb{N}$, there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability such that f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation and $f(n) > g(n)$ for every $n \in \mathbb{N}$. Both (1) and (2) justify the title of the article.

Key words and phrases: current mathematical knowledge, eventual domination, limit-computable function, single-fold Diophantine representation, time-dependent truth in mathematics.

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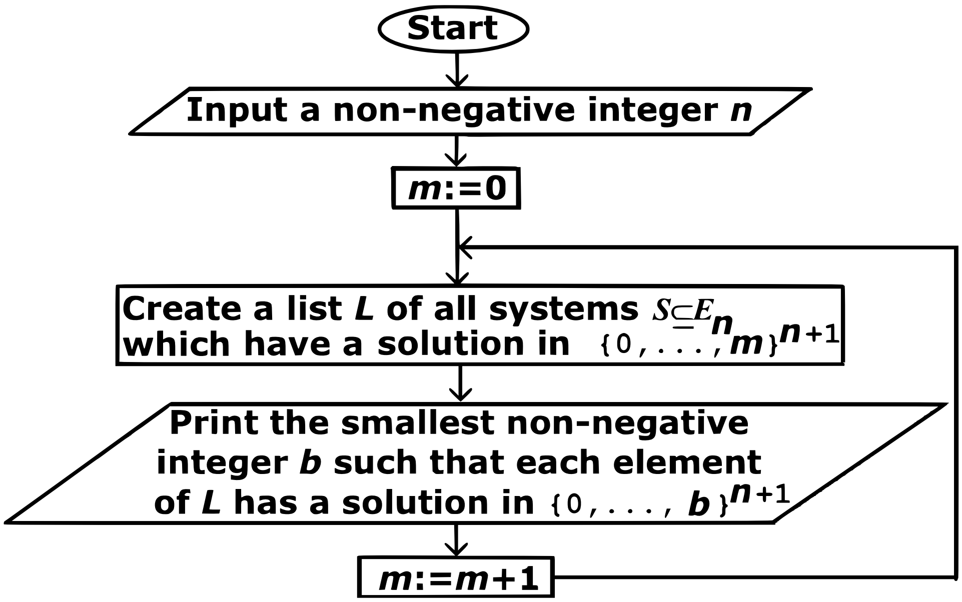
Let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$$

1. CLASSICAL COMPUTABILITY THEORY

Theorem 1. *([4, p. 118]). There exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The function $\beta_1 : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$, see [5]. Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit.



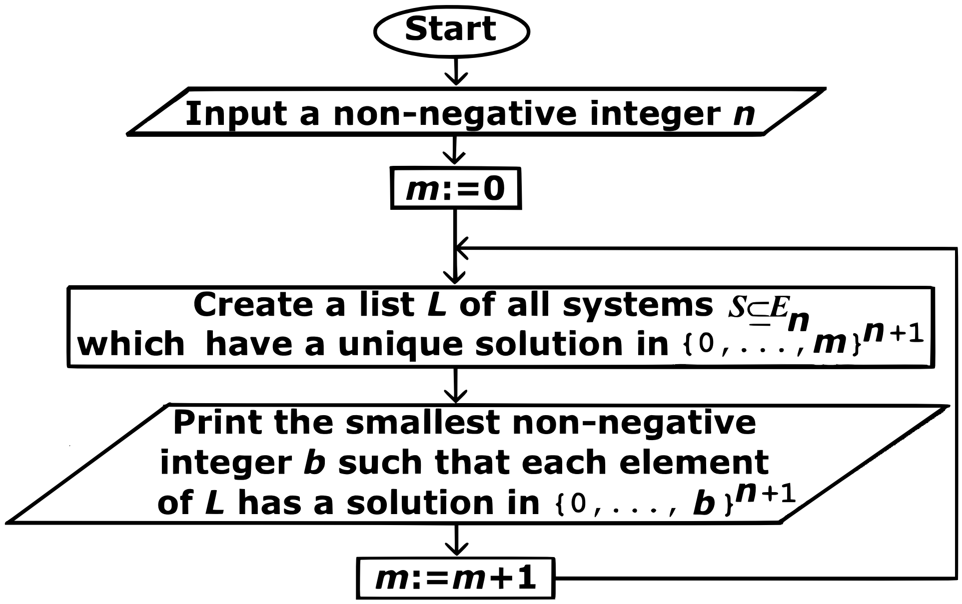
Flowchart 1
A semi-algorithm which computes $\beta_1(n)$ in the limit

Conjecture 1. *(*[1, pp. 341–342]*,* [2, p. 42]*,* [3]*). Every listable set* $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ *has a single-fold Diophantine representation.*

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \dots, b\}^{n+1}$.

Theorem 2. *The function* $\beta : \mathbb{N} \to \mathbb{N}$ *is computable in the limit and eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. This is proved in [5]. The term *"dominated"* in the title of [5] means *"eventually dominated"*. Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 2

A semi-algorithm which computes $\beta(n)$ in the limit

□

2. COMPUTABILITY WITH THE PREDICATE $\mathcal{K}$ OF THE CURRENT MATHEMATICAL KNOWLEDGE

$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. The statement

$$\neg\mathcal{K}(P \neq NP) \land \neg\mathcal{K}(P = NP)$$

does not justify the title of the article.

Statement 1. *There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. It follows from Theorem 2 by taking $f = \beta$ and the following conjunction:

$$((\text{The function } \beta \text{ is computable}) \notin \mathcal{K}) \land ((\text{The function } \beta \text{ is uncomputable}) \notin \mathcal{K})$$

□

Statement 2. *For every computable function $g : \mathbb{N} \to \mathbb{N}$, there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability such that f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation and $f(n) > g(n)$ for every $n \in \mathbb{N}$.*

Proof. It follows from Theorem 2 by taking $f = g + 1 + \beta$ and the following conjunction:

$$((\text{The function } \beta \text{ is computable}) \notin \mathcal{K}) \wedge ((\text{The function } \beta \text{ is uncomputable}) \notin \mathcal{K})$$

□

Both Statement 1 and Statement 2 justify the title of the article. Statements 1 and 2 are non-trivially true, contain the predicate $\mathcal{K}$, and will be false when someone proves Conjecture 1. Since the function β_1 in Theorem 1 is not computable, Statements 1 and 2 do not follow from Theorem 1. Ignoring the epistemic condition in Statements 1 and 2, they follow from Theorem 1 by taking $f = \beta_1$ or $f = g + 1 + \beta_1$.

In [7], the author showed that the predicate $\mathcal{K}$ non-trivially extends constructive mathematics, see also [6], [8], [9].

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