

A true and falsifiable statement Ψ with the predicate \mathcal{K} of the current mathematical knowledge, where Ψ does not express what is proved or unproved in mathematics without \mathcal{K}

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Abstract. The theorem of Royer and Case states that there exists a limit-computable function $\beta_1: \mathbb{N} \rightarrow \mathbb{N}$ which eventually dominates every computable function $\delta_1: \mathbb{N} \rightarrow \mathbb{N}$. We present an alternative proof of this theorem. \mathcal{K} denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set \mathcal{K} is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to \mathcal{K} . We prove: (1) there exists a limit-computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability which eventually dominates every function $\delta: \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation. We present both constructive and non-constructive proof of (1). Statement (1) claims that there exists a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $(f \text{ is computable in the limit}) \wedge (\neg \mathcal{K}(f \text{ is computable})) \wedge (\neg \mathcal{K}(f \text{ is uncomputable})) \wedge (f \text{ eventually dominates every function } \delta: \mathbb{N} \rightarrow \mathbb{N} \text{ with a single-fold Diophantine representation})$. Since Martin Davis' conjecture on single-fold Diophantine representations disproves Statement (1), Statement (1) justifies the title of the article.

Key words and phrases: eventual domination, limit-computable function, predicate \mathcal{K} of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate \mathcal{K} .

2020 Mathematics Subject Classification: 03D20.

\mathcal{K} denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set \mathcal{K} is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to \mathcal{K} . For $n \in \mathbb{N}$, let

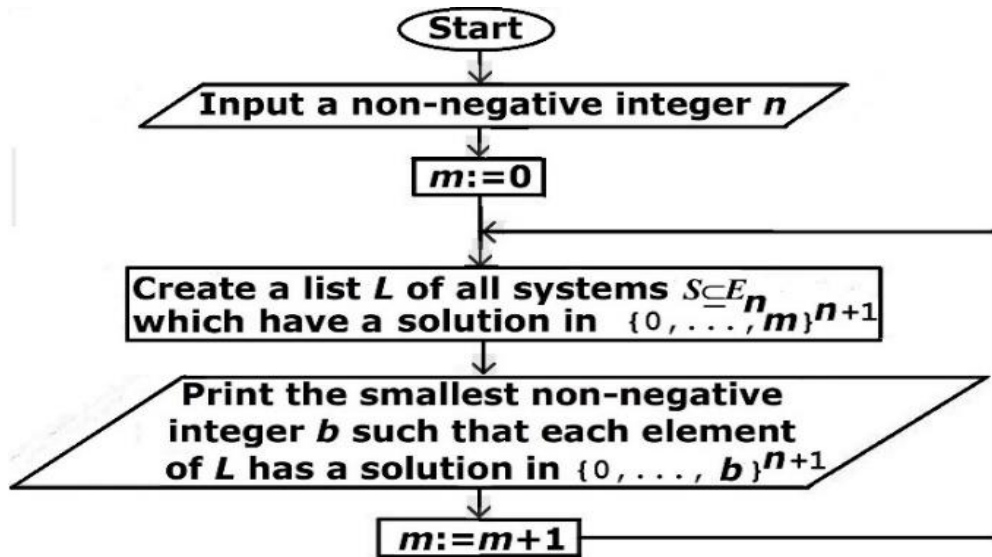
$$E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \dots, n\}\}$$

1. Classical mathematics

Theorem 1. ([4, p. 118]). *There exists a limit-computable function $\beta_1: \mathbb{N} \rightarrow \mathbb{N}$ which eventually dominates every computable function $\delta_1: \mathbb{N} \rightarrow \mathbb{N}$.*

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in \mathbb{N}^{n+1} , then this solution belongs to $\{0, \dots, b\}^{n+1}$. The function $\beta_1: \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1: \mathbb{N} \rightarrow \mathbb{N}$, see [5].

Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit.



Flowchart 1

A semi-algorithm which computes $\beta_1(n)$ in the limit

Conjecture 1. ([1, pp. 341-342], [2, p. 42], [3, p. 745]). Every listable set $X \subseteq \mathbb{N}^k$ ($k \in \mathbb{N} \setminus \{0\}$) has a single-fold Diophantine representation.

Let Φ denote the following statement: *the function $\mathbb{N} \ni n \rightarrow 2^n \in \mathbb{N}$ eventually dominates every function $\delta: \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.* For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \beta_1(n), & \text{otherwise} \end{cases}$$

The function $g: \mathbb{N} \rightarrow \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg \mathcal{K}(\Phi)) \wedge (\neg \mathcal{K}(\neg \Phi)) \wedge (\neg \mathcal{K}(g \text{ is computable})) \wedge (\neg \mathcal{K}(g \text{ is uncomputable}))$$

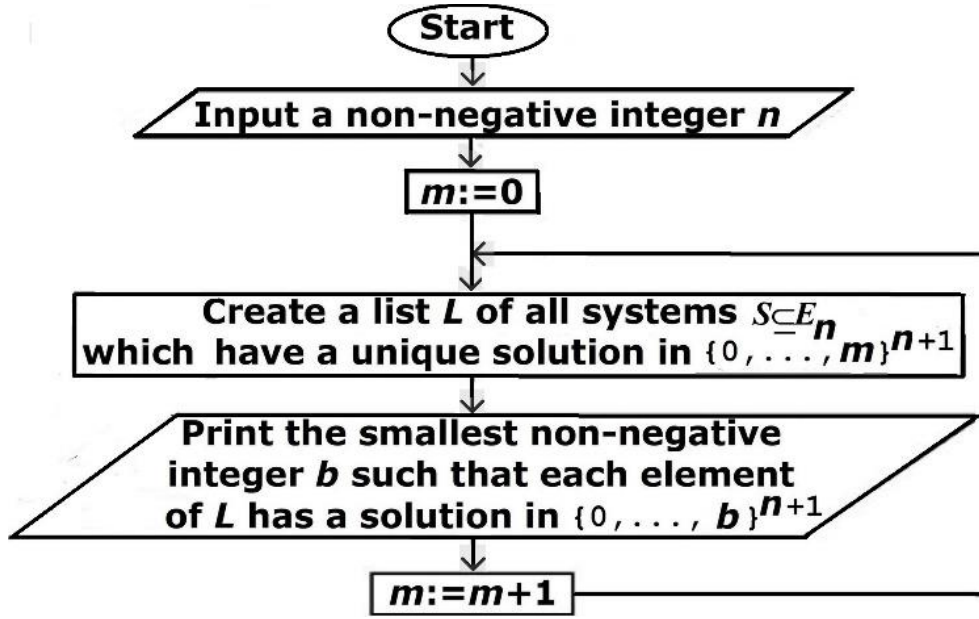
Lemma 1. *The function g is computable in the limit and eventually dominates every function $\delta: \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. It follows from Theorem 1. \square

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in \mathbb{N}^{n+1} , then this solution belongs to $\{0, \dots, b\}^{n+1}$.

Theorem 2. *The function $\beta: \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta: \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [5]. The term "dominated" in the title of [5] means "eventually dominated". Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 2

A semi-algorithm which computes $\beta(n)$ in the limit \square

2. Mathematics with the predicate \mathcal{K} of the current mathematical knowledge

Let T denote the set of twin primes. Let Γ denote the following true statement: *it is unknown whether or not there are infinitely many twin primes.*

Proposition 1. *The statement Γ does not justify the title of the article.*

Proof. The statement Γ expresses that $(\neg\mathcal{K}(\text{card}(T)=\omega)) \wedge (\neg\mathcal{K}(\text{card}(T)<\omega))$. This conjunction expresses the current knowledge about the provability of the statements $\text{card}(T)=\omega$ and $\text{card}(T)<\omega$. \square

Statement 1. *There exists a limit-computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability which eventually dominates every function $\delta: \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 1 follows constructively from Theorem 2 by taking $f = \beta$ and the following conjunction:

$$(\neg\mathcal{K}(\beta \text{ is computable})) \wedge (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

Statement 1 follows non-constructively from Lemma 1 by taking $f = g$ and the following conjunction:

$$(\neg\mathcal{K}(g \text{ is computable})) \wedge (\neg\mathcal{K}(g \text{ is uncomputable})) \square$$

Proposition 2. *Statement 1 justifies the title of the article.*

Proof. Statement 1 claims that there exists a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that (f is computable in the limit) \wedge ($\neg \mathcal{K}(f$ is computable)) \wedge ($\neg \mathcal{K}(f$ is uncomputable)) \wedge (f eventually dominates every function $\delta: \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation). Conjecture 1 disproves Statement 1. \square

Since the function β_1 in Theorem 1 is not computable, Statement 1 does not follow from Theorem 1. Ignoring the epistemic condition in Statement 1, Statement 1 follows from Theorem 1 by taking $f = \beta_1$.

In [7], the author showed that the predicate \mathcal{K} non-trivially extends constructive mathematics, see also [6], [8], [9].

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