A falsifiable statement Ψ of the form "$\exists f : \mathbb{N} \to \mathbb{N}$ of unknown computability such that ..." which significantly strengthens a non-trivial theorem

Apoloniusz Tyszka

Abstract

We present a new constructive proof of the following theorem: there exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$. We prove: (1) there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation, (2) statement (1) significantly strengthens a non-trivial mathematical theorem, (3) Martin Davis' conjecture on single-fold Diophantine representations disproves (1). We present both constructive and non-constructive proof of (1).

Key words and phrases: eventual domination, limit-computable function, predicate $\mathcal{K}$ of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate $\mathcal{K}$ of the current mathematical knowledge.

1 The goal of the article

We formulate a falsifiable statement Ψ of the form:

$\exists f : \mathbb{N} \to \mathbb{N}$ such that f satisfies a mathematical condition $\mathcal{C}$ and

it is unknown whether or not f satisfies a mathematical condition $\mathcal{D}$.

We prove that Ψ significantly strengthens a non-trivial mathematical theorem. There is no widely known theorem from which we can draw Ψ. Ignoring the epistemic condition in Ψ, Ψ follows from a known theorem.

2 Predicate $\mathcal{K}$ of the current mathematical knowledge

$\mathcal{K}$ denotes both the predicate satisfied by every currently known theorem and the set of all currently known theorems. Any theorem of any mathematician from past or present belongs to $\mathcal{K}$. The set $\mathcal{K}$ is time-dependent. $\mathcal{K}$ contains all written down theorems and their particular cases. Hence,

$$\{0+1=1,\ 1+1=2,\ 2+1=3,\ \ldots\} \subseteq \mathcal{K}$$

$\mathcal{K}$ contains every particular case of any written down schema of theorems. Hence, every axiom of ZFC belongs to $\mathcal{K}$.

Proposition 1. *If $\mathcal{T}$ denotes the set of twin primes, then the statement*

$$(\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) = \omega)) \wedge (\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) < \omega))$$

is true, falsifiable, and expresses what is currently unproved in mathematics.

Statement 1. *There exists a non-zero integer n such that*

$$(\neg\mathcal{K}(n<0)) \wedge (\neg\mathcal{K}(n>0)) \tag{1}$$

Proof. It holds for

$$n = \begin{cases} -1, & \text{if } Continuum\ Hypothesis \text{ holds} \\ 1, & \text{otherwise} \end{cases}$$

□

Proposition 2. *Statement 1 holds forever.*

Proof. Since $Continuum\ Hypothesis$ is independent from ZFC, conjunction (1) holds forever for the above n. □

Proposition 3. *Statement 1 does not express what is currently unproved in mathematics.*

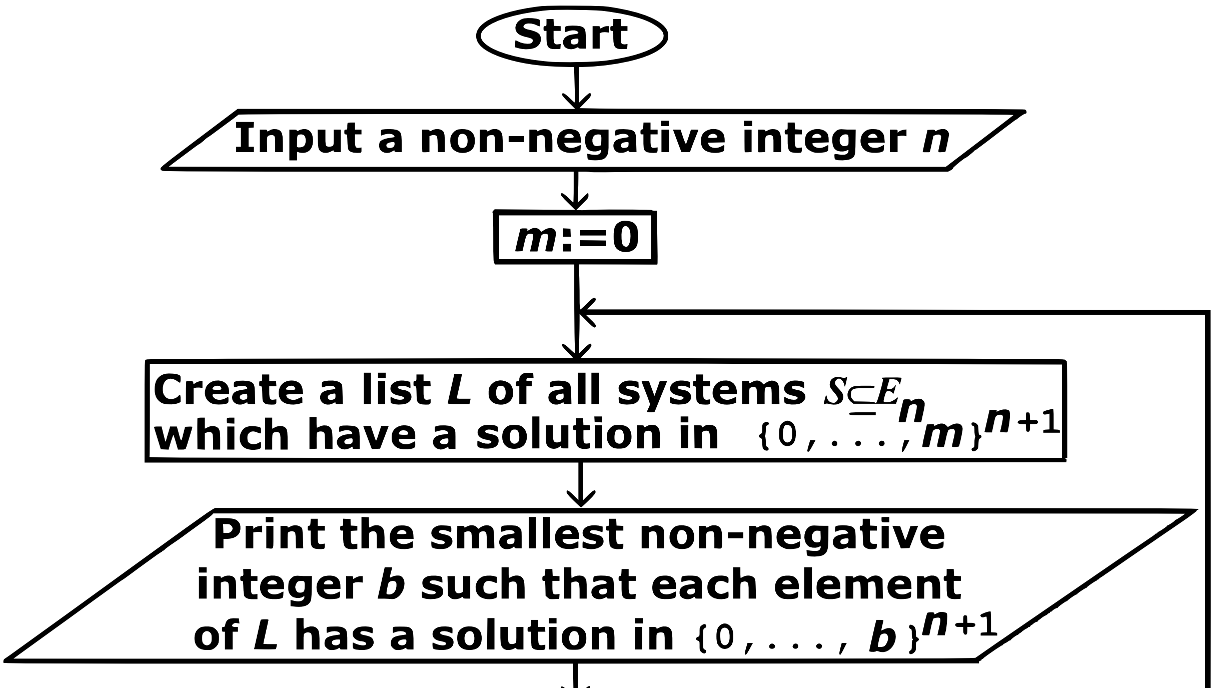
3 Classical computability theory

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i,j,k \in \{0,\ldots,n\}\}$$

Theorem 1. *([4, p. 118]). There exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0,\ldots,b\}^{n+1}$. The function $\beta_1 : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$, see [5]. Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit, see [5].



Flowchart 1

A semi-algorithm which computes $\beta_1(n)$ in the limit

Proposition 4. *If $k \in \mathbb{N}$, then the statement "the function $\mathbb{N} \ni n \to k + \beta_1(n) \in \mathbb{N}$ is uncomputable" belongs to $\mathcal{K}$.*

Conjecture 1. *([1, pp. 341–342], [2, p. 42], [3, p. 745]). Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ has a single-fold Diophantine representation.*

Let Φ denote the following statement: *the function $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \beta_1(n), & \text{otherwise} \end{cases}$$

The function $g : \mathbb{N} \to \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge (\neg\mathcal{K}(g \textit{ is computable})) \wedge (\neg\mathcal{K}(g \textit{ is uncomputable}))$$

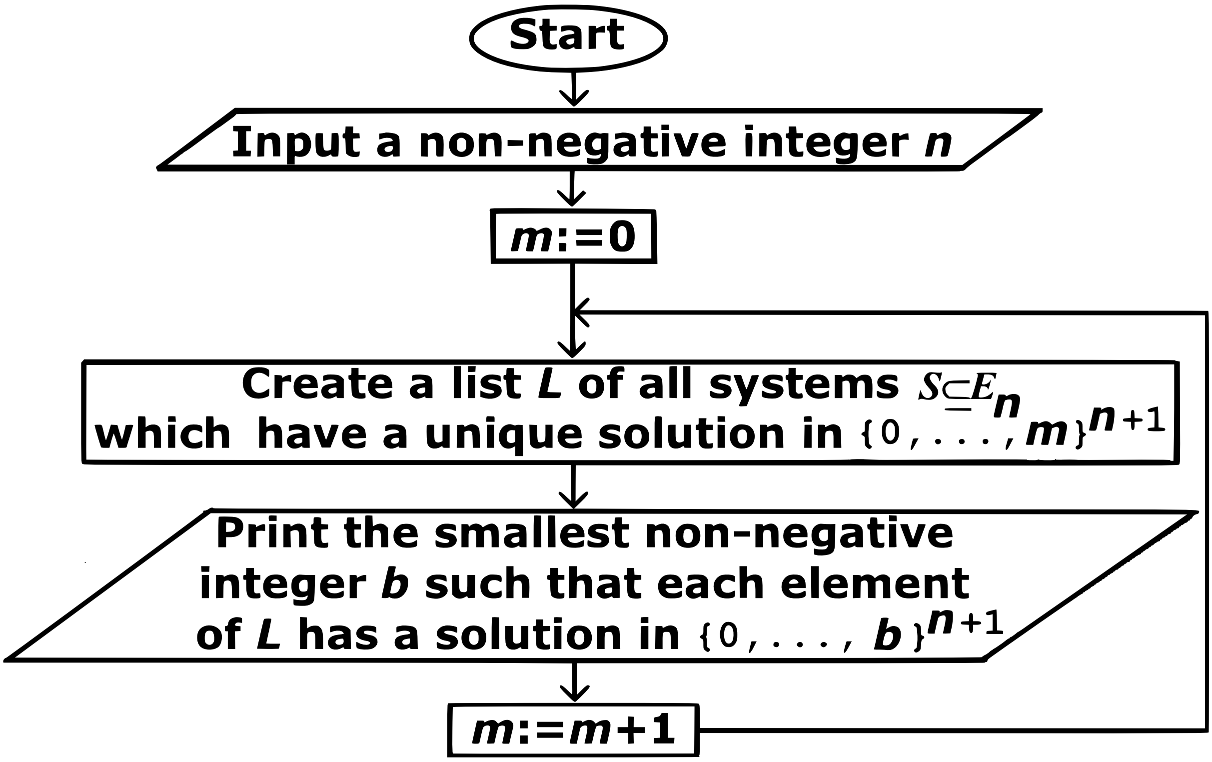
Lemma 1. *The function g is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$.

Theorem 2. *The function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [5]. The term *"dominated"* in the title of [5] means *"eventually dominated"*. Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit, see [5].



Flowchart 2

A semi-algorithm which computes $\beta(n)$ in the limit

□

4 Main results

Statement 2. *There exists a limit-computable function $f:\mathbb{N}\to\mathbb{N}$ of unknown computability which eventually dominates every function $\delta:\mathbb{N}\to\mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 2 follows constructively from Theorem 2 by taking $f=\beta$ and the following conjunction:

$$(\neg\mathcal{K}(\beta \text{ is computable})) \ \wedge\ (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

Statement 2 follows non-constructively from Lemma 1 by taking $f=g$ and the following conjunction:

$$(\neg\mathcal{K}(g \text{ is computable})) \ \wedge\ (\neg\mathcal{K}(g \text{ is uncomputable}))$$

□

Since the function β_1 in Theorem 1 is not computable, Statement 2 does not follow from Theorem 1.

Proposition 5. *Statement 2 significantly strengthens a non-trivial mathematical theorem. Statement 2 refers to the current mathematical knowledge and may be false in the future. Statement 2 does not express what is currently unproved in mathematics.*

Proof. Statement 2 strengthens Statement 2 without the epistemic condition. The weakened Statement 2 is a theorem which follows from Theorem 1. Statement 2 claims that there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that

(f is computable in the limit) $\wedge$ ($\neg\mathcal{K}$(f is computable)) $\wedge$ ($\neg\mathcal{K}$(f is uncomputable)) $\wedge$

(f eventually dominates every function $\delta:\mathbb{N}\to\mathbb{N}$ with a single-fold Diophantine representation)

Conjecture 1 disproves Statement 2. □

5 Predicate $\mathcal{K}$ of the written down mathematical knowledge

In this section, $\mathcal{K}$ denotes both the predicate satisfied by every written down theorem and the finite set of all written down theorems. It changes what is taken as known in mathematics.

Proposition 6. *Since $\mathcal{K}$ is finite, there exists $k\in\mathbb{N}$ such that the computability of the function*

$$\mathbb{N}\ni n\to k+\beta_1(n)\in\mathbb{N}$$

is unknown. For this k, Statement 2 holds when $f(n)=k+\beta_1(n)$.

Proposition 7 is of little use because Proposition 6 contradicts Proposition 4 with the right definition of $\mathcal{K}$.

Proposition 7. *ZFC expresses Statement 2 at any time.*

Proof. Let $\mathcal{K}=\{T_1,\ldots,T_n\}$. For $i\in\{1,\ldots,n\}$, let

$$A_i=\begin{cases}(f:\mathbb{N}\to\mathbb{N})\wedge T_i\wedge(f\neq g_i), & \text{if } T_i \text{ states that a function } g_i:\mathbb{N}\to\mathbb{N} \text{ is computable}\\ (f:\mathbb{N}\to\mathbb{N})\wedge T_i\wedge(f\neq h_i), & \text{if } T_i \text{ states that a function } h_i:\mathbb{N}\to\mathbb{N} \text{ is uncomputable}\\ f:\mathbb{N}\to\mathbb{N}, & \text{in other cases}\end{cases}$$

The conjunction $A_1\wedge\ldots\wedge A_n$ expresses that

$$(f:\mathbb{N}\to\mathbb{N})\wedge(\neg\mathcal{K}(f \text{ is computable})) \ \wedge\ (\neg\mathcal{K}(f \text{ is uncomputable}))$$

□

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Apoloniusz Tyszka
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: *rttyszka@cyf-kr.edu.pl*