A true and falsifiable statement Ψ with the predicate $\mathcal{K}$ of the current mathematical knowledge, where Ψ strengthens a theorem and does not express what is currently unproved in mathematics

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Abstract

The theorem of Royer and Case states that there exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$. We present an alternative proof of this theorem. $\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the set of all currently known theorems. The set $\mathcal{K}$ is time-dependent. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We prove: `(1)` there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation, `(2)` statement `(1)` strengthens a theorem. We present both constructive and non-constructive proof of `(1)`. Statement `(1)` claims that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that (f is computable in the limit) $\wedge$ ($\neg\mathcal{K}$(f is computable)) $\wedge$ ($\neg\mathcal{K}$(f is uncomputable)) $\wedge$ (f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation). Since Martin Davis' conjecture on single-fold Diophantine representations disproves statement `(1)`, statement `(1)` has all properties from the title of the article.

Key words and phrases: eventual domination, limit-computable function, predicate $\mathcal{K}$ of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate $\mathcal{K}$.

1 The goal of the article

$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the set of all currently known theorems. The set $\mathcal{K}$ is time-dependent. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We formulate statements of the form:

there exists a mathematical object $\mathcal{X}$ such that

$\mathcal{X}$ satisfies a mathematical condition $\mathcal{C}$ and

it is unknown whether or not $\mathcal{X}$ satisfies a mathematical condition $\mathcal{D}$.

We present a statement Ψ of the above form which has all properties from the title of the article.

2 Statements with the predicate $\mathcal{K}$ which do not have all properties from the title of the article

Let $\mathcal{T}$ denote the set of twin primes.

Proposition 1. *The statement*

$$(\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) = \omega)) \wedge (\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) < \omega))$$

is true, falsifiable, and expresses what is currently unproved in mathematics.

Statement 1. *There exists a non-zero integer n such that*

$$(\neg\mathcal{K}(n<0)) \wedge (\neg\mathcal{K}(n>0)) \tag{1}$$

Proof. It holds for

$$n = \begin{cases} -1, & \text{if } Continuum\ Hypothesis \text{ holds} \\ 1, & \text{otherwise} \end{cases}$$

□

Proposition 2. *Statement 1 holds forever.*

Proof. Since $Continuum\ Hypothesis$ is independent from ZFC, conjunction (1) holds forever for the above n. □

Proposition 3. *Statement 1 does not express what is currently unproved in mathematics.*

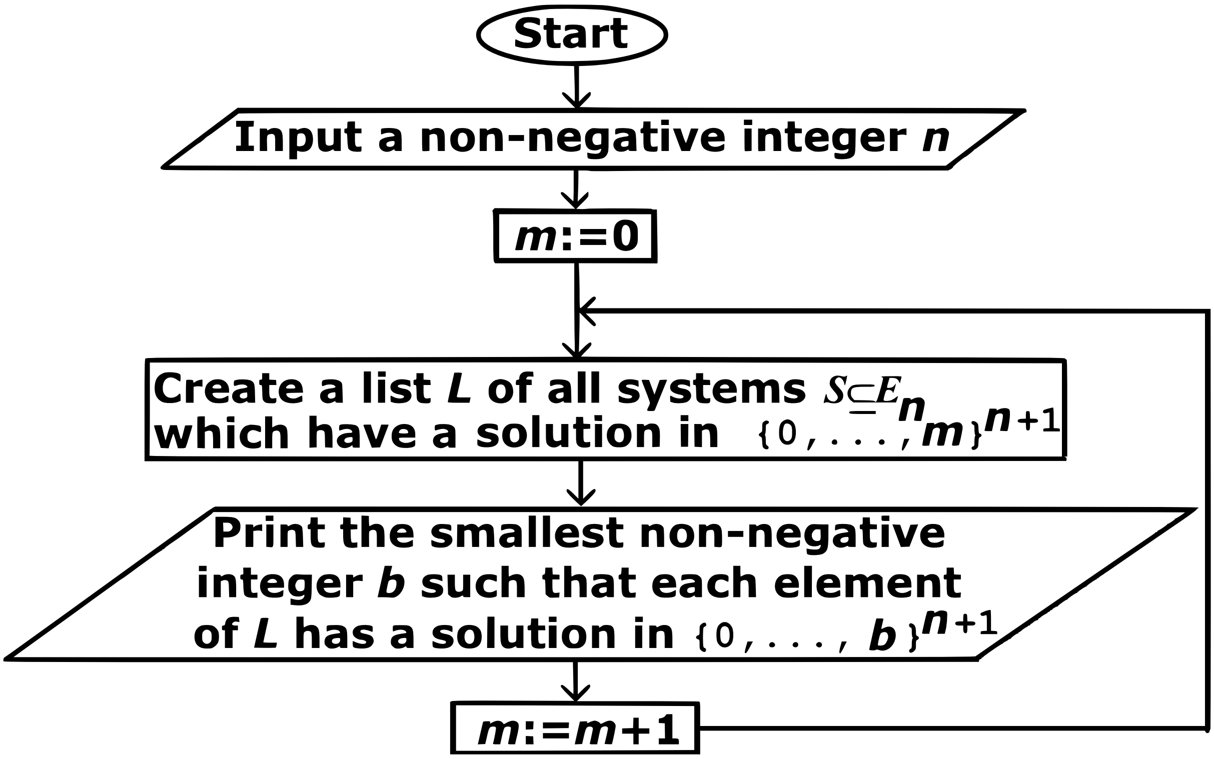
3 Classical computability theory

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i,j,k \in \{0,\dots,n\}\}$$

Theorem 1. *([4, p. 118]). There exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0,\dots,b\}^{n+1}$. The function $\beta_1 : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$, see [5]. Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit.



Flowchart 1

A semi-algorithm which computes $\beta_1(n)$ in the limit

Conjecture 1. *([1, pp. 341–342], [2, p. 42], [3, p. 745]). Every listable set* $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ *has a single-fold Diophantine representation.*

Let Φ denote the following statement: *the function* $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ *eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \beta_1(n), & \text{otherwise} \end{cases}$$

The function $g : \mathbb{N} \to \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge (\neg\mathcal{K}(g \textit{ is computable})) \wedge (\neg\mathcal{K}(g \textit{ is uncomputable}))$$

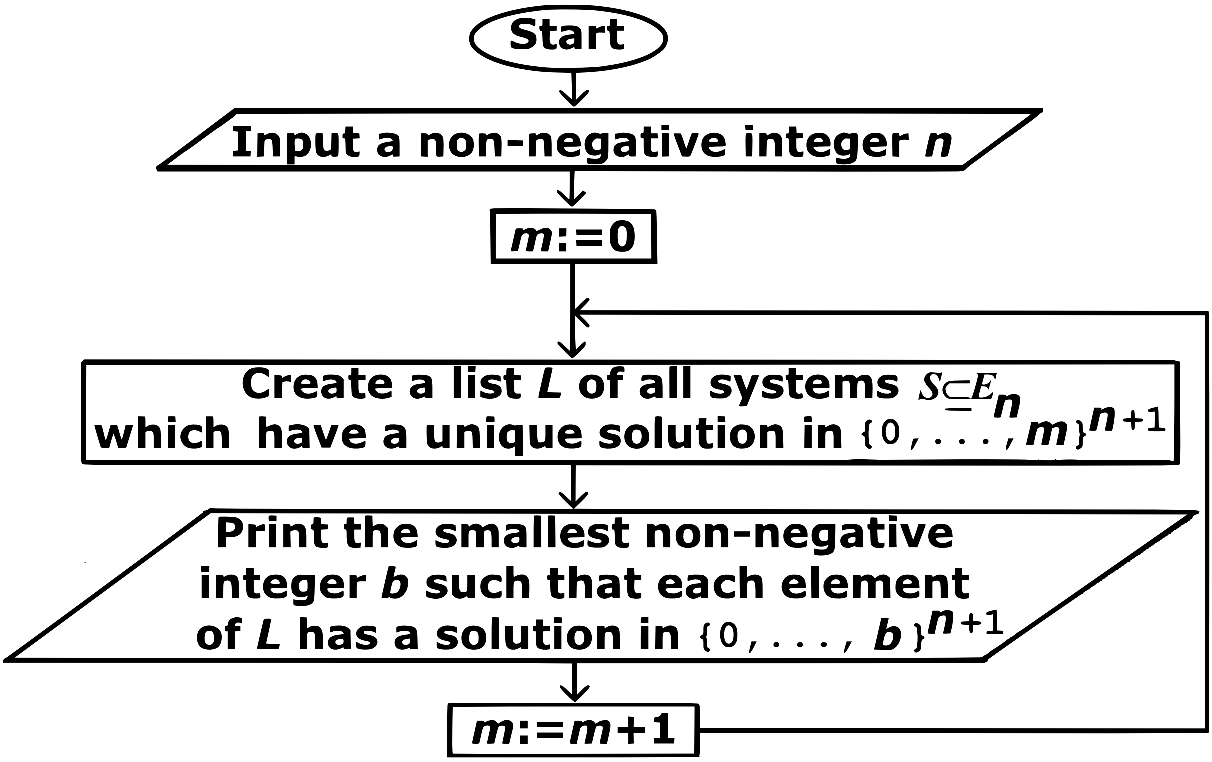
Lemma 1. *The function* g *is computable in the limit and eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$.

Theorem 2. *The function* $\beta : \mathbb{N} \to \mathbb{N}$ *is computable in the limit and eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. This is proved in [5]. The term *"dominated"* in the title of [5] means *"eventually dominated"*. Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 2

A semi-algorithm which computes $\beta(n)$ in the limit

□

4 A statement described by the title of the article

Statement 2. *There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 2 follows constructively from Theorem 2 by taking $f = \beta$ and the following conjunction:

$$(\neg\mathcal{K}(\beta \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

Statement 2 follows non-constructively from Lemma 1 by taking $f = g$ and the following conjunction:

$$(\neg\mathcal{K}(g \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(g \text{ is uncomputable}))$$

□

Proposition 4. *Statement 2 has all properties from the title of the article.*

Proof. Statement 2 claims that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

(f is computable in the limit) $\wedge$ ($\neg\mathcal{K}$(f is computable)) $\wedge$ ($\neg\mathcal{K}$(f is uncomputable)) $\wedge$

(f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation)

Conjecture 1 disproves Statement 2. Statement 2 without the epistemic condition is a theorem. □

Since the function β_1 in Theorem 1 is not computable, Statement 2 does not follow from Theorem 1. Ignoring the epistemic condition in Statement 2, Statement 2 follows from Theorem 1 by taking $f = \beta_1$.

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