A true and falsifiable statement Ψ with the predicate $\mathcal{K}$ of the current mathematical knowledge, where Ψ non-trivially strengthens a non-trivial theorem and does not express what is currently unproved in mathematics

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Abstract

We present a new constructive proof of the following theorem: there exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$. $\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the set of all currently known theorems. The set $\mathcal{K}$ is time-dependent. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We prove: `(1)` there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation. We present both constructive and non-constructive proof of `(1)`. Statement `(1)` claims that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that (f is computable in the limit) $\wedge$ ($\neg\mathcal{K}$(f is computable)) $\wedge$ ($\neg\mathcal{K}$(f is uncomputable)) $\wedge$ (f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation). Since Martin Davis' conjecture on single-fold Diophantine representations disproves statement `(1)`, statement `(1)` has all properties from the title of the article.

Key words and phrases: eventual domination, limit-computable function, predicate $\mathcal{K}$ of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate $\mathcal{K}$.

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1 The goal of the article

We formulate a statement Ψ of the form:

there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

f satisfies a mathematical condition $\mathcal{C}$ and

it is unknown whether or not f satisfies a mathematical condition $\mathcal{D}$.

We prove that Ψ has all properties from the title of the article. There is no widely known theorem from which we can draw Ψ. Ignoring the epistemic condition in Ψ, Ψ follows from a known theorem.

2 Statements with the predicate $\mathcal{K}$ which do not have all properties from the title of the article

Let $\mathcal{T}$ denote the set of twin primes.

Proposition 1. *The statement*

$$(\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) = \omega)) \wedge (\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) < \omega))$$

is true, falsifiable, and expresses what is currently unproved in mathematics.

Statement 1. *There exists a non-zero integer n such that*

$$(\neg\mathcal{K}(n < 0)) \wedge (\neg\mathcal{K}(n > 0)) \tag{1}$$

Proof. It holds for

$$n = \begin{cases} -1, & \text{if } Continuum\ Hypothesis \text{ holds} \\ 1, & \text{otherwise} \end{cases}$$

□

Proposition 2. *Statement 1 holds forever.*

Proof. Since $Continuum\ Hypothesis$ is independent from ZFC, conjunction (1) holds forever for the above n. □

Proposition 3. *Statement 1 does not express what is currently unproved in mathematics.*

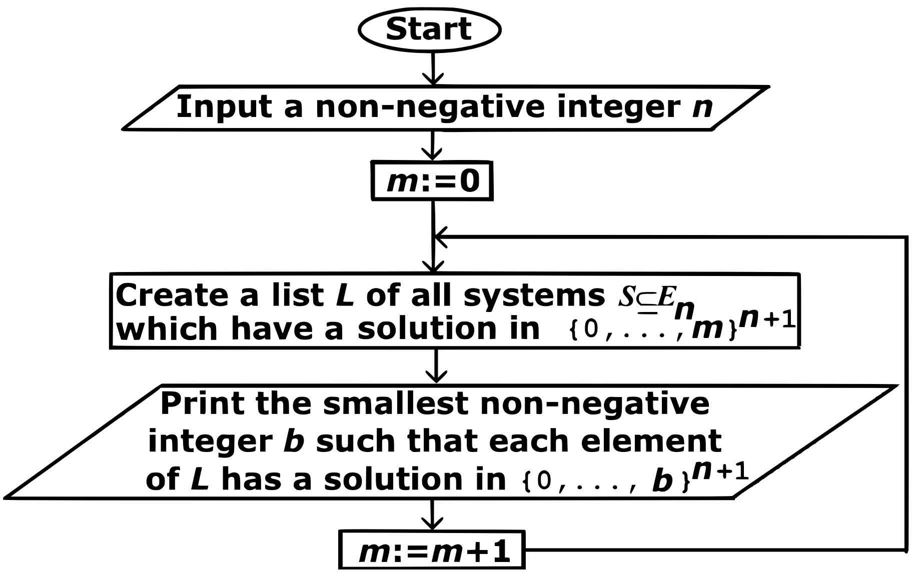
3 Classical computability theory

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$$

Theorem 1. *([4, p. 118]). There exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The function $\beta_1 : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$, see [5]. Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit.



Flowchart 1

A semi-algorithm which computes $\beta_1(n)$ in the limit

Conjecture 1. *([1, pp. 341–342], [2, p. 42], [3, p. 745]). Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ has a single-fold Diophantine representation.*

Let Φ denote the following statement: *the function $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \beta_1(n), & \text{otherwise} \end{cases}$$

The function $g : \mathbb{N} \to \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge (\neg\mathcal{K}(g \textit{ is computable})) \wedge (\neg\mathcal{K}(g \textit{ is uncomputable}))$$

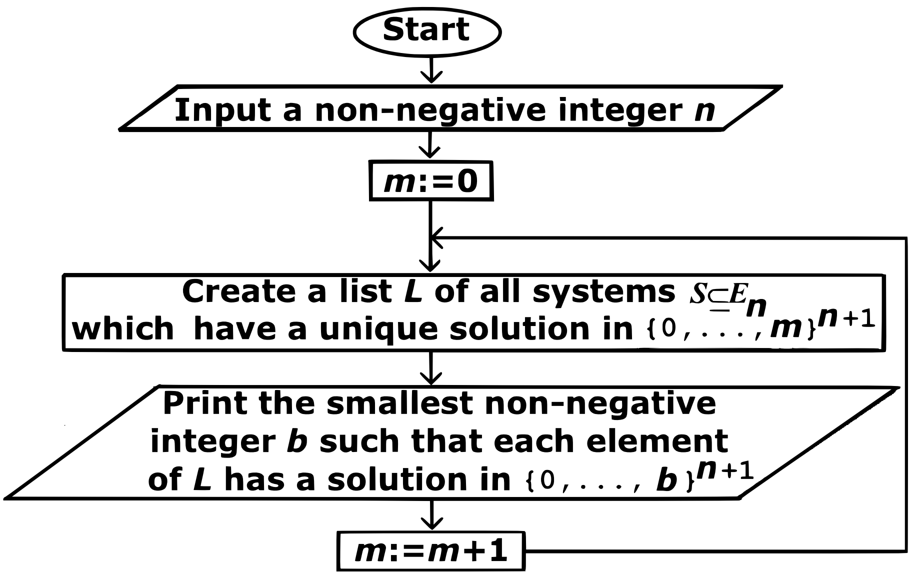
Lemma 1. *The function g is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$.

Theorem 2. *The function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [5]. The term *"dominated"* in the title of [5] means *"eventually dominated"*. Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 2

A semi-algorithm which computes $\beta(n)$ in the limit

□

4 A statement described by the title of the article

Statement 2. *There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 2 follows constructively from Theorem 2 by taking $f = \beta$ and the following conjunction:

$$(\neg\mathcal{K}(\beta \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

Statement 2 follows non-constructively from Lemma 1 by taking $f = g$ and the following conjunction:

$$(\neg\mathcal{K}(g \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(g \text{ is uncomputable}))$$

□

Since the function β_1 in Theorem 1 is not computable, Statement 2 does not follow from Theorem 1

Proposition 4. *Statement 2 has all properties from the title of the article.*

Proof. Statement 2 claims that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$(f \text{ is computable in the limit}) \wedge (\neg\mathcal{K}(f \text{ is computable})) \wedge (\neg\mathcal{K}(f \text{ is uncomputable})) \wedge$$

$$(f \text{ eventually dominates every function } \delta : \mathbb{N} \to \mathbb{N} \text{ with a single-fold Diophantine representation})$$

Conjecture 1 disproves Statement 2. Statement 2 strengthens Statement 2 without the epistemic condition. The weakened Statement 2 is a theorem which follows from Theorem 1. □

For a mathematical statement Θ, let $\mathcal{U}(\Theta)$ denote $(\neg\mathcal{K}(\Theta)) \wedge (\neg\mathcal{K}(\neg\Theta))$. The predicate $\mathcal{U}$ is time-dependent. $\mathcal{U}(\Theta)$ states that it is currently unknown whether or not Θ holds. Hence, $\mathcal{U}(\Theta)$ holds forever when Θ is independent from ZFC. Statement 2 provides a falsifiable statement of the form

$$\exists f : \mathbb{N} \to \mathbb{N} \ ((f \text{ satisfies a mathematical condition } \mathcal{C}) \wedge$$

$$\mathcal{U}(f \text{ satisfies a mathematical condition } \mathcal{D}))$$

which non-trivially strengthens a non-trivial theorem.

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