A true and falsifiable statement Ψ with the predicate $\mathcal{K}$ of the current mathematical knowledge, where Ψ does not express what is proved or unproved in mathematics without $\mathcal{K}$

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Abstract. The theorem of Royer and Case states that there exists a limit-computable function $\beta_1\colon \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1\colon \mathbb{N} \to \mathbb{N}$. We present an alternative proof of this theorem. $\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We prove: (1) there exists a limit-computable function $f\colon \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta\colon \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation. We present both constructive and non-constructive proof of (1). Statement (1) claims that there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that (f is computable in the limit) $\wedge$ ($\neg\mathcal{K}$(f is computable)) $\wedge$ ($\neg\mathcal{K}$(f is uncomputable)) $\wedge$ (f eventually dominates every function $\delta\colon \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation). Since Martin Davis' conjecture on single-fold Diophantine representations disproves Statement (1), Statement (1) justifies the title of the article.

Key words and phrases: eventual domination, limit-computable function, predicate $\mathcal{K}$ of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate $\mathcal{K}$.

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$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. For $n \in \mathbb{N}$, let

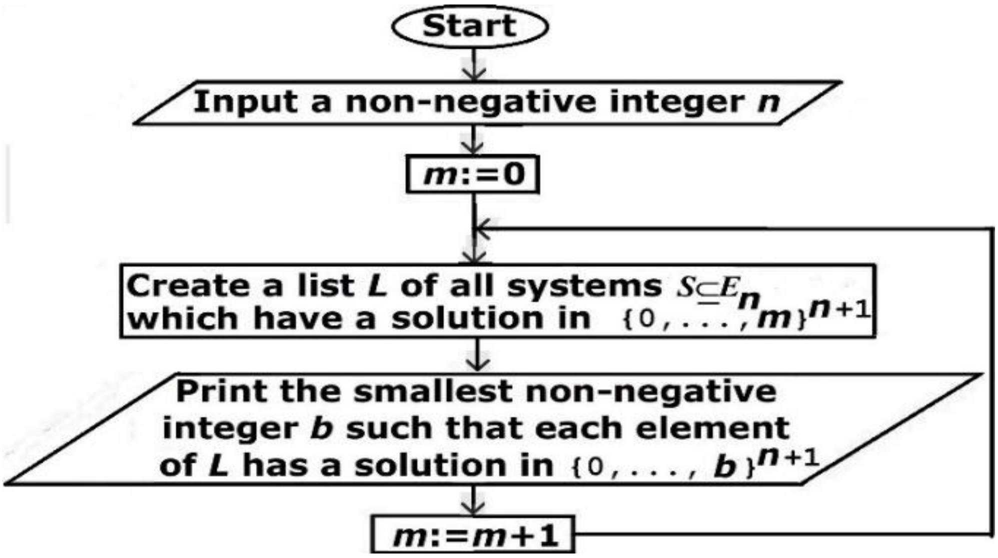
$$E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k \colon i, j, k \in \{0, \dots, n\}\}$$

1. Classical mathematics

Theorem 1. *([4, p. 118]). There exists a limit-computable function* $\beta_1\colon \mathbb{N} \to \mathbb{N}$ *which eventually dominates every computable function* $\delta_1\colon \mathbb{N} \to \mathbb{N}$.

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \dots, b\}^{n+1}$. The function $\beta_1\colon \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1\colon \mathbb{N} \to \mathbb{N}$, see [5].

Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit.



Flowchart 1
A semi-algorithm which computes $\beta_1(n)$ in the limit

Conjecture 1. *([1, pp. 341-342], [2, p. 42], [3, p. 745]). Every listable set* $\mathcal{X} \subseteq \mathbb{N}^k$ ($k \in \mathbb{N}\setminus\{0\}$) *has a single-fold Diophantine representation.*

Let Φ denote the following statement: *the function* $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ *eventually dominates every function* $\delta\colon \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & if\ \Phi\ holds \\ \beta_1(n), & \mathrm{otherwise} \end{cases}$$

The function $g\colon \mathbb{N} \to \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge \big(\neg\mathcal{K}(g\ is\ computable)\big) \wedge \big(\neg\mathcal{K}(g\ is\ uncomputable)\big)$$

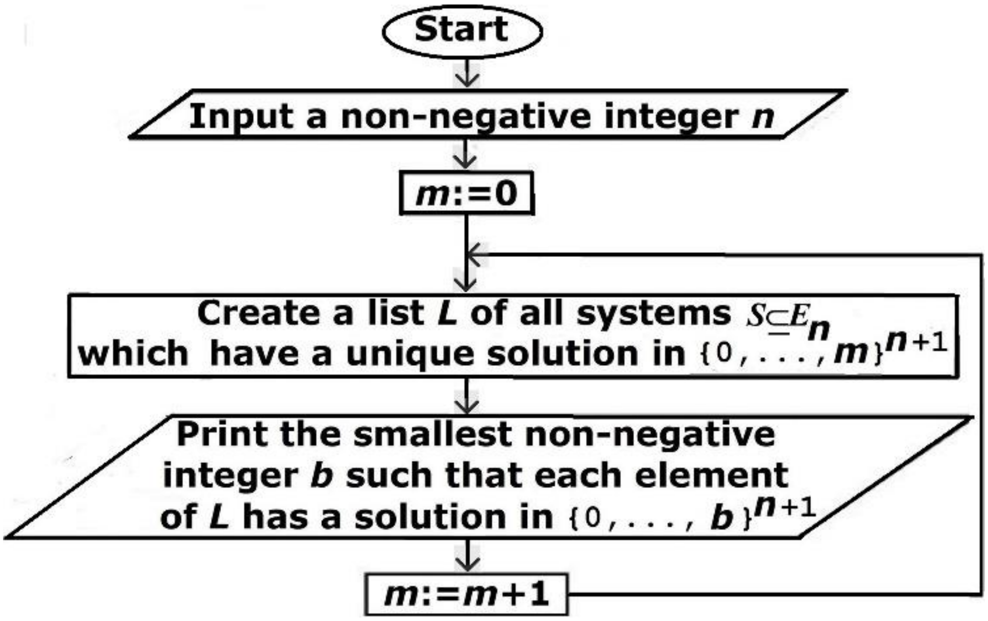
Lemma 1. *The function* g *is computable in the limit and eventually dominates every function* $\delta\colon \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \dots, b\}^{n+1}$.

Theorem 2. *The function* $\beta\colon \mathbb{N} \to \mathbb{N}$ *is computable in the limit and eventually dominates every function* $\delta\colon \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. This is proved in [5]. The term *"dominated"* in the title of [5] means *"eventually dominated"*. Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 2
A semi-algorithm which computes $\beta(n)$ in the limit □

2. Mathematics with the predicate $\mathcal{K}$ of the current mathematical knowledge

Let $\boldsymbol{\Gamma}$ denote the following true statement: *β is computable in the limit, the computability of β is unknown, and β eventually dominates every function $\delta\colon \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proposition 1. *The statement $\boldsymbol{\Gamma}$ does not justify the title of the article.*

Proof. The statement $\boldsymbol{\Gamma}$ expresses that

$$\mathcal{K}(\beta \textit{ is computable in the limit}) \wedge$$
$$\big(\neg\mathcal{K}(\beta \textit{ is computable})\big) \wedge \big(\neg\mathcal{K}(\beta \textit{ is uncomputable})\big) \wedge$$
$$\mathcal{K}(\beta \textit{ eventually dominates every function } \delta\colon \mathbb{N} \to \mathbb{N} \textit{ with a single-fold Diophantine representation}) \; \square$$

The analogous proposition holds for the function g instead of β.

Statement 1. *There exists a limit-computable function $f\colon \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta\colon \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 1 follows constructively from Theorem 2 by taking $f = \beta$ and the following conjunction:

$$\left(\neg\mathcal{K}(\beta \text{ is computable})\right) \wedge \left(\neg\mathcal{K}(\beta \text{ is uncomputable})\right)$$

Statement 1 follows non-constructively from Lemma 1 by taking $f = g$ and the following conjunction:

$$\left(\neg\mathcal{K}(g \text{ is computable})\right) \wedge \left(\neg\mathcal{K}(g \text{ is uncomputable})\right) \square$$

Proposition 2. *Statement 1 justifies the title of the article.*

Proof. Statement 1 claims that there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that (f is computable in the limit) $\wedge$ ($\neg\mathcal{K}(f$ is computable)) $\wedge$ ($\neg\mathcal{K}(f$ is uncomputable)) $\wedge$ (f eventually dominates every function $\delta\colon \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation). Conjecture 1 disproves Statement 1. $\square$

Since the function β_1 in Theorem 1 is not computable, Statement 1 does not follow from Theorem 1. Ignoring the epistemic condition in Statement 1, Statement 1 follows from Theorem 1 by taking $f = \beta_1$.

In [7], the author showed that the predicate $\mathcal{K}$ non-trivially extends constructive mathematics, see also [6], [8], [9].

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