On true and falsifiable statements Ψ with the predicate $\mathcal{K}$ of the current mathematical knowledge, where Ψ does not express what is proved or unproved in mathematics without $\mathcal{K}$

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Abstract

The theorem of Royer and Case states that there exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$. We present an alternative proof of this theorem. $\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We prove: `(1)` there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation. We present both constructive and non-constructive proof of `(1)`. Statement `(1)` claims that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that (f is computable in the limit) $\wedge$ ($\neg\mathcal{K}$(f is computable)) $\wedge$ ($\neg\mathcal{K}$(f is uncomputable)) $\wedge$ (f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation). Since Martin Davis' conjecture on single-fold Diophantine representations disproves Statement `(1)`, Statement `(1)` has all properties from the title of the article. Let Λ denote the statement: *there exists a positive integer b such that it is unknown whether or not the equation $x! + b = y^2$ has infinitely many solutions in positive integers x and y*. We prove that Λ has all properties from the title of the article.

Key words and phrases: Diophantine equation $x! + b = y^2$, eventual domination, limit-computable function, predicate $\mathcal{K}$ of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate $\mathcal{K}$.

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$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent and publicly available. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. In [8], the author showed that the predicate $\mathcal{K}$ non-trivially extends constructive mathematics, see also [7], [9], [10].

1 Statements with the predicate $\mathcal{K}$ which do not have all properties from the title of the article

Let $\mathcal{T}$ denote the set of twin primes.

Proposition 1. *The statement*

$$(\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) = \omega)) \wedge (\neg\mathcal{K}(\mathrm{card}(\mathcal{T}) < \omega))$$

is true, falsifiable, and expresses what is unproved in mathematics without $\mathcal{K}$.

Statement 1. *There exists a non-zero integer n such that*

$$(\neg\mathcal{K}(n < 0)) \wedge (\neg\mathcal{K}(n > 0)) \tag{1}$$

Proof. It holds for

$$n = \begin{cases} -1, & \text{if } Continuum\ Hypothesis \text{ holds} \\ 1, & \text{otherwise} \end{cases}$$

□

Proposition 2. *Statement 1 holds forever.*

Proof. Since $Continuum\ Hypothesis$ is independent from ZFC, conjunction (1) holds forever for the above n. □

Proposition 3. *Statement 1 does not express what is proved or unproved in mathematics without $\mathcal{K}$.*

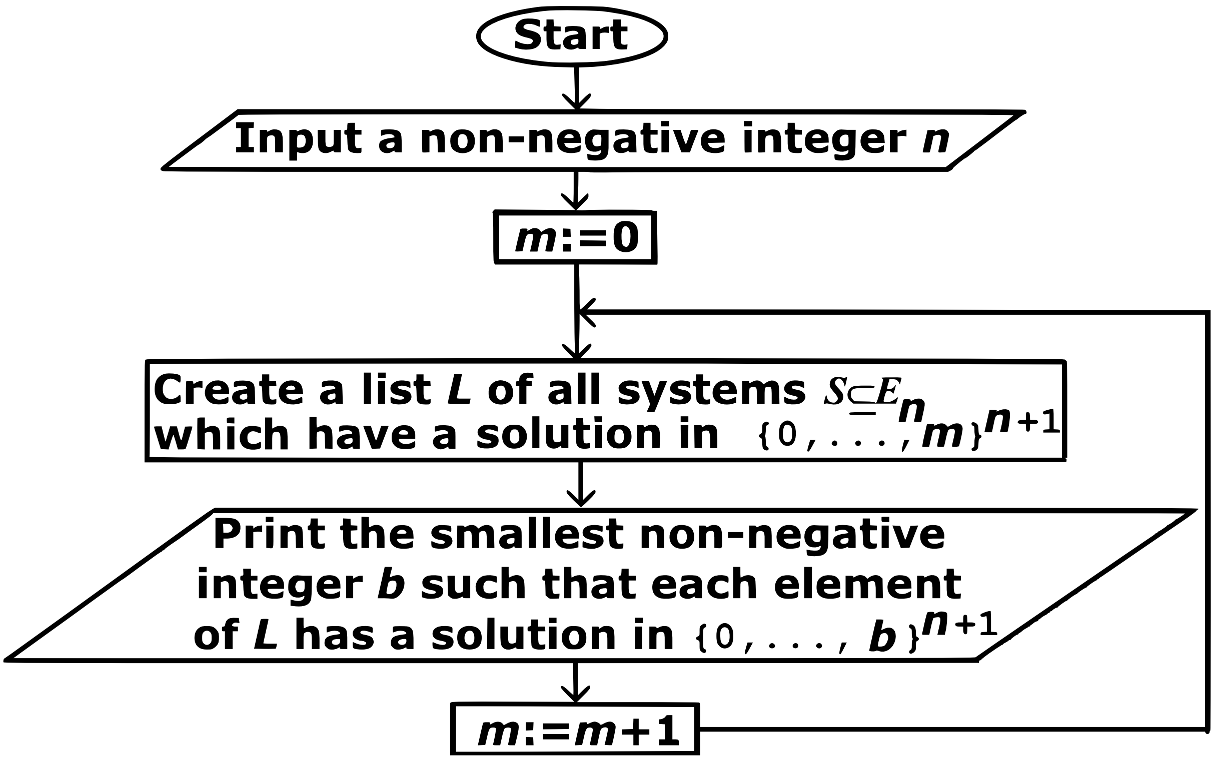
2 Classical computability theory

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$$

Theorem 1. *([5, p. 118]). There exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For every $n \in \mathbb{N}$, we define $\beta_1(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The function $\beta_1 : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$, see [6]. Flowchart 1 describes a semi-algorithm which computes $\beta_1(n)$ in the limit.



Flowchart 1

A semi-algorithm which computes $\beta_1(n)$ in the limit

Conjecture 1. *([1, pp. 341–342], [3, p. 42], [4, p. 745]). Every listable set* $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ *has a single-fold Diophantine representation.*

Let Φ denote the following statement: *the function* $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ *eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \beta_1(n), & \text{otherwise} \end{cases}$$

The function $g : \mathbb{N} \to \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge (\neg\mathcal{K}(g \textit{ is computable})) \wedge (\neg\mathcal{K}(g \textit{ is uncomputable}))$$

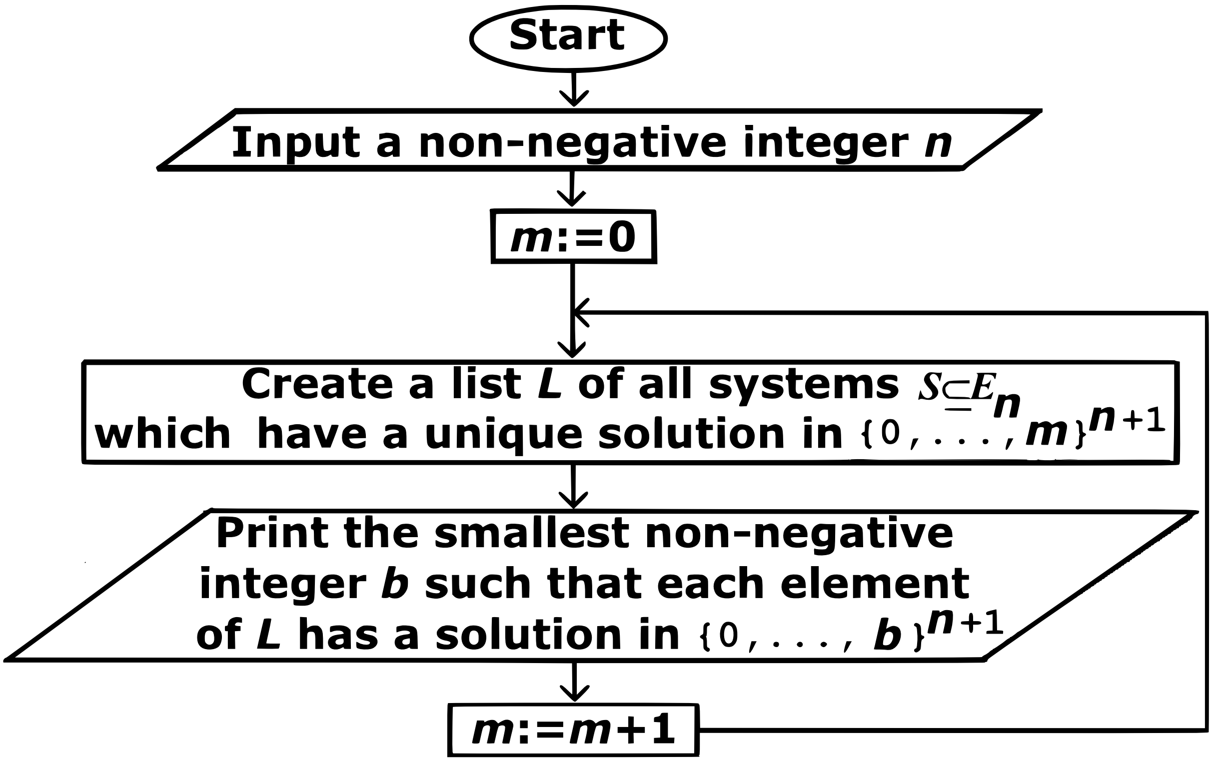
Lemma 1. *The function* g *is computable in the limit and eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For every $n \in \mathbb{N}$, we define $\beta(n)$ as the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$.

Theorem 2. *The function* $\beta : \mathbb{N} \to \mathbb{N}$ *is computable in the limit and eventually dominates every function* $\delta : \mathbb{N} \to \mathbb{N}$ *with a single-fold Diophantine representation.*

Proof. This is proved in [6]. The term *"dominated"* in the title of [6] means *"eventually dominated"*. Flowchart 2 describes a semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 2

A semi-algorithm which computes $\beta(n)$ in the limit

□

3 Statements described by the title of the article

Statement 2. *There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 2 follows constructively from Theorem 2 by taking $f = \beta$ and the following conjunction:

$$(\neg\mathcal{K}(\beta \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

Statement 2 follows non-constructively from Lemma 1 by taking $f = g$ and the following conjunction:

$$(\neg\mathcal{K}(g \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(g \text{ is uncomputable}))$$

□

Proposition 4. *Statement 2 has all properties from the title of the article.*

Proof. Statement 2 claims that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$(f$ is computable in the limit$) \wedge (\neg\mathcal{K}(f$ is computable$)) \wedge (\neg\mathcal{K}(f$ is uncomputable$)) \wedge$

$(f$ eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation$)$

Conjecture 1 disproves Statement 2. □

Since the function β_1 in Theorem 1 is not computable, Statement 2 does not follow from Theorem 1. Ignoring the epistemic condition in Statement 2, Statement 2 follows from Theorem 1 by taking $f = \beta_1$.

Statement 3. *There exists a positive integer b such that it is unknown whether or not the equation $x! + b = y^2$ has infinitely many solutions in positive integers x and y.*

Proof. It holds for $b = 1$, see [2]. □

Proposition 5. *Statement 3 has all properties from the title of the article.*

Proof. Statement 3 claims that there exists a positive integer b such that

$\neg\mathcal{K}($the equation $x! + b = y^2$ has infinitely many solutions in positive integers x and $y) \wedge$

$\neg\mathcal{K}($the equation $x! + b = y^2$ has at most finitely many solutions in positive integers x and $y)$

A weak form of Szpiro's conjecture implies that the equation $x! + b = y^2$ has at most finitely many solutions in positive integers x and y when $b \in \mathbb{N} \setminus \{0\}$, see [2]. □

References

[1] M. Davis, Yu. Matiyasevich, J. Robinson, *Hilbert's tenth problem, Diophantine equations: positive aspects of a negative solution;* in: Mathematical developments arising from Hilbert problems (ed. F. E. Browder), Proc. Sympos. Pure Math., vol. 28, Part 2, Amer. Math. Soc., Providence, RI, 1976, 323–378, *http://doi.org/10.1090/pspum/028.2*; reprinted in: The collected works of Julia Robinson (ed. S. Feferman), Amer. Math. Soc., Providence, RI, 1996, 269–324.

[2] A. Dąbrowski, *On the Diophantine equation* $x! + A = y^2$, Nieuw Arch. Wisk. IV. Ser. 14 (1996), no. 3, 321–324.

[3] Yu. Matiyasevich, *Hilbert's tenth problem: what was done and what is to be done,* in: Proceedings of the Workshop on Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), Contemp. Math. 270, Amer. Math. Soc., Providence, RI, 2000, 1–47, *http://doi.org/10.1090/conm/270*.

[4] Yu. Matiyasevich, *Towards finite-fold Diophantine representations,* J. Math. Sci. (N. Y.) vol. 171, no. 6, 2010, 745–752, *http://doi.org/10.1007%2Fs10958-010-0179-4*.

[5] J. S. Royer and J. Case, *Subrecursive Programming Systems: Complexity and Succinctness,* Birkhäuser, Boston, 1994.

[6] A. Tyszka, *All functions $g:\mathbb{N} \to \mathbb{N}$ which have a single-fold Diophantine representation are dominated by a limit-computable function $f:\mathbb{N} \setminus \{0\} \to \mathbb{N}$ which is implemented in MuPAD and whose computability is an open problem*, in: Computation, cryptography, and network security (eds. N. J. Daras, M. Th. Rassias), Springer, Cham, 2015, 577–590, *http://doi.org/10.1007/978-3-319-18275-9_24*.

[7] A. Tyszka, *Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that refer to the current knowledge on $\mathcal{X}$,* Journal of Applied Computer Science $\&$ Mathematics 16 (2022), no. 2, 31–35, *http://doi.org/10.4316/JACSM.202202005*.

[8] A. Tyszka, *In constructive and informal mathematics, in contradistinction to any empirical science, the predicate of the current knowledge in the subject is necessary,* Asian Research Journal of Mathematics 19 (2023), no. 12, 69–79, *http://doi.org/10.9734/arjom/2023/v19i12773*.

[9] A. Tyszka, *Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$,* Pi Mu Epsilon J. 15 (2023), no. 8, 493–504.

[10] A. Tyszka, *Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that contain informal notions and refer to the current knowledge on $\mathcal{X}$,* Creat. Math. Inform. 32 (2023), no. 2, 247–253, *http://semnul.com/creative-mathematics/wp-content/uploads/2023/07/creative_2023_32_2_247_253.pdf*.

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