Are there a naturally defined set $X \subseteq \mathbb{N}$ and a constructively defined integer $n$ such that $(\text{card}(X) < \omega \Rightarrow X \subseteq (\neg\infty, n)]) \land (\text{the infiniteness of } X \text{ is conjectured and cannot be decided by any known method}) \land (\text{a constructively defined algorithm decides } X)$?

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Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau’s conjecture states that the set $P_{n^2 + 1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$: $\text{card}(P_{n^2 + 1}) < \omega \Rightarrow P_{n^2 + 1} \subseteq \{2, f(7)\}$. Let $B$ denote the system of equations:

$$\{x_i! = x_k : i, k \in \{1, \ldots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\}. \text{ We write down a system } \mathcal{U} \subseteq B \text{ of } 9 \text{ equations which has exactly two solutions in positive integers } x_1, \ldots, x_9, \text{ namely } (1, \ldots, 1) \text{ and } (f(1), \ldots, f(9)). \text{ We write down a system } \mathcal{A} \subseteq B \text{ of } 8 \text{ equations. Let } \Lambda \text{ denote the statement: } \text{if the system } \mathcal{A} \text{ has at most finitely many solutions in positive integers } x_1, \ldots, x_9, \text{ then each such solution } (x_1, \ldots, x_9) \text{ satisfies } x_1, \ldots, x_9 \leq f(9). \text{ The statement } \Lambda \text{ is equivalent to the statement } \Phi. \text{ This proof does not yield that } \text{card}(P_{n^2 + 1}) = \omega. \text{ Algorithms always terminate. We explain the distinction between existing algorithms } (\text{i.e. algorithms whose existence is provable in ZFC}) \text{ and known algorithms } (\text{i.e. algorithms whose definition is constructive and currently known to us}). \text{ A definition of an integer } n \text{ is called constructive, if it provides a known algorithm with no input that returns } n. \text{ Conditions } (1) \text{--} (5) \text{ concern sets } X \subseteq \mathbb{N}. \text{ (1) There are many elements of } X \text{ and it is conjectured that } X \text{ is infinite. (2) No known algorithm with no input returns the logical value of the statement } \text{card}(X) = \omega. \text{ (3) A known algorithm for every } k \in \mathbb{N} \text{ decides whether or not } k \in X. \text{ (4) A known algorithm with no input returns an integer } n \text{ satisfying } \text{card}(X) < \omega \Rightarrow X \subseteq (\neg\infty, n]. \text{ (5) } X \text{ is naturally defined } \text{i.e. } X \text{ has the simplest definition among known sets } Y \subseteq \mathbb{N} \text{ with the same set of known elements. The set } X = \{k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7), k) \cap P_{n^2 + 1} \neq \emptyset\} \text{ satisfies conditions } (1) \text{--} (4). \text{ No set } X \subseteq \mathbb{N} \text{ will satisfy conditions } (1) \text{--} (4) \text{ forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. The statement } \Phi \text{ implies that conditions } (1) \text{--} (5) \text{ hold for } X = P_{n^2 + 1}.

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1
1. Definitions and the distinction between existing algorithms and known algorithms

Algorithms always terminate. Semi-algorithms may not terminate. Examples [1] and the proof of Statement 1 explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known to us). A definition of an integer \( n \) is called constructive, if it provides a known algorithm with no input that returns \( n \).

**Definition 1.** Conditions (1)–(5) concern sets \( X \subseteq \mathbb{N} \).

1. There are many elements of \( X \) and it is conjectured that \( X \) is infinite.
2. No known algorithm with no input returns the logical value of the statement \( \text{card}(X) = \omega \).
3. A known algorithm for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \).
4. A known algorithm with no input returns an integer \( n \) satisfying \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \).
5. \( X \) is naturally defined i.e. \( X \) has the simplest definition among known sets \( Y \subseteq \mathbb{N} \) with the same set of known elements.

Condition (2) implies that no known proof shows the finiteness/infiniteness of \( X \).

**Definition 2.** Let \( \beta = (((24!)!)!)! \).

**Lemma 1.** \( \log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298 \).

**Proof.** We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com) \( \square \)

Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2 + 1} \) of primes of the form \( n^2 + 1 \) is infinite, see [6]–[8]. Let \( [\cdot] \) denote the integer part function.

**Example 1.** The set \( X = \mathcal{P}_{n^2 + 1} \) satisfies condition (2).

**Example 2.** The set \( X = \begin{cases} \mathbb{N}, & \text{if } \lceil \frac{\beta}{\pi} \rceil \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \) does not satisfy condition (2) because we know an algorithm with no input that computes \( \lceil \frac{\beta}{\pi} \rceil \).

**Example 3.** ([1], [4], [5, p. 9]). The function
\[
\mathbb{N} \ni n \mapsto h \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}
\]
is computable because \( h = \mathbb{N} \times \{1\} \) or there exists \( k \in \mathbb{N} \) such that
\[
h = ((0, \ldots, k) \times \{1\}) \cup ((k + 1, k + 2, k + 3, \ldots) \times \{0\})
\]
No known algorithm computes the function \( h \).

**Example 4.** The set
\[
X = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis is true} \\ 0, & \text{otherwise} \end{cases}
\]
is decidable. No constructively existing algorithm decides \( X \), which holds forever.
Definition 3. Let $\Phi$ denote the following unproven statement:

$$\text{card}(P_{n^2+1}) < \omega \Rightarrow P_{n^2+1} \subseteq [2, \beta]$$

Landau’s conjecture implies the statement $\Phi$. In Section 4, we heuristically prove the statement $\Phi$. This proof does not yield that $\text{card}(P_{n^2+1}) = \omega$.

**Statement 1.** Condition (4) fails for $X = P_{n^2+1}$.

*Proof.* For every set $X \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(X)$ with no input that returns

$$n = \begin{cases} 
0, & \text{if } \text{card}(X) \in \{0, \omega\} \\
\max(X), & \text{otherwise}
\end{cases}$$

This $n$ satisfies the implication in condition (4), but the algorithm $\text{Alg}(P_{n^2+1})$ is unknown for us because its definition is ineffective. □

Proving the statement $\Phi$ will disprove Statement 1. Statement 1 cannot be formalized in mathematics because it refers to the current mathematical knowledge. The same is true for Open Problem 1 and Statements 2 and 3.

**Definition 4.** We say that an integer $n$ is a threshold number of a set $X \subseteq \mathbb{N}$, if

$$\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n].$$

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $[\max(X), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1

**Open Problem 1.** Is there a set $X \subseteq \mathbb{N}$ which satisfies conditions (1)–(5)?

Open Problem 1 asks: Are there a set $X \subseteq \mathbb{N}$ and a constructively defined integer $n$ such that $(\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n])$ and (there are many elements of $X$) and (the infiniteness of $X$ is conjectured and cannot be decided by any known method) $\land$ (a constructively defined algorithm decides $X$) $\land$ (the set $X$ equals among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements)?

**Statement 2.** The set

$$X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$$

satisfies conditions (1)–(4). Condition (5) fails for $X$.

*Proof.* Condition (1) holds as $X \supseteq [0, \ldots, \beta]$ and the set $P_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $P_{n^2+1}$ is greater than $\beta$, see [3]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus $X$ satisfies condition (4). Condition (5) fails for $X$ as the set of known elements of $X$ equals $\{0, \ldots, \beta\}$. □

Proving Landau’s conjecture will disprove Statement 2.
Theorem 1. No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

Proof. The proof goes by contradiction. We fix an integer $n$ that satisfies condition (4). Since conditions (2)-(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$(T) \quad n + 1 \notin X, \ n + 2 \notin X, \ n + 3 \notin X, \ldots$$

The sentences from the sequence $(T)$ and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n,m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $X$ is finite, contrary to the conjecture in condition (1). \hfill $\Box$

The physical limits of computation ([3]) disprove the assumption of Theorem 1.

3. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} 
  x_1! = x_1 \\
  x_1 \cdot x_1 = x_2 \\
  \forall i \in \{2,\ldots,n-1\} \ x_i! = x_{i+1}
\end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

![Fig. 2 Construction of the system $\mathcal{U}_n$](image)

Lemma 2. For every positive integer $n$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1,\ldots,1)$ and $(f(1),\ldots,f(n))$. 

![Fig. 1 Semi-algorithm that terminates if and only if the set $X$ is infinite](image)
Let $B_n$ denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

**Theorem 2.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

**Proof.** It follows from Lemma 2 because $U_n \subseteq B_n$. □

**Theorem 3.** For every integer $n \geq 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.

**Proof.** If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement $\Psi_{n+1}$ implies that $x_i! = x_{n+1} \leq f(n + 1) = f(n)!$. Hence, $x_i \leq f(n)$. □

**Theorem 4.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

4. A conjectural solution to Open Problem

**Lemma 3.** For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if $$(x + 1 = y) \lor (x = y = 1)$$

**Lemma 4.** (Wilson’s theorem, [2 p. 89]). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$. 
Lemma 5. For every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
  x_2 &= x_1^2, \\
  x_3 &= (x_1^2)! \\
  x_4 &= ((x_1^2)!)! \\
  x_5 &= x_1^2 + 1 \\
  x_6 &= (x_1^2 + 1)! \\
  x_7 &= (x_1^2)! + 1 \\
  x_8 &= (x_1^2)! + 1 \\
  x_9 &= ((x_1^2)! + 1)! 
\end{align*}
\]

Proof. By Lemma 3, for every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \qed

Lemma 6. There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \{\frac{1}{2}, 1, 2\} \cap \mathbb{N} = \{1, 2\}$. \qed
Conjecture 1. The statement $\Psi$ is true when is restricted to the system $\mathcal{A}$.

Theorem 5. Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 > f(7)$. Hence, $(x_1^2)! > f(7)! = f(8)$. Consequently, $x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$.

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite.

Theorem 6. Conjecture 1 implies the statement $\Phi$.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$.

Theorem 7. The statement $\Phi$ implies Conjecture 1.

Proof. By Lemmas 5 and 6 if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then $(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})$ or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement $\Phi$ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every $x_i$ does not exceed $f(9)$.

Statement 3. The statement $\Phi$ implies that conditions (1) - (5) hold for $X = \mathcal{P}_{n^2+1}$.

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28}]$, see [7]. These two facts imply condition (1). By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [8]. Thus condition (2) holds. Conditions (3) and (5) hold trivially. The statement $\Phi$ implies that $\beta$ is a threshold number of $X = \mathcal{P}_{n^2+1}$. Thus condition (4) holds.

Proving Landau’s conjecture will disprove Statement 5.

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