

On sets $\mathcal{X} \subseteq \mathbb{N}$ for which we know an algorithm that computes a threshold number $t(\mathcal{X}) \in \mathbb{N}$ such that \mathcal{X} is infinite if and only if \mathcal{X} contains an element greater than $t(\mathcal{X})$

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Abstract

Let $\Gamma_{\boxed{n}}(k)$ denote $(k-1)!$, where $n \in \{3, \dots, 16\}$ and $k \in \{2\} \cup [2^{n-3} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \dots, 16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq \{\Gamma_{\boxed{n}}(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ with Γ instead of $\Gamma_{\boxed{n}}$ has only finitely many solutions in positive integers x_1, \dots, x_n , then every tuple $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system \mathcal{S} satisfies $x_1, \dots, x_n \leq 2^{2^{n-2}}$. Our hypothesis claims that the statements $\Sigma_3, \dots, \Sigma_{16}$ are true. The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(1, 2), (2, 3)\}$. The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$. The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$. The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geq 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$. The statement Σ_{14} implies the infinitude of twin primes. The statement Σ_{16} implies the infinitude of Sophie Germain primes. A modified statement Σ_7 implies the infinitude of Wilson primes.

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation $x! + 1 = y^2$, composite Fermat numbers, Erdős' equation $x(x+1) = y!$, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, Richert's lemma, Sophie Germain primes, Wilson primes, twin prime conjecture.

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1 Introduction

We consider sets $\mathcal{X} \subseteq \mathbb{N}$ for which we know an algorithm that computes a threshold number $t(\mathcal{X}) \in \mathbb{N}$ such that \mathcal{X} is infinite if and only if \mathcal{X} contains an element greater than $t(\mathcal{X})$, cf. [35] and [36]. We assume here that the sets $\mathcal{X} \subseteq \mathbb{N}$ are defined by formulae in the language of ZF whereas the algorithm that computes $t(\mathcal{X})$ is written specifically for \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer m is a threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of \mathcal{X} form the set $\{\max(\mathcal{X}), \max(\mathcal{X}) + 1, \max(\mathcal{X}) + 2, \dots\}$.

2 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

Theorem 1. ([4, p. 35]). *There exists a polynomial $D(x_1, \dots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation $D(x_1, \dots, x_m) = 0$ is solvable in non-negative integers" and "The equation $D(x_1, \dots, x_m) = 0$ is not solvable in non-negative integers" are not provable in ZFC.*

Let \mathcal{Y} denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has no solutions in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Let $\gamma: \mathbb{N}^{m+1} \rightarrow \mathbb{N}$ be a computable bijection, and let $\mathcal{E} \subseteq \mathbb{N}^{m+1}$ be the solution set of the equation $D(x_1, \dots, x_m) + 0 \cdot x_{m+1} = 0$. Theorem 1 implies Theorems 2 and 3.

Theorem 2. *If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences " n is a threshold number of \mathcal{Y} " and " n is not a threshold number of \mathcal{Y} " are not provable in ZFC.*

Theorem 3. *We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \gamma(\mathcal{E})$. The set $\gamma(\mathcal{E})$ is empty or infinite. In both cases, every non-negative integer n is a threshold number of $\gamma(\mathcal{E})$. If ZFC is arithmetically consistent, then the sentences " $\gamma(\mathcal{E})$ is empty", " $\gamma(\mathcal{E})$ is not empty", " $\gamma(\mathcal{E})$ is finite", and " $\gamma(\mathcal{E})$ is infinite" are not provable in ZFC.*

In Figure 1, $D(x_1, \dots, x_m)$ stands for the polynomial described in Theorem 1. Let \mathcal{K} denote the set of all positive integers k such that the algorithm in Figure 1 halts for k on the input. If ZFC is consistent, then $\mathcal{K} = \emptyset$. Otherwise, $\text{card}(\mathcal{K}) = 1$.

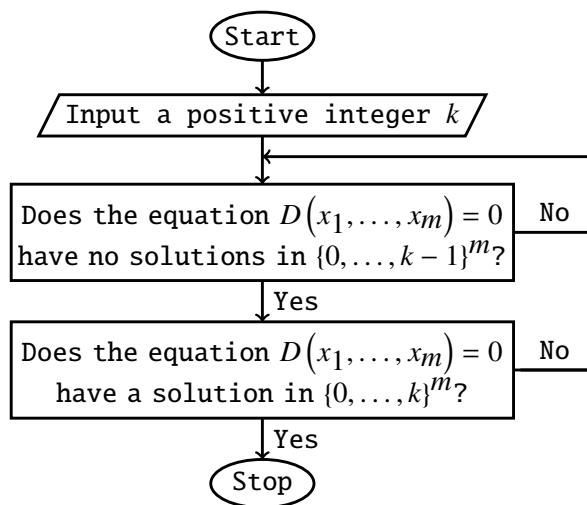


Fig. 1 The algorithm which may halt only when ZFC is inconsistent

Theorem 4. *If ZFC is consistent, then for every positive integer n , the inclusion $\mathcal{K} \subseteq \{1, \dots, n\}$ is not provable in ZFC.*

Proof. It follows from Gödel's second incompleteness theorem because the inclusion $\mathcal{K} \subseteq \{1, \dots, n\}$ implies $\mathcal{K} = \emptyset$ and the consistency of ZFC. \square

Theorem 5. *(cf. Theorem 29). If ZFC is consistent and a computer program halts for at most finitely many positive integers k on the input, then not always we can write the decimal expansion of a positive integer n which is not smaller than every such number k .*

Proof. We write a computer program which implements the algorithm in Figure 1. This program halts exactly for elements of \mathcal{K} on the input. The set \mathcal{K} is finite as $\text{card}(\mathcal{K}) \leq 1$. By Theorem 4, if ZFC is consistent, then for every positive integer n , the inclusion $\mathcal{K} \subseteq \{1, \dots, n\}$ is not provable in ZFC. \square

3 Hypothetical statements Ψ_3, \dots, Ψ_{16} and number-theoretic lemmas

For a positive integer n , let $\Gamma(n)$ denote $(n-1)!$. Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let $h(1) = 1$, and let $h(n+1) = 2^{2^{h(n)}}$ for every positive integer n . Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geq 3$. For an integer $n \geq 3$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{2\} & x_i! = x_{i+1} \\ & x_1 \cdot x_2 = x_3 \\ & x_2 \cdot x_2 = x_3 \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system \mathcal{U}_n .

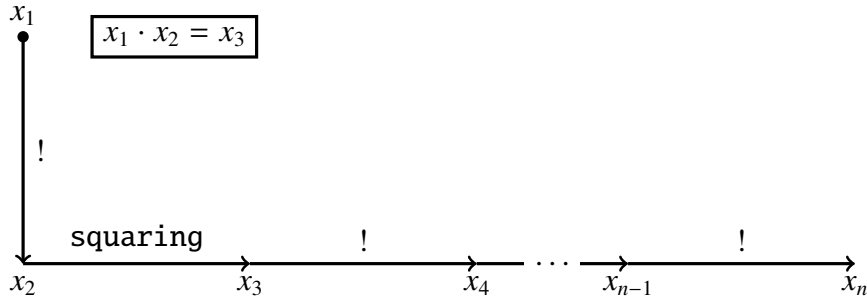


Fig. 2 Construction of the system \mathcal{U}_n

Lemma 1. For every integer $n \geq 3$, the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(2, 2, g(3), \dots, g(n))$.

Let

$$B_n = \{x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer $n \geq 3$, let Ψ_n denote the following statement: if a system $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq g(n)$. The statement Ψ_n says that for subsystems of B_n the largest known solution is indeed the largest possible.

Hypothesis 1. The statements Ψ_3, \dots, Ψ_{16} are true.

Theorem 6. Every statement Ψ_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

Theorem 7. For every statement Ψ_n , the bound $g(n)$ cannot be decreased.

Proof. It follows from Lemma 1 because $\mathcal{U}_n \subseteq B_n$. □

Lemma 2. For every positive integers x and y , $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 3. For every positive integers x and y , $x \cdot \Gamma(x) = \Gamma(y)$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. For every positive integers x and y , $x + 1 = y$ if and only if

$$(1 \neq y) \wedge (x! \cdot y = y!)$$

Lemma 5. For every non-negative integers b and c , $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.

Let \mathcal{P} denote the set of prime numbers.

Lemma 6. (Wilson's theorem, [7, p. 89]). For every positive integer x , x divides $(x - 1)! + 1$ if and only if $x \in \{1\} \cup \mathcal{P}$.

4 Heuristic arguments against the statement $\forall n \in \mathbb{N} \setminus \{0, 1, 2\} \Psi_n$

Let

$$G_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{x_i + 1 = x_k : i, k \in \{1, \dots, n\}\}$$

Hypothesis 2. ([30, p. 109].) *If a system $\mathcal{S} \subseteq G_n$ has only finitely many solutions in non-negative integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq h(2n)$.*

Hypothesis 3. *If a system $\mathcal{S} \subseteq G_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(2n)$.*

Observations 1 and 2 heuristically justify Hypothesis 3.

Observation 1. (cf. [30, p. 110, Observation 1]). *For every system $\mathcal{S} \subseteq G_n$ which involves all the variables x_1, \dots, x_n , the following new system*

$$\left(\bigcup_{x_i \cdot x_j = x_k \in \mathcal{S}} \{x_i \cdot x_j = x_k\} \right) \cup \{x_k! = y_k : k \in \{1, \dots, n\}\} \cup \left(\bigcup_{x_i + 1 = x_k \in \mathcal{S}} \{1 \neq x_k, y_i \cdot x_k = y_k\} \right)$$

is equivalent to \mathcal{S} . If the system \mathcal{S} has only finitely many solutions in positive integers x_1, \dots, x_n , then the new system has only finitely many solutions in positive integers $x_1, \dots, x_n, y_1, \dots, y_n$.

Proof. It follows from Lemma 4. □

Observation 2. *The equation $x_1! = x_1$ has exactly two solutions in positive integers, namely $x_1 = 1$ and $x_1 = f(1)$. The system $\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \end{cases}$ has exactly two solutions in positive integers, namely $(1, 1)$ and $(f(1), f(2))$. For every integer $n \geq 3$, the following system*

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.

For a positive integer n , let Φ_n denote the following statement: *if a system*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{1 \neq x_k : k \in \{1, \dots, n\}\}$$

has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$.

Theorem 8. *The statement $\forall n \in \mathbb{N} \setminus \{0\} \Phi_n$ implies Hypothesis 3.*

Proof. It follows from Lemma 4. □

Let \mathcal{Rng} denote the class of all rings \mathbf{K} that extend \mathbb{Z} , and let

$$E_n = \{1 = x_k : k \in \{1, \dots, n\}\} \cup \{x_i + x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

Th. Skolem proved that every Diophantine equation can be algorithmically transformed into an equivalent system of Diophantine equations of degree at most 2, see [22, pp. 2–3] and [13, pp. 3–4]. The following result strengthens Skolem's theorem.

Lemma 7. ([28, p. 720]). Let $D(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$. Assume that $\deg(D, x_i) \geq 1$ for each $i \in \{1, \dots, p\}$. We can compute a positive integer $n > p$ and a system $T \subseteq E_n$ which satisfies the following two conditions:

Condition 1. If $\mathbf{K} \in \mathcal{Rng} \cup \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$, then

$$\forall \tilde{x}_1, \dots, \tilde{x}_p \in \mathbf{K} \left(D(\tilde{x}_1, \dots, \tilde{x}_p) = 0 \iff \exists \tilde{x}_{p+1}, \dots, \tilde{x}_n \in \mathbf{K} (\tilde{x}_1, \dots, \tilde{x}_p, \tilde{x}_{p+1}, \dots, \tilde{x}_n) \text{ solves } T \right)$$

Condition 2. If $\mathbf{K} \in \mathcal{Rng} \cup \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$, then for each $\tilde{x}_1, \dots, \tilde{x}_p \in \mathbf{K}$ with $D(\tilde{x}_1, \dots, \tilde{x}_p) = 0$, there exists a unique tuple $(\tilde{x}_{p+1}, \dots, \tilde{x}_n) \in \mathbf{K}^{n-p}$ such that the tuple $(\tilde{x}_1, \dots, \tilde{x}_p, \tilde{x}_{p+1}, \dots, \tilde{x}_n)$ solves T .

Conditions 1 and 2 imply that for each $\mathbf{K} \in \mathcal{Rng} \cup \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$, the equation $D(x_1, \dots, x_p) = 0$ and the system T have the same number of solutions in \mathbf{K} .

Let α, β , and γ denote variables.

Lemma 8. ([20, p. 100]) For each positive integers x, y, z , $x + y = z$ if and only if

$$(zx + 1)(zy + 1) = z^2(xy + 1) + 1$$

Corollary 1. We can express the equation $x + y = z$ as an equivalent system \mathcal{F} , where \mathcal{F} involves x, y, z and 9 new variables, and where \mathcal{F} consists of equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$.

Proof. The new 9 variables express the following polynomials:

$$zx, \quad zx + 1, \quad zy, \quad zy + 1, \quad z^2, \quad xy, \quad xy + 1, \quad z^2(xy + 1), \quad z^2(xy + 1) + 1$$

□

Lemma 9. (cf. [30, p. 110, Lemma 4]). Let $D(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$. Assume that $\deg(D, x_i) \geq 1$ for each $i \in \{1, \dots, p\}$. We can compute a positive integer $n > p$ and a system $T \subseteq G_n$ which satisfies the following two conditions:

Condition 3. For every positive integers $\tilde{x}_1, \dots, \tilde{x}_p$,

$$D(\tilde{x}_1, \dots, \tilde{x}_p) = 0 \iff \exists \tilde{x}_{p+1}, \dots, \tilde{x}_n \in \mathbb{N} \setminus \{0\} (\tilde{x}_1, \dots, \tilde{x}_p, \tilde{x}_{p+1}, \dots, \tilde{x}_n) \text{ solves } T$$

Condition 4. If positive integers $\tilde{x}_1, \dots, \tilde{x}_p$ satisfy $D(\tilde{x}_1, \dots, \tilde{x}_p) = 0$, then there exists a unique tuple $(\tilde{x}_{p+1}, \dots, \tilde{x}_n) \in (\mathbb{N} \setminus \{0\})^{n-p}$ such that the tuple $(\tilde{x}_1, \dots, \tilde{x}_p, \tilde{x}_{p+1}, \dots, \tilde{x}_n)$ solves T .

Conditions 3 and 4 imply that the equation $D(x_1, \dots, x_p) = 0$ and the system T have the same number of solutions in positive integers.

Proof. Let the system T be given by Lemma 7. We replace in T each equation of the form $1 = x_k$ by the equation $x_k \cdot x_k = x_k$. Next, we apply Corollary 1 and replace in T each equation of the form $x_i + x_j = x_k$ by an equivalent system of equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$. □

Theorem 9. Hypothesis 3 implies that there is an algorithm which takes as input a Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set.

Proof. It follows from Lemma 9. □

Open Problem 1. Is there an algorithm which takes as input a Diophantine equation, and returns an integer such that this integer is greater than the moduli of integer (non-negative integer, positive integer) solutions, if the solution set is finite?

Matiyasevich's conjecture on finite-fold Diophantine representations ([15]) implies a negative answer to Open Problem 1, see [14, p. 42].

The statement $\forall n \in \mathbb{N} \setminus \{0\} \Phi_n$ implies that there is an algorithm which takes as input a factorial Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is a bit strange because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [12, p. 300].

5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let \mathcal{A} denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system \mathcal{A} .

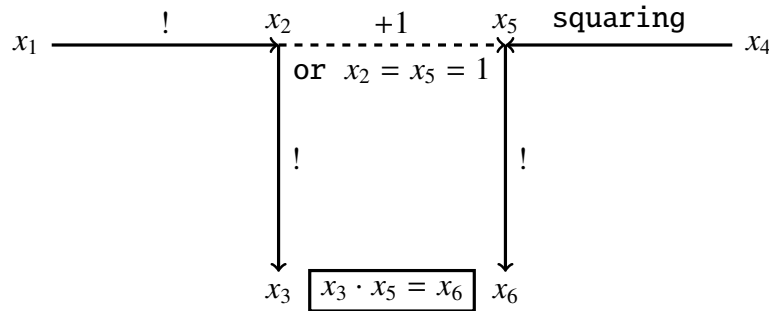


Fig. 3 Construction of the system \mathcal{A}

Lemma 10. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system \mathcal{A} is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [31, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [17].

Theorem 10. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 10, the system \mathcal{A} is solvable in positive integers x_2, x_3, x_5, x_6 . Since $\mathcal{A} \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$. Hence, $x_1! + 1 \leq g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \dots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

6 Are there infinitely many prime numbers of the form $n^2 + 1$?

Let \mathcal{B} denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system \mathcal{B} .

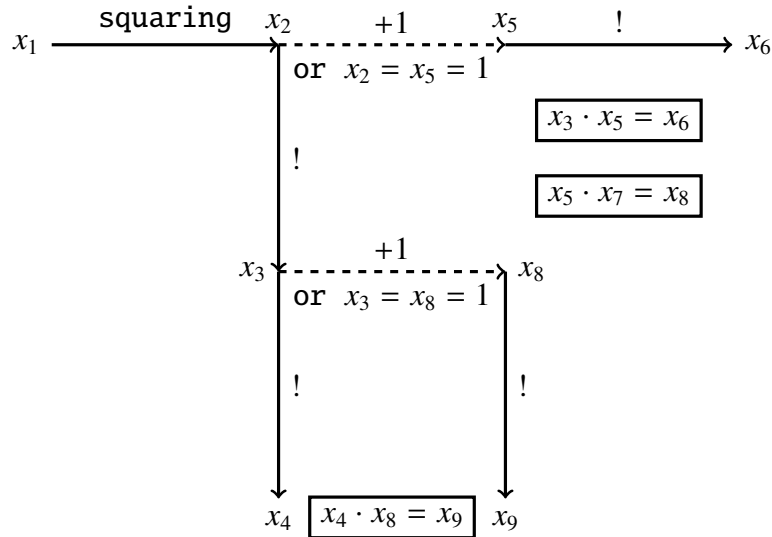


Fig. 4 Construction of the system \mathcal{B}

Lemma 11. For every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 2, for every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 11 follows from Lemma 6. \square

Lemma 12. There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system \mathcal{B} and satisfy $x_1 = 1$.

Proof. If a tuple $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system \mathcal{B} and $x_1 = 1$, then $x_1, \dots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. \square

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [16, pp. 37–38].

Theorem 11. The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.

Proof. Suppose that the antecedent holds. By Lemma 11, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{B} . Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geq g(7)$. Hence, $(x_1^2)! \geq g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq \mathcal{B}_9$, the statement Ψ_9 and the inequality $x_9 > g(9)$ imply that the system \mathcal{B} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 11 and 12, there are infinitely many primes of the form $n^2 + 1$. \square

7 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [3, p. 443] and [23].

Theorem 12. (cf. Theorem 17). *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. We leave the analogous proof to the reader. □

8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [16, p. 39]. Let C denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{array} \right.$$

Lemma 2 and the diagram in Figure 5 explain the construction of the system C .

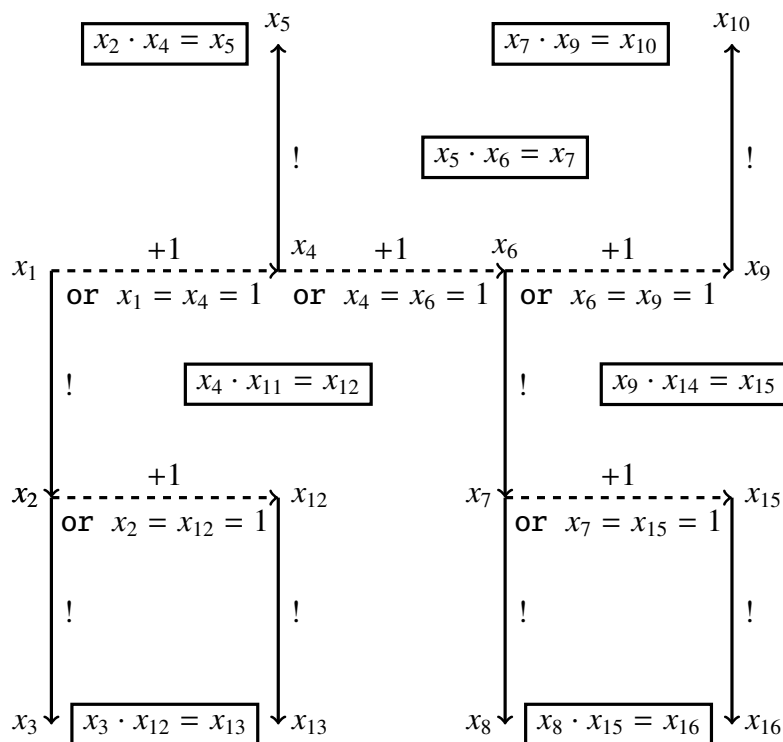


Fig. 5 Construction of the system C

Lemma 13. For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

Proof. By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | ((x_4 - 1)! + 1)) \wedge (x_9 | ((x_9 - 1)! + 1))$$

Hence, the claim of Lemma 13 follows from Lemma 6. \square

Lemma 14. There are only finitely many tuples $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system C and satisfy

$$(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$$

Proof. If a tuple $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system C and

$$(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$$

then $x_1, \dots, x_{16} \leq 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. \square

Theorem 13. The statement Ψ_{16} proves the following implication: (*) if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.

Proof. Suppose that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma 13, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ such that the tuple (x_1, \dots, x_{16}) solves the system C . Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geq g(14)$. Therefore, $(x_9 - 1)! \geq g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement Ψ_{16} and the inequality $x_{16} > g(16)$ imply that the system C has infinitely many solutions in positive integers x_1, \dots, x_{16} . According to Lemmas 13 and 14, there are infinitely many twin primes. \square

Let $\mathbb{P}(x)$ denote the predicate " x is a prime number". Dickson's conjecture ([16, p. 36], [33, p. 109]) implies that the existential theory of $(\mathbb{N}, =, +, \mathbb{P})$ is decidable, see [33, Theorem 2, p. 109]. For a positive integer n , let Θ_n denote the following statement: for every system $\mathcal{S} \subseteq \{x_i + 1 = x_k : i, k \in \{1, \dots, n\}\} \cup \{\mathbb{P}(x_i) : i \in \{1, \dots, n\}\}$ the solvability of \mathcal{S} in non-negative integers is decidable.

Lemma 15. *If the existential theory of $(\mathbb{N}, =, +, \mathbb{P})$ is decidable, then the statements Θ_n are true.*

Proof. For every non-negative integers x and y , $x + 1 = y$ if and only if

$$\exists u, v \in \mathbb{N} ((u + u = v) \wedge \mathbb{P}(v) \wedge (x + u = y))$$

□

Theorem 14. *The conjunction of the implication (*) and the statement $\Theta_{g(14)+2}$ implies that the twin prime conjecture is decidable.*

Proof. By the statement $\Theta_{g(14)+2}$, we can decide the truth of the sentence

$$\exists x_1 \dots \exists x_{g(14)+2} ((\forall i \in \{1, \dots, g(14) + 1\} x_i + 1 = x_{i+1}) \wedge \mathbb{P}(x_{g(14)}) \wedge \mathbb{P}(x_{g(14)+2})) \quad (1)$$

If sentence (1) is false, then the twin prime conjecture is false. If sentence (1) is true, then there exists a twin prime greater than $g(14)$. In this case, the twin prime conjecture follows from Theorem 13. □

9 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ about the Gamma function and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n + 1) = \Gamma(\lambda(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let \mathcal{J}_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n - 1\} \setminus \{3\} \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{cases}$$

Lemma 3 and the diagram in Figure 6 explain the construction of the system \mathcal{J}_n .

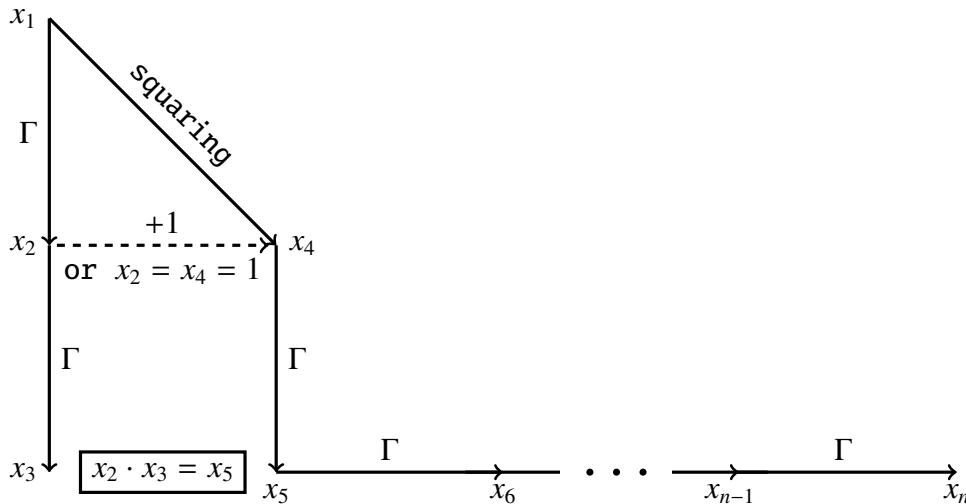


Fig. 6 Construction of the system \mathcal{J}_n

Observation 3. *For every integer $n \geq 5$, the system \mathcal{J}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \dots, \lambda(n))$.*

For an integer $n \geq 5$, let Δ_n denote the following statement: *if a system $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq \lambda(n)$.*

Hypothesis 4. *The statements $\Delta_5, \dots, \Delta_{14}$ are true.*

Lemmas 3 and 6 imply that the statements Δ_n have similar consequences as the statements Ψ_n .

Theorem 15. *The statement Δ_6 implies that any prime number $p \geq 25$ proves the infinitude of primes.*

Proof. It follows from Lemmas 3 and 6. We leave the details to the reader. □

10 Hypothetical statements $\Sigma_3, \dots, \Sigma_{16}$ about the Gamma function and their consequences

Let $\Gamma_{\boxed{n}}(k)$ denote $(k - 1)!$, where $n \in \{3, \dots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \dots, 16\}$, let

$$Q_n = \{\Gamma_{\boxed{n}}(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer $n \in \{3, \dots, 16\}$, let P_n denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \Gamma_{\boxed{n}}(x_2) = x_1 \\ \forall i \in \{2, \dots, n-1\} x_i \cdot x_i = x_{i+1} \end{cases}$$

Lemma 16. For every integer $n \in \{3, \dots, 16\}$, $P_n \subseteq Q_n$ and the system P_n with Γ instead of $\Gamma_{\boxed{n}}$ has exactly one solution in positive integers x_1, \dots, x_n , namely $(1, 2^{2^0}, 2^{2^1}, 2^{2^2}, \dots, 2^{2^{n-2}})$.

For an integer $n \in \{3, \dots, 16\}$, let Σ_n denote the following statement: if a system of equations $S \subseteq Q_n$ with Γ instead of $\Gamma_{\boxed{n}}$ has only finitely many solutions in positive integers x_1, \dots, x_n , then every tuple $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system S satisfies $x_1, \dots, x_n \leq 2^{2^{n-2}}$.

Hypothesis 5. The statements $\Sigma_3, \dots, \Sigma_{16}$ are true.

Lemma 17. (cf. Lemma 3). For every integer $n \in \{4, \dots, 16\}$ and for every positive integers x and y , $x \cdot \Gamma_{\boxed{n}}(x) = \Gamma_{\boxed{n}}(y)$ if and only if $(x + 1 = y) \wedge (x \geq 2^{2^{n-3}} + 1)$.

Let $Z_9 \subseteq Q_9$ be the system of equations in Figure 7.

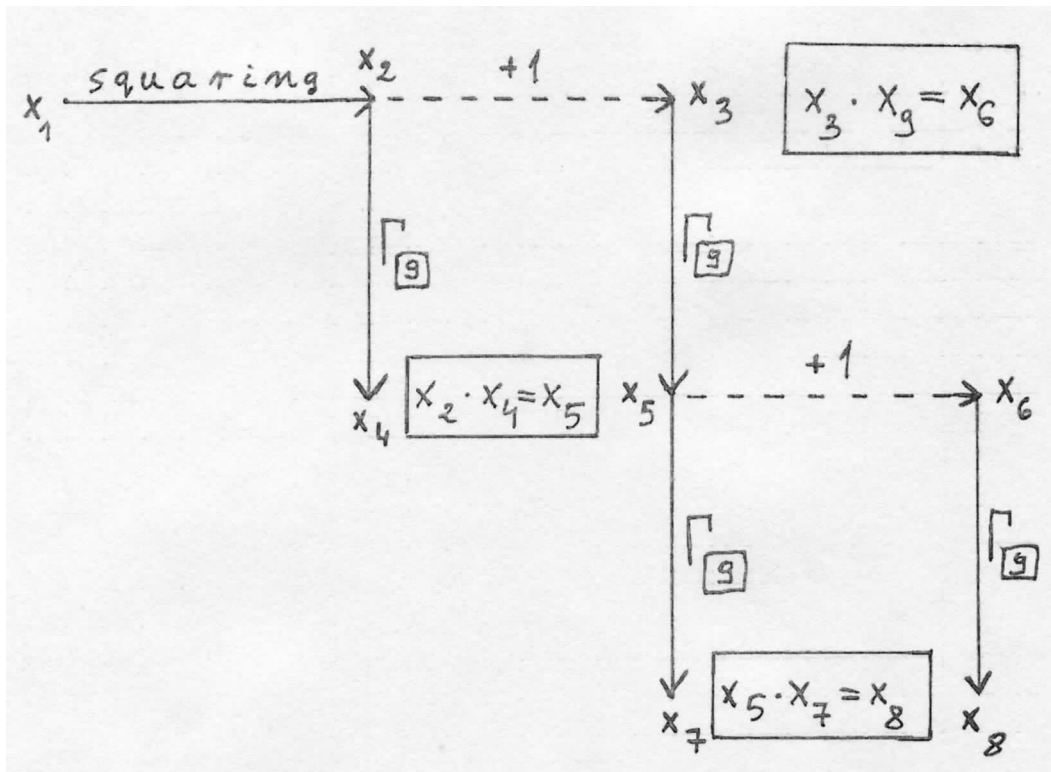


Fig. 7 Construction of the system Z_9

Lemma 18. For every positive integer x_1 , the system \mathcal{Z}_9 is solvable in positive integers x_2, \dots, x_9 if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers x_2, \dots, x_9 are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with n and solve the system \mathcal{Z}_9 with Γ instead of $\Gamma_{\boxed{9}}$.

Proof. It follows from Lemmas 3, 6, and 17. □

Lemma 19. ([26]). The number $(13!)^2 + 1 = 38775788043632640001$ is prime.

Lemma 20. $\left((13!)^2 \geq 2^{2^{9-3}} + 1 = 18446744073709551617 \right) \wedge \left(\Gamma_{\boxed{9}}((13!)^2) > 2^{2^{9-2}} \right)$.

Theorem 16. The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$.

Proof. It follows from Lemmas 18–20. □

Theorem 17. (cf. Theorem 12). The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geq 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$.

Proof. We leave the proof to the reader. □

Let $\mathcal{Z}_{14} \subseteq \mathcal{Q}_{14}$ be the system of equations in Figure 8.

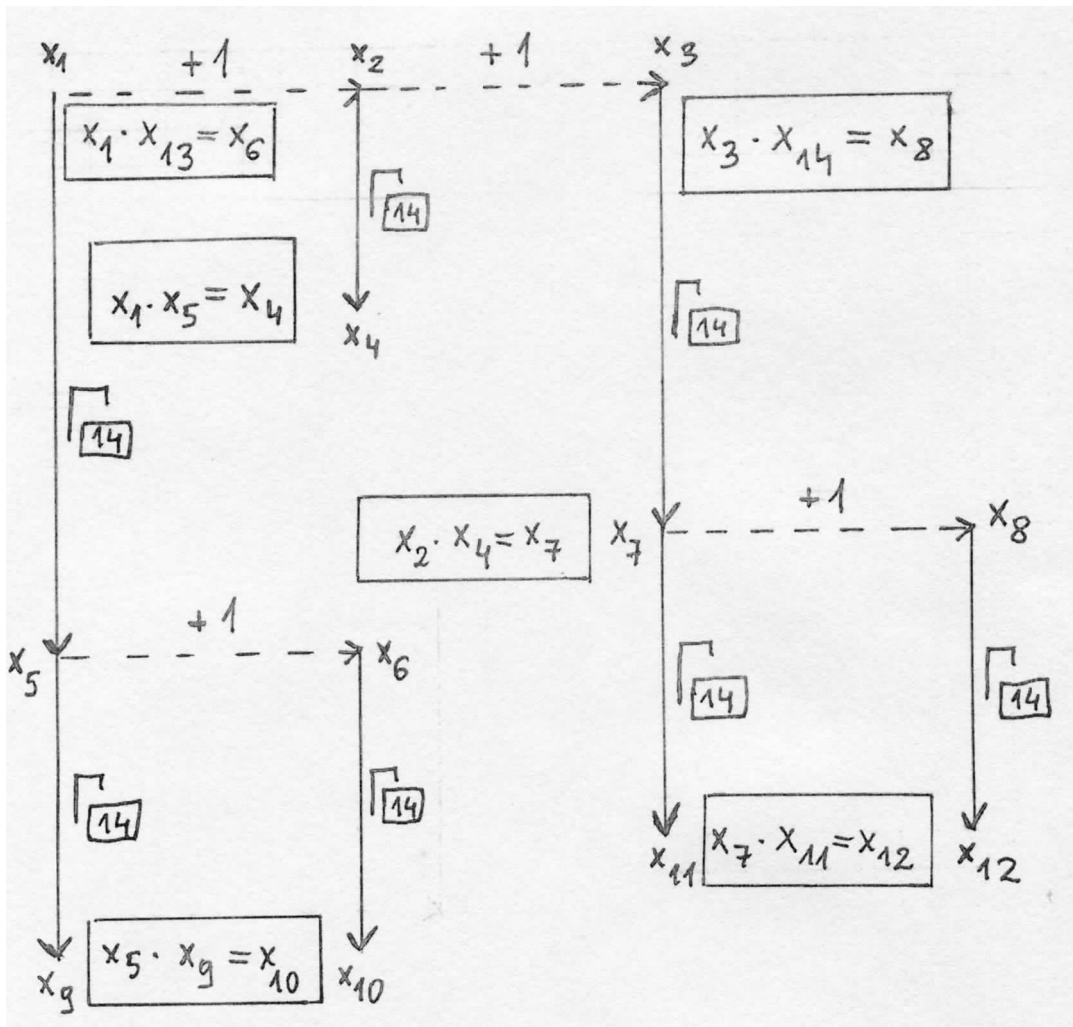


Fig. 8 Construction of the system \mathcal{Z}_{14}

Lemma 21. For every positive integer x_1 , the system \mathcal{Z}_{14} is solvable in positive integers x_2, \dots, x_{14} if and only if x_1 and $x_1 + 2$ are prime and $x_1 \geq 2^{2^{14-3}} + 1$. In this case, positive integers x_2, \dots, x_{14} are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with n and solve the system \mathcal{Z}_{14} with Γ instead of Γ_{14} .

Proof. It follows from Lemmas 3, 6, and 17. □

Lemma 22. ([34, p. 87]). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).

Lemma 23. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 18. The statement Σ_{14} implies the infinitude of twin primes.

Proof. It follows from Lemmas 21–23. □

A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [32]. Let $\mathcal{Z}_{16} \subseteq Q_{16}$ be the system of equations in Figure 9.

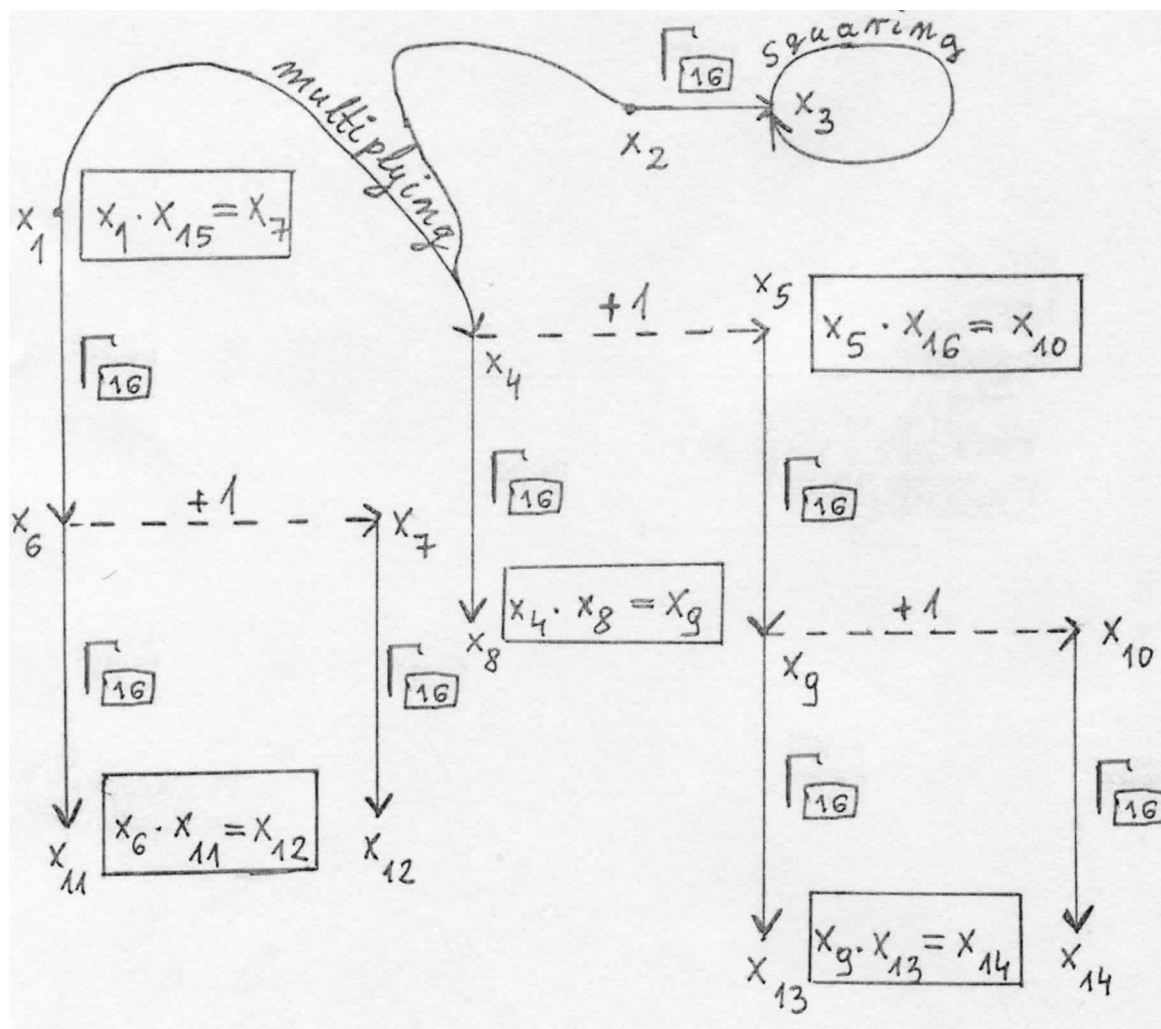


Fig. 9 Construction of the system \mathcal{Z}_{16}

Lemma 24. For every positive integer x_1 , the system \mathcal{Z}_{16} is solvable in positive integers x_2, \dots, x_{16} if and only if x_1 is a Sophie Germain prime and $x_1 \geq 2^{2^{16-3}} + 1$. In this case, positive integers x_2, \dots, x_{16} are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with n and solve the system \mathcal{Z}_{16} with Γ instead of Γ_{16} .

Proof. It follows from Lemmas 3, 6, and 17. □

Lemma 25. ([18, p. 330]). $8069496435 \cdot 10^{5072} - 1$ is a Sophie Germain prime (Harvey Dubner).

Lemma 26. $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$.

Theorem 19. The statement Σ_{16} implies the infinitude of Sophie Germain primes.

Proof. It follows from Lemmas 24–26. □

Theorem 20. The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(1, 2), (2, 3)\}$.

Proof. We leave the proof to the reader. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdős, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [11].

Theorem 21. The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. We leave the proof to the reader. □

11 A hypothesis which implies the infinitude of Wilson primes

Let

$$\mathcal{V}_7 = \{\Gamma_{\boxed{5}}(x_i) = x_k : i, k \in \{1, \dots, 7\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, 7\}\}$$

Let \mathcal{I}_7 denote the following system of equations:

$$\left\{ \begin{array}{l} x_1 \cdot x_1 = x_1 \\ \Gamma_{\boxed{5}}(x_2) = x_1 \\ x_2 \cdot x_2 = x_3 \\ x_3 \cdot x_3 = x_4 \\ x_4 \cdot x_4 = x_5 \\ \Gamma_{\boxed{5}}(x_5) = x_6 \\ \Gamma_{\boxed{5}}(x_6) = x_7 \end{array} \right.$$

Lemma 27. $\mathcal{I}_7 \subseteq \mathcal{V}_7$ and the system \mathcal{I}_7 with Γ instead of $\Gamma_{\boxed{5}}$ has exactly one solution in positive integers x_1, \dots, x_7 , namely $(1, 2, 4, 16, 256, 255!, (255! - 1)!)$.

Let Ξ_7 denote the following statement: if a system $\mathcal{S} \subseteq \mathcal{V}_7$ with Γ instead of $\Gamma_{\boxed{5}}$ has only finitely many solutions in positive integers x_1, \dots, x_7 , then every tuple $(x_1, \dots, x_7) \in (\mathbb{N} \setminus \{0\})^7$ that solves the original system \mathcal{S} satisfies $x_1, \dots, x_7 \leq (255! - 1)!$.

Hypothesis 6. The statement Ξ_7 is true.

Lemma 28. (cf. Lemma 3). For every positive integers x and y , $x \cdot \Gamma_{\boxed{5}}(x) = \Gamma_{\boxed{5}}(y)$ if and only if $(x+1 = y) \wedge (x \geq 17)$.

A Wilson prime is a prime number p such that p^2 divides $(p - 1)! + 1$, see [2], [18, p. 346], and [27]. It is conjectured that the set of Wilson primes is infinite, see [2]. Let $\mathcal{Z}_7 \subseteq \mathcal{V}_7$ be the system of equations in Figure 10.

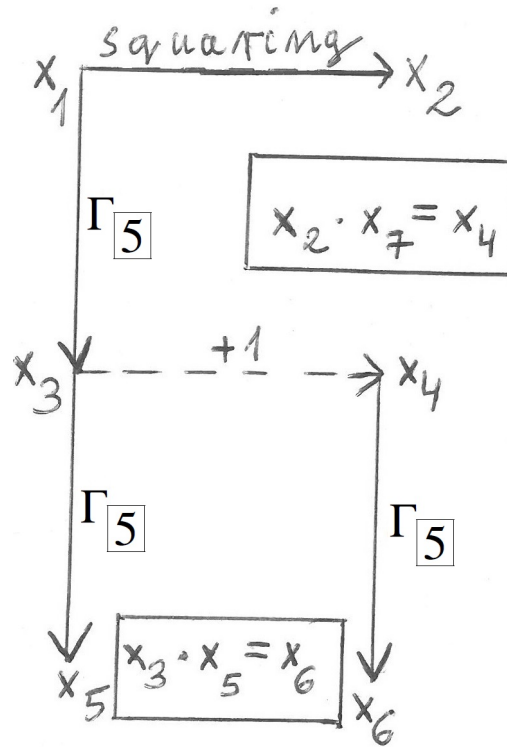


Fig. 10 Construction of the system \mathcal{Z}_7

Lemma 29. For every positive integer x_1 , the system \mathcal{Z}_7 is solvable in positive integers x_2, \dots, x_7 if and only if x_1 is a Wilson prime and $x_1 \geq 17$. In this case, positive integers x_2, \dots, x_7 are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_7) \in (\mathbb{N} \setminus \{0\})^7$ begin with n and solve the system \mathcal{Z}_7 with Γ instead of Γ_5 .

Proof. It follows from Lemmas 3, 6, and 28. □

Lemma 30. ([2], [18, p. 346], [27]). 563 is a Wilson prime.

Lemma 31. $\Gamma_5(\Gamma_5(563) + 1) > (255! - 1)!$.

Theorem 22. The statement Ξ_7 implies the infinitude of Wilson primes.

Proof. It follows from Lemmas 29–31. □

Let $\widehat{\Xi}_7$ denote the following statement: if a system of equations

$$\mathcal{S} \subseteq \{\Gamma_6(x_i) = x_k : i, k \in \{1, \dots, 7\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, 7\}\}$$

with Γ instead of Γ_6 has only finitely many solutions in positive integers x_1, \dots, x_7 , then every tuple $(x_1, \dots, x_7) \in (\mathbb{N} \setminus \{0\})^7$ that solves the original system \mathcal{S} satisfies $x_1, \dots, x_7 \leq (256^2 - 1)!$.

Theorem 23. The statement $\widehat{\Xi}_7$ implies the infinitude of Wilson primes.

Proof. We leave the analogous proof to the reader. □

12 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [10, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [10, p. 1].

Open Problem 2. ([10, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [9, p. 23].

Theorem 24. ([29]). *An unproven inequality stated in [29] implies that $2^{2^n} + 1$ is composite for every integer $n \geq 5$.*

Let

$$H_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Lemma 32. *The following subsystem of H_n*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \dots, h(n))$.

For a positive integer n , let ξ_n denote the following statement: *if a system $S \subseteq H_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq h(n)$.* The statement ξ_n says that for subsystems of H_n the largest known solution is indeed the largest possible.

Hypothesis 7. *The statements ξ_1, \dots, ξ_{13} are true.*

The truth of the statement $\forall n \in \mathbb{N} \setminus \{0\} \xi_n$ is doubtful because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [12, p. 300].

Theorem 25. *Every statement ξ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system H_n has a finite number of subsystems. □

Theorem 26. *The statement ξ_{13} proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers z .*

Proof. Let us consider the equation

$$(x+1)(y+1) = 2^{2^z} + 1 \tag{2}$$

in positive integers. By Lemma 5, we can transform equation (2) into an equivalent system \mathcal{G} which has 13 variables (x, y, z , and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 11.

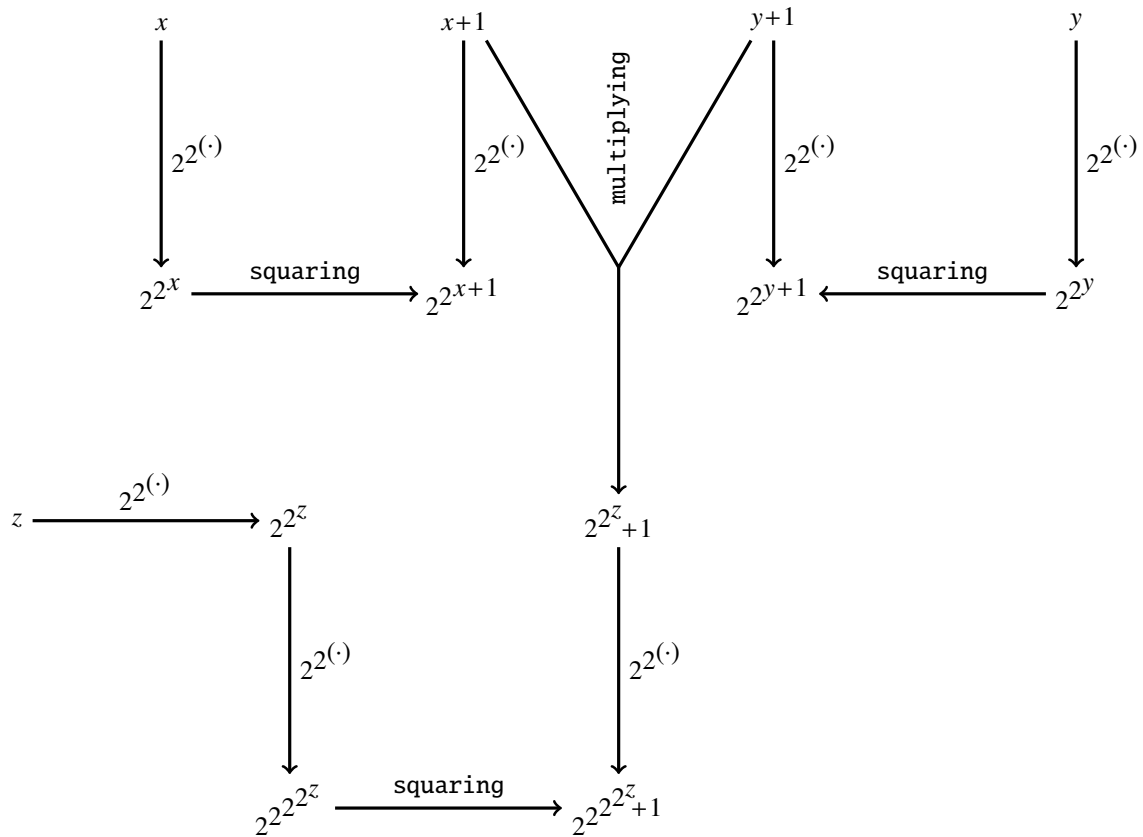


Fig. 11 Construction of the system \mathcal{G}

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z} + 1}} > h(13)$. By this, the statement ξ_{13} implies that the system \mathcal{G} has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. \square

13 Subsets of \mathbb{N} whose infinitude is unconditionally equivalent to the halting of a Turing machine

The following lemma is known as Richert's lemma.

Lemma 33. ([6], [19], [21, p. 152]). *Let $\{m_i\}_{i=1}^\infty$ be an increasing sequence of positive integers such that for some positive integer k the inequality $m_{i+1} \leq 2m_i$ holds for all $i > k$. Suppose there exists a non-negative integer b such that the numbers $b + 1, b + 2, b + 3, \dots, b + m_{k+1}$ are all expressible as sums of one or more distinct elements of the set $\{m_1, \dots, m_k\}$. Then every integer greater than b is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$.*

Let \mathcal{T} denote the set of all positive integers i such that every integer $j \geq i$ is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$. Obviously, $\mathcal{T} = \emptyset$ or $\mathcal{T} = [d, \infty) \cap \mathbb{N}$ for some positive integer d .

Corollary 2. *If the sequence $\{m_i\}_{i=1}^\infty$ is computable and the algorithm in Figure 12 terminates, then almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$. In particular, if the sequence $\{m_i\}_{i=1}^\infty$ is computable and the algorithm in Figure 12 terminates, then the set \mathcal{T} is infinite. In this case, the algorithm in Figure 12 prints all positive integers which are not expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$.*

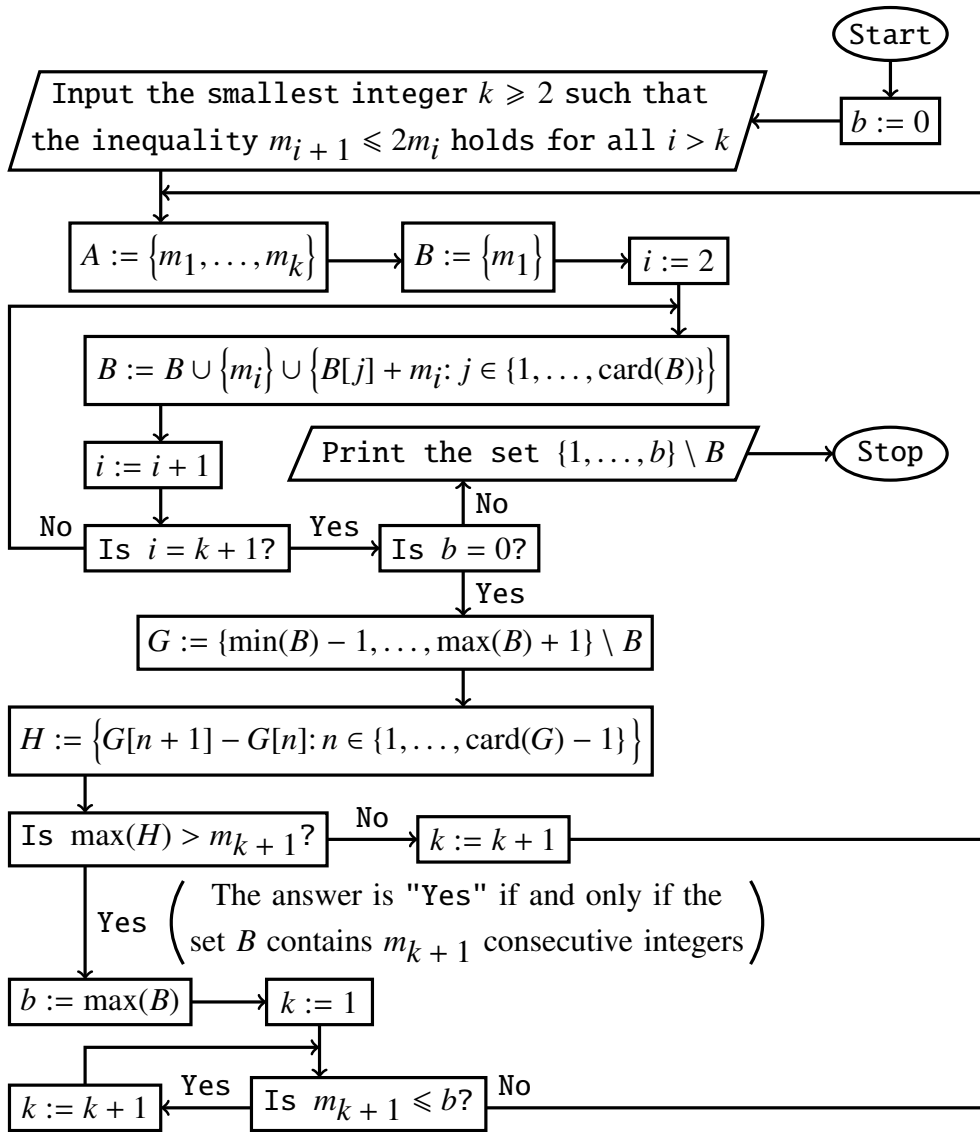


Fig. 12 The algorithm which uses Richert's lemma

Theorem 27. ([8, Theorem 2.3]). *If there exists $\varepsilon > 0$ such that the inequality $m_{i+1} \leq (2 - \varepsilon) \cdot m_i$ holds for every sufficiently large i , then the algorithm in Figure 12 terminates if and only if almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$.*

Corollary 3. *If there exists $\varepsilon > 0$ such that the inequality $m_{i+1} \leq (2 - \varepsilon) \cdot m_i$ holds for every sufficiently large i , then the algorithm in Figure 12 terminates if and only if the set \mathcal{T} is infinite.*

We show how the algorithm in Figure 12 works for a concrete sequence $\{m_i\}_{i=1}^\infty$. Let $[\cdot]$ denote the integer part function. For a positive integer i , let $t_i = \frac{(i + 19)^{i + 19}}{(i + 19)! \cdot 2^i + 19}$, and let $m_i = [t_i]$.

Lemma 34. *The inequality $m_{i+1} \leq 2m_i$ holds for every positive integer i .*

Proof. For every positive integer i ,

$$\frac{m_i}{m_{i+1}} = \frac{[t_i]}{[t_{i+1}]} > \frac{t_i - 1}{t_{i+1}} = \frac{t_i}{t_{i+1}} - \frac{1}{t_{i+1}} \geq \frac{t_i}{t_{i+1}} - \frac{1}{t_2} =$$

$$2 \cdot \frac{i + 20}{i + 19} \cdot \left(1 - \frac{1}{i + 20}\right)^{i+20} - \frac{21! \cdot 2^{21}}{21^{21}} > 2 \cdot \left(1 - \frac{1}{21}\right)^{21} - \frac{21! \cdot 2^{21}}{21^{21}} = \frac{4087158528442715204485120000}{5842587018385982521381124421}$$

The last fraction was computed by MuPAD and is greater than $\frac{1}{2}$. □

Theorem 28. *The algorithm in Figure 12 terminates for the sequence $\{m_i\}_{i=1}^{\infty}$.*

Proof. By Lemma 34, we take $k = 2$ as the initial value of k . The following *MuPAD* code

```

k:=2:
repeat
A:={floor((i+19)^(i+19)/((i+19)!*2^(i+19))) $i=1..k+1}:
B:={A[1]}:
for i from 2 to nops(A)-1 do
B:=B union {A[i]} union {B[j]+A[i] $j=1..nops(B)}:
end_for:
G:={y $y=B[1]-1..B[nops(B)]+1} minus B:
H:={G[n+1]-G[n] $n=1..nops(G)-1}:
k:=k+1:
until H[nops(H)]>A[nops(A)] end_repeat:
b:=B[nops(B)]:
k:=1:
while floor((k+20)^(k+20)/((k+20)!*2^(k+20)))<=b do
k:=k+1:
end_while:
A:={floor((i+19)^(i+19)/((i+19)!*2^(i+19))) $i=1..k}:
B:={A[1]}:
for i from 2 to nops(A)-1 do
B:=B union {A[i]} union {B[j]+A[i] $j=1..nops(B)}:
end_for:
print({n $n=1..b} minus B):

```

implements the algorithm in Figure 12 because *MuPAD* automatically orders every finite set of integers and the inequality $H[nops(H)] > A[nops(A)]$ holds true if and only if the set B contains m_{k+1} consecutive integers. The code returns the following output:

```

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
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 1647, 1667, 1679, 1681, 1699, 1701, 1721, 1753, 1773, 1775, 1780, 1795,
 1817, 1832, 1849, 1852, 1869, 1871, 1886, 1923, 1925, 1943, 1945, 1950,
 1997, 2022, 2039, 2073, 2120, 2174, 2221, 2246, 2297, 2369, 2416, 2591,
 2761}

□

Corollary 4. $\mathcal{T} = [2762, \infty) \cap \mathbb{N}$.

MuPAD is a general-purpose computer algebra system. *MuPAD* is no longer available as a stand-alone computer program, but only as the *Symbolic Math Toolbox* of *MATLAB*. Fortunately, the presented code can be executed by *MuPAD Light*, which was offered for free for research and education until autumn 2005.

14 A hypothetical infinitude of various classes of primes via computer programs which halt for at most finitely many positive integers on the input

Let $\text{fact}^{-1}: \{1, 2, 6, 24, \dots\} \rightarrow \mathbb{N} \setminus \{0\}$ denote the inverse function to the factorial function. For positive integers x and y , let $\text{rem}(x, y)$ denote the remainder from dividing x by y .

Definition. For a positive integer n , by a program of length n we understand any sequence of terms x_1, \dots, x_n such that x_1 is defined as the variable x , and for every integer $i \in \{2, \dots, n\}$, x_i is defined as $\Gamma(x_{i-1})$, or $\text{fact}^{-1}(x_{i-1})$, or $\text{rem}(x_{i-1}, x_{i-2})$ – but only if $i \geq 3$ and x_{i-1} is defined as $\Gamma(x_{i-2})$.

Let $\delta(4) = 3$, and let $\delta(n+1) = \delta(n)!$ for every integer $n \geq 4$. For an integer $n \geq 4$, let Ω_n denote the following statement: if a program of length n returns positive integers x_1, \dots, x_n for at most finitely many positive integers x , then every such x does not exceed $\delta(n)$.

Theorem 29. (cf. Theorem 5). For every integer $n \geq 4$, the statement Ω_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , there are only finitely many programs of length n . \square

Lemma 35. ([21, pp. 214–215]). For every positive integer x , $\text{rem}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$.

Theorem 30. For every integer $n \geq 4$ and for every positive integer x , the following program \mathcal{H}_n

$$\left\{ \begin{array}{l} x_1 := x \\ \forall i \in \{2, \dots, n-3\} x_i := \text{fact}^{-1}(x_{i-1}) \\ x_{n-2} := \Gamma(x_{n-3}) \\ x_{n-1} := \Gamma(x_{n-2}) \\ x_n := \text{rem}(x_{n-1}, x_{n-2}) \end{array} \right.$$

returns positive integers x_1, \dots, x_n if and only if $x = \delta(n)$.

Proof. We make three observations.

Observation 4. If $x_{n-3} = 3$, then $x_1, \dots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x = x_1 = \delta(n)$.

If $x = \delta(n)$, then $x_1, \dots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x_{n-3} = 3$.

Hence, $x_{n-2} = \Gamma(x_{n-3}) = 2$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 1$.

Observation 5. If $x_{n-3} = 2$, then $x = x_1 = \dots = x_{n-3} = 2$.

If $x = 2$, then $x_1 = \dots = x_{n-3} = 2$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 1$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$.

Therefore, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.

Observation 6. If $x_{n-3} = 1$, then $x_{n-2} = \Gamma(x_{n-3}) = 1$. Hence, $x_{n-1} = \Gamma(x_{n-2}) = 1$.

Therefore, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.

Observations 4–6 cover the case when $x_{n-3} \in \{1, 2, 3\}$. If $x_{n-3} \geq 4$, then $x_{n-2} = \Gamma(x_{n-3})$ is greater than 4 and composite. By Lemma 35, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = \text{rem}(\Gamma(x_{n-2}), x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$. \square

Corollary 5. For every integer $n \geq 4$, the bound $\delta(n)$ in the statement Ω_n cannot be decreased.

Lemma 36. If $x \in \mathcal{P}$, then $\text{rem}(\Gamma(x), x) = x - 1$.

Proof. It follows from Lemma 6. \square

Lemma 37. For every positive integer x , the following program \mathcal{A}

$$\left\{ \begin{array}{l} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \text{rem}(x_2, x_1) \\ x_4 := \text{fact}^{-1}(x_3) \end{array} \right.$$

returns positive integers x_1, \dots, x_4 if and only if $x = 4$ or x is a prime number of the form $n! + 1$.

Proof. For an integer $i \in \{1, \dots, 4\}$, let A_i denote the set of positive integers x such that the first i instructions of the program \mathcal{A} returns positive integers x_1, \dots, x_i . We show that

$$A_4 = \{4\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P} \quad (4)$$

For every positive integer x , the terms x_1 and x_2 belong to $\mathbb{N} \setminus \{0\}$. By Lemma 35, the term x_3 (which equals $\text{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$. Hence, $A_3 = \{4\} \cup \mathcal{P}$. If $x = 4$, then $x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in A_4$. If $x \in \mathcal{P}$, then Lemma 36 implies that $x_3 = \text{rem}(\Gamma(x), x) = x - 1 \in \mathbb{N} \setminus \{0\}$. Therefore, for every $x \in \mathcal{P}$, the term $x_4 = \text{fact}^{-1}(x_3)$ belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\}$. This proves equality (4). \square

Theorem 31. The statement Ω_4 implies that the set of primes of the form $n! + 1$ is infinite.

Proof. The number $3! + 1 = 7$ is prime. By Lemma 37, for $x = 7$ the program \mathcal{A} returns positive integers x_1, \dots, x_4 . Since $x = 7 > 3 = \delta(4)$, the statement Ω_4 guarantees that the program \mathcal{A} returns positive integers x_1, \dots, x_4 for infinitely many positive integers x . By Lemma 37, there are infinitely many primes of the form $n! + 1$. \square

Lemma 38. *If $x \in \mathbb{N} \setminus \{0, 1\}$, then $\text{fact}^{-1}(\Gamma(x)) = x - 1$.*

Theorem 32. *If the set of primes of the form $n! + 1$ is infinite, then the statement Ω_4 is true.*

Proof. There exist exactly 10 programs of length 4 that differ from \mathcal{H}_4 and \mathcal{A} , see Figure 13. For every such program \mathcal{F}_i , we determine the set S_i of all positive integers x such that the program \mathcal{F}_i outputs positive integers x_1, \dots, x_4 on input x . We omit 10 easy proofs which use Lemmas 35 and 38. The sets S_i are infinite, see Figure 13.

\mathcal{F}_1	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \mathbb{N} \setminus \{0\} = S_1$
\mathcal{F}_2	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \text{fact}^{-1}(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \mathbb{N} \setminus \{0\} = S_2$
\mathcal{H}_4	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \text{rem}(x_3, x_2)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x = 3$
\mathcal{F}_3	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \text{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \mathbb{N} \setminus \{0\} = S_3$
\mathcal{F}_4	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \text{fact}^{-1}(x_2)$	$x_4 := \text{fact}^{-1}(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{1\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} = S_4$
\mathcal{F}_5	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \text{rem}(x_2, x_1)$	$x_4 := \Gamma(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{4\} \cup \mathcal{P} = S_5$
\mathcal{A}	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \text{rem}(x_2, x_1)$	$x_4 := \text{fact}^{-1}(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{4\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P}$
\mathcal{F}_6	$x_1 := x$	$x_2 := \text{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{n! : n \in \mathbb{N} \setminus \{0\}\} = S_6$
\mathcal{F}_7	$x_1 := x$	$x_2 := \text{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \text{fact}^{-1}(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{n! : n \in \mathbb{N} \setminus \{0\}\} = S_7$
\mathcal{F}_8	$x_1 := x$	$x_2 := \text{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \text{rem}(x_3, x_2)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{4!\} \cup \{p! : p \in \mathcal{P}\} = S_8$
\mathcal{F}_9	$x_1 := x$	$x_2 := \text{fact}^{-1}(x_1)$	$x_3 := \text{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{(n!)! : n \in \mathbb{N} \setminus \{0\}\} = S_9$
\mathcal{F}_{10}	$x_1 := x$	$x_2 := \text{fact}^{-1}(x_1)$	$x_3 := \text{fact}^{-1}(x_2)$	$x_4 := \text{fact}^{-1}(x_3)$	$x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{((n!)!)! : n \in \mathbb{N} \setminus \{0\}\} = S_{10}$

Fig. 13 12 programs of length 4, $x \in \mathbb{N} \setminus \{0\}$

This completes the proof. \square

Hypothesis 8. *The statements $\Omega_4, \dots, \Omega_7$ are true.*

Lemma 39. *For every positive integer x , the following program \mathcal{B}*

$$\begin{cases} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \text{rem}(x_2, x_1) \\ x_4 := \text{fact}^{-1}(x_3) \\ x_5 := \Gamma(x_4) \\ x_6 := \text{rem}(x_5, x_4) \end{cases}$$

returns positive integers x_1, \dots, x_6 if and only if $x \in \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}$

Proof. For an integer $i \in \{1, \dots, 6\}$, let B_i denote the set of positive integers x such that the first i instructions of the program \mathcal{B} returns positive integers x_1, \dots, x_i . Since the programs \mathcal{A} and \mathcal{B} have the same first four instructions, the equality $B_i = A_i$ holds for every $i \in \{1, \dots, 4\}$. In particular,

$$B_4 = \{4\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P}$$

We show that

$$B_6 = \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P} \quad (5)$$

If $x = 4$, then $x_1, \dots, x_6 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in B_6$. Let $x \in \mathcal{P}$, and let $x = n! + 1$, where $n \in \mathbb{N} \setminus \{0\}$. Hence, $n \neq 4$. Lemma 36 implies that $x_3 = \text{rem}(\Gamma(x), x) = x - 1 = n!$. Hence, $x_4 = \text{fact}^{-1}(x_3) = n$ and $x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\}$. By Lemma 35, the term x_6 (which equals $\text{rem}(\Gamma(n), n)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $n \in \{4\} \cup \mathcal{P}$. This proves equality (5) as $n \neq 4$. \square

Theorem 33. *The statement Ω_6 implies that for infinitely many primes p the number $p! + 1$ is prime.*

Proof. The numbers 11 and $11! + 1$ are prime, see [3, p. 441] and [25]. By Lemma 39, for $x = 11! + 1$ the program \mathcal{B} returns positive integers x_1, \dots, x_6 . Since $x = 11! + 1 > 6! = \delta(6)$, the statement Ω_6 guarantees that the program \mathcal{B} returns positive integers x_1, \dots, x_6 for infinitely many positive integers x . By Lemma 39, for infinitely many primes p the number $p! + 1$ is prime. \square

Lemma 40. *For every positive integer x , the following program \mathcal{C}*

$$\left\{ \begin{array}{l} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \Gamma(x_2) \\ x_4 := \text{fact}^{-1}(x_3) \\ x_5 := \Gamma(x_4) \\ x_6 := \text{rem}(x_5, x_4) \end{array} \right.$$

returns positive integers x_1, \dots, x_6 if and only if $(x - 1)! - 1$ is prime.

Proof. For an integer $i \in \{1, \dots, 6\}$, let C_i denote the set of positive integers x such that the first i instructions of the program \mathcal{C} returns positive integers x_1, \dots, x_i . If $x \in \{1, 2, 3\}$, then $x_6 = 0$. Therefore, $C_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 38, for every integer $x \geq 4$, $x_4 = (x - 1)! - 1$, $x_5 = \Gamma((x - 1)! - 1)$, and $x_1, \dots, x_5 \in \mathbb{N} \setminus \{0\}$. By Lemma 35, for every integer $x \geq 4$,

$$x_6 = \text{rem}(\Gamma((x - 1)! - 1), (x - 1)! - 1)$$

belongs to $\mathbb{N} \setminus \{0\}$ if and only if $(x - 1)! - 1 \in \{4\} \cup \mathcal{P}$. The last condition equivalently expresses that $(x - 1)! - 1$ is prime as $(x - 1)! - 1 \geq 5$ for every integer $x \geq 4$. Hence,

$$C_6 = (\mathbb{N} \setminus \{0, 1, 2, 3\}) \cap \{x \in \mathbb{N} \setminus \{0, 1, 2, 3\} : (x - 1)! - 1 \in \mathcal{P}\} = \{x \in \mathbb{N} \setminus \{0\} : (x - 1)! - 1 \in \mathcal{P}\}$$

\square

It is conjectured that there are infinitely many primes of the form $n! - 1$, see [3, p. 443] and [24].

Theorem 34. *The statement Ω_6 implies that there are infinitely many primes of the form $x! - 1$.*

Proof. The number $(975 - 1)! - 1$ is prime, see [3, p. 441] and [24]. By Lemma 40, for $x = 975$ the program \mathcal{C} returns positive integers x_1, \dots, x_6 . Since $x = 975 > 720 = \delta(6)$, the statement Ω_6 guarantees that the program \mathcal{C} returns positive integers x_1, \dots, x_6 for infinitely many positive integers x . By Lemma 40, the set $\{x \in \mathbb{N} \setminus \{0\} : (x - 1)! - 1 \in \mathcal{P}\}$ is infinite. \square

Lemma 41. *For every positive integer x , the following program \mathcal{D}*

$$\left\{ \begin{array}{l} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \text{rem}(x_2, x_1) \\ x_4 := \Gamma(x_3) \\ x_5 := \text{fact}^{-1}(x_4) \\ x_6 := \Gamma(x_5) \\ x_7 := \text{rem}(x_6, x_5) \end{array} \right.$$

returns positive integers x_1, \dots, x_7 if and only if both x and $x - 2$ are prime.

Proof. For an integer $i \in \{1, \dots, 7\}$, let D_i denote the set of positive integers x such that the first i instructions of the program \mathcal{D} returns positive integers x_1, \dots, x_i . If $x = 1$, then $x_3 = 0$. Hence, $D_7 \subseteq D_3 \subseteq \mathbb{N} \setminus \{0, 1\}$. If $x \in \{2, 3, 4\}$, then $x_7 = 0$. Therefore,

$$D_7 \subseteq (\mathbb{N} \setminus \{0, 1\}) \cap (\mathbb{N} \setminus \{0, 2, 3, 4\}) = \mathbb{N} \setminus \{0, 1, 2, 3, 4\}$$

By Lemma 35, for every integer $x \geq 5$, the term x_3 (which equals $\text{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \mathcal{P} \setminus \{2, 3\}$. By Lemma 36, for every $x \in \mathcal{P} \setminus \{2, 3\}$, $x_3 = x - 1 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 38, for every $x \in \mathcal{P} \setminus \{2, 3\}$, the terms x_4 and x_5 belong to $\mathbb{N} \setminus \{0\}$ and $x_5 = x_3 - 1 = x - 2$. By Lemma 35, for every $x \in \mathcal{P} \setminus \{2, 3\}$, the term x_7 (which equals $\text{rem}(\Gamma(x_5), x_5)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x_5 = x - 2 \in \{4\} \cup \mathcal{P}$. From these facts, we obtain that

$$D_7 = (\mathbb{N} \setminus \{0, 1, 2, 3, 4\}) \cap (\mathcal{P} \setminus \{2, 3\}) \cap (\{6\} \cup \{p + 2 : p \in \mathcal{P}\}) = \{p \in \mathcal{P} : p - 2 \in \mathcal{P}\}$$

□

Theorem 35. *The statement Ω_7 implies that there are infinitely many twin primes.*

Proof. Harvey Dubner proved that the numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime, see [34, p. 87]. By Lemma 41, for $x = 459 \cdot 2^{8529} + 1$ the program \mathcal{D} returns positive integers x_1, \dots, x_7 . Since $x > 720! = \delta(7)$, the statement Ω_7 guarantees that the program \mathcal{D} returns positive integers x_1, \dots, x_7 for infinitely many positive integers x . By Lemma 41, there are infinitely many twin primes. □

We can transform every program of length n into a computer program with n instructions which for every $x \in \mathbb{N} \setminus \{0\}$ does the same if $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, and never halts if $(x_1, \dots, x_n) \notin (\mathbb{N} \setminus \{0\})^n$ or the tuple (x_1, \dots, x_n) is undefined. To do so, we perform the following steps:

a) We replace the instruction $x_1 := x$ by the following instruction:

$$x_1 := x \ \& \ \text{PRINT}(x_1)$$

b) We replace every instruction of the form $x_i = \Gamma(x_{i-1})$ by the following instruction:

$$x_i := \Gamma(x_{i-1}) \ \& \ \text{PRINT}(x_i)$$

c) We replace every instruction of the form $x_i := \text{fact}^{-1}(x_{i-1})$ by the following instruction:

$$\text{IF } \text{fact}^{-1}(x_{i-1}) \in \mathbb{N} \setminus \{0\} \ \text{THEN } x_i := \text{fact}^{-1}(x_{i-1}) \ \& \ \text{PRINT}(x_i) \ \text{ELSE GOTO Instruction 1}$$

d) We replace every instruction of the form $x_i := \text{rem}(x_{i-1}, x_{i-2})$ by the following instruction:

$$\text{IF } \text{rem}(x_{i-1}, x_{i-2}) \in \mathbb{N} \setminus \{0\} \ \text{THEN } x_i := \text{rem}(x_{i-1}, x_{i-2}) \ \& \ \text{PRINT}(x_i) \ \text{ELSE GOTO Instruction 1}$$

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