The physical limits of computation inspire an open problem that concerns decidable sets $X \subseteq \mathbb{N}$ and cannot be formalized in $ZFC$ as it refers to the current knowledge on $X$.

Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2 + 1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$:

Let $B$ denote the system of equations:

\[
\begin{align*}
\{ x_i! = x_k : i, k \in \{1, \ldots, 9\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\} \}.
\end{align*}
\]

We write down a system $U \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1, \ldots, x_9$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. We write down a system $A \subseteq B$ of 8 equations.

Let $\Lambda$ denote the statement: if the system $A$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \leq f(9)$. The statement $\Lambda$ is equivalent to the statement $\Phi$. It heuristically proves the statement $\Phi$. This proof does not yield that $\text{card}(\mathcal{P}_{n^2 + 1}) = \omega$. The following problem is open: Is there a set $X \subseteq \mathbb{N}$ such that (there is a constructively defined integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n)$) $\wedge$ (there are many elements of $X$) $\wedge$ (the infiniteness of $X$ is conjectured and cannot be decided by any known method) $\wedge$ (there is the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements)? Let $\mathcal{F}(X)$ denote the conjunction of the first four conditions of the problem. The set $X = \{ k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7), k) \cap \mathcal{P}_{n^2 + 1} = \emptyset \}$ satisfies the formula $\mathcal{F}(X)$. No set $X \subseteq \mathbb{N}$ will satisfy the formula $\mathcal{F}(X)$ forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. The set $X = \mathcal{P}_{n^2 + 1}$ satisfies the conjunction of the last four conditions of the problem. The statement $\Phi$ implies that $X = \mathcal{P}_{n^2 + 1}$ solves the problem. It seems that the conjunction from the problem implies that the set $X$ is naturally defined, where this term has only informal meaning.

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1. Definitions and the distinction between existing algorithms and known algorithms

Algorithms always terminate. Semi-algorithms may not terminate. Examples [1–4] and the proof of Statement 1 explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known to us). A definition of an integer \( n \) is called constructive, if it provides a known algorithm with no input that returns \( n \).

**Definition 1.** Conditions (1)–(5) concern sets \( X \subseteq \mathbb{N} \).

1. There are many elements of \( X \) and it is conjectured that \( X \) is infinite.
2. No known algorithm with no input returns the logical value of the statement \( \text{card}(X) = \omega \).
3. A known algorithm for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \).
4. A known algorithm with no input returns an integer \( n \) satisfying \( \text{card}(X) < \omega \Rightarrow X \subseteq (−\infty, n] \).
5. \( X \) has the simplest definition among known sets \( Y \subseteq \mathbb{N} \) with the same set of known elements.

Condition (2) implies that no known proof shows the finiteness/infiniteness of \( X \). It seems that the conjunction of conditions (1)–(5) implies that the set \( X \) is naturally defined, where this term has only informal meaning.

**Definition 2.** Let \( \beta = (((24!)!)!)! \).

**Lemma 1.** \( \log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta)))))))))))}) \approx 1.42298 \).

**Proof.** We ask Wolfram Alpha at \[\text{http://wolframalpha.com}\].

Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2+1} \) of primes of the form \( n^2 + 1 \) is infinite, see [6]–[8]. Let \( \lfloor \cdot \rfloor \) denote the integer part function.

**Example 1.** The set \( X = \mathcal{P}_{n^2+1} \) satisfies condition (2).

**Example 2.** The set \( X = \begin{cases} \{0\}, & \text{if } \lfloor \frac{\beta}{\pi} \rfloor \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases} \) does not satisfy condition (2) because we know an algorithm with no input that computes \( \lfloor \frac{\beta}{\pi} \rfloor \).

**Example 3.** ([1], [4], [5 p. 9]). The function

\[ \mathbb{N} \ni n \rightarrow h \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases} \]

is computable because \( h = \mathbb{N} \times \{1\} \) or there exists \( k \in \mathbb{N} \) such that

\[ h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1, k+2, k+3, \ldots\} \times \{0\}) \]

No known algorithm computes the function \( h \).

**Example 4.** The set \[ X = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis is true} \\ 0, & \text{otherwise} \end{cases} \]

is decidable. No constructively existing algorithm decides \( X \), which holds forever.
Definition 3. Let $\Phi$ denote the following unproven statement:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau’s conjecture implies the statement $\Phi$. In Section 4, we heuristically prove the statement $\Phi$. This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. Condition (4) remains unproven for $X = \mathcal{P}_{n^2+1}$.

Proof. For every set $X \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(X)$ with no input that returns

$$n = \begin{cases} 
0, & \text{if card}(X) \in \{0, \omega\} \\
\max(X), & \text{otherwise}
\end{cases}$$

This $n$ satisfies the implication in condition (4), but the algorithm $\text{Alg}(\mathcal{P}_{n^2+1})$ is unknown for us because its definition is ineffective. □

Proving the statement $\Phi$ will disprove Statement 1. Statement 1 cannot be formalized in mathematics because it refers to the current mathematical knowledge. The same is true for Open Problem 1 and Statements 2 and 3.

Definition 4. We say that an integer $n$ is a threshold number of a set $X \subseteq \mathbb{N}$, if

$$\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$$

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $[\max(X), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1.

Open Problem 1. Is there a set $X \subseteq \mathbb{N}$ which satisfies conditions (1)–(5)?

Open Problem 1 asks: Are there a set $X \subseteq \mathbb{N}$ and a constructively defined integer $n$ such that (card($X$) < $\omega$ ⇒ $X \subseteq (\infty, n]$) ∧ ($X$ is decidable by a constructively defined algorithm) ∧ (there are many elements of $X$) ∧ (the infiniteness of $X$ is conjectured and cannot be decided by any known method) ∧ ($X$ has the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements)?

Statement 2. The set

$$X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions (1)–(4). Condition (5) fails for $X$.

Proof. Condition (1) holds as $X \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $\beta$, see [3]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus $X$ satisfies condition (4). Condition (5) fails for $X$ as the set of known elements of $X$ equals $\{0, \ldots, \beta\}$. □

Proving Landau’s conjecture will disprove Statement 2.
**Theorem 1.** No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

**Proof.** The proof goes by contradiction. We fix an integer $n$ that satisfies condition (4). Since conditions (2)-(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

\[(T) \quad n+1 \notin X, \quad n+2 \notin X, \quad n+3 \notin X, \ldots\]

Fig. 1 Semi-algorithm that terminates if and only if the set $X$ is infinite

The sentences from the sequence $(T)$ and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n,m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $X$ is finite, contrary to the conjecture in condition (1). $\square$

The physical limits of computation ([3]) disprove the assumption of Theorem 1.

3. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

\[
\begin{align*}
x_1! &= x_1 \\
x_1 \cdot x_3 &= x_2 \\
\forall i \in \{2, \ldots, n-1\} \quad x_i! &= x_{i+1}
\end{align*}
\]

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

Fig. 2 Construction of the system $\mathcal{U}_n$

**Lemma 2.** For every positive integer $n$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$. 
Let $B_n$ denote the following system of equations:

$$\left\{ x_i! = x_k : i, k \in \{1, \ldots, n\} \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\}$$

For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

**Theorem 2.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

**Proof.** It follows from Lemma 2 because $U_n \subseteq B_n$. □

**Theorem 3.** For every integer $n \geq 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.

**Proof.** If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement $\Psi_{n+1}$ implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)$! Hence, $x_i \leq f(n)$. □

**Theorem 4.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

## 4. A conjectural solution to Open Problem

**Lemma 3.** For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \lor (x = y = 1)$$

**Lemma 4.** ([Wilson’s theorem, [2, p. 89]]). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.

Let $A$ denote the following system of equations:

$$\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $A$. 
Lemma 5. For every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2, \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= (x_1^2)! + 1 \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)! 
\end{align*}
\]

Proof. By Lemma 4, for every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_2^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4.

Lemma 6. There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \{1, 2, \frac{1}{2}, \frac{3}{2}\} \cap \mathbb{N} = \{1, 2\}$.
Conjecture 1. The statement $\Psi_9$ is true when is restricted to the system $\mathcal{A}$.

Theorem 5. Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)!$. Consequently, $x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$.

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. Conjecture 1 implies the statement $\Phi$.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. The statement $\Phi$ implies Conjecture 1.

Proof. By Lemmas 5 and 6 if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement $\Phi$ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every $x_i$ does not exceed $f(9)$. □

Statement 3. Conditions (1)–(3) and (5) hold for $X = \mathcal{P}_{n^2+1}$. The statement $\Phi$ implies that condition (4) holds for $X = \mathcal{P}_{n^2+1}$.

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{25}]$, see [7]. These two facts imply condition (1). By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$. Thus condition (2) holds. Conditions (3) and (5) hold trivially. The statement $\Phi$ implies that $\beta$ is a threshold number of $\mathcal{P}_{n^2+1}$. Hence, the statement $\Phi$ implies that condition (4) holds for $X = \mathcal{P}_{n^2+1}$. □

Proving Landau’s conjecture will disprove Statement 3.

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