Does a sufficiently large twin prime prove that the set of twin primes is infinite?

Apoloniusz Tyszka

Abstract

Let $f(3) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 3$. For an integer $n \geqslant 3$, let Φ_n denote the following statement: if a system $\mathcal{S} \subseteq \Big\{x_i! = x_k:~(i,k \in \{1,\ldots,n\}) \wedge (i \neq k)\Big\} \cup \Big\{x_i \cdot x_j = x_k:~i,j,k \in \{1,\ldots,n\}\Big\}$ has at most finitely many solutions in integers $x_1,\ldots,x_n$ greater than 1, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(n)$. We conjecture that the statements $\Phi_3,\ldots,\Phi_{16}$ are true. We prove: (1) the statement Φ_6 proves the implication: if there exists an integer $x \geqslant 24$ such that $x!+1$ is a perfect square, then the equation $x!+1=y^2$ has infinitely many solutions in integers greater than 1; (2) if the equation $x!+1=y^2$ has only finitely many solutions in positive integers, then the statement Φ_6 implies that each such solution (x,y) belongs to the set $\{(4,5),(5,11),(7,71)\}$; (3) the statement Φ_9 proves the implication: if there exists an integer x such that x^2+1 is prime and $x^2+1 > f(7)$, then there are infinitely many primes of the form n^2+1; (4) the statement Φ_{16} proves the implication: if there exists a twin prime greater than $f(14)$, then there are infinitely many twin primes.

Key words and phrases: Brocard's problem, Brocard-Ramanujan Diophantine equation, equation $x!+1=y^2$, equation $x(x+1)=y!$, prime numbers of the form n^2+1, single query to an oracle for the halting problem, twin prime conjecture.

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1. Introduction and basic lemmas

In this article, we propose a conjecture which provides a common approach to Brocard's problem, the problem of solving the equation $x(x+1)=y!$, the problem of the infinitude of primes of the form n^2+1, and the twin prime problem. Let $f(3)=4$, and let $f(n+1)=f(n)!$ for every integer $n \geqslant 3$. For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1,\ldots,n-1\} \setminus \{2\}~x_i! &= x_{i+1} \\ x_1 \cdot x_1 &= x_3 \\ x_2 \cdot x_2 &= x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

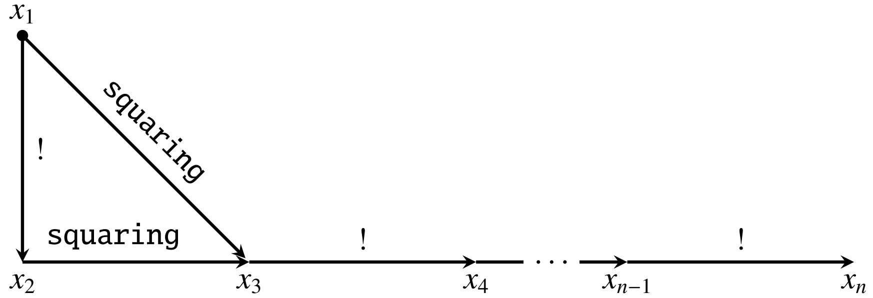

Fig. 1 Construction of the system $\mathcal{U}_n$

Lemma 1. *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly one solution in integers greater than* 1, *namely* $\big(2, 2, f(3), \ldots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k : (i, k \in \{1, \ldots, n\}) \wedge (i \neq k)\Big\} \cup \Big\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\Big\}$$

For an integer $n \geqslant 3$, let Φ_n denote the following statement: if a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in integers $x_1, \ldots, x_n$ greater than 1, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$. We conjecture that the statements $\Phi_3, \ldots, \Phi_{16}$ are true. For every integer $n \geqslant 3$, the system B_n has a finite number of subsystems. Therefore, every statement Φ_n is true with an integer bound that depends on n.

Lemma 2. *For every statement Φ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 1 because $\mathcal{U}_n \subseteq B_n$. □

Lemma 3. *For every integers x and y greater than* 1, *$x! \cdot y = y!$ if and only if $x + 1 = y$.*

Lemma 4. *If $x \geqslant 4$, then* $\dfrac{(x-1)!+1}{x} > 1$.

Lemma 5. *(Wilson's theorem, [3, p. 89]) For every integer $x \geqslant 2$, x is prime if and only if x divides* $(x-1)!+1$.

2. Brocard's problem and Erdös' problem

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

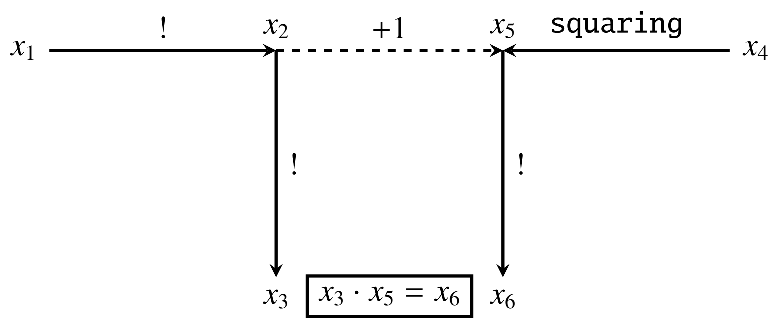

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 6. *For every integers x_1 and x_4 greater than* 1, *the system $\mathcal{A}$ is solvable in integers x_2, x_3, x_5, x_6 greater than* 1 *if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 3. □

Theorem 1. *The statement* Φ_6 *proves the implication: if there exists an integer* $x_1 \geqslant 24$ *such that* $x_1! + 1$ *is a perfect square, then the equation* $x_1! + 1 = x_4^2$ *has infinitely many solutions in integers greater than* 1.

Proof. Assume that the antecedent holds. Assume that there exists an integer $x_1 \geqslant 24$ such that $x_1! + 1$ equals x_4^2 for some non-negative integer x_4. Then, $x_4 > 1$. By Lemma 6, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0, 1\})^4$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Since $x_1 \geqslant 24 = f(4)$, we obtain that $x_5 = x_1! + 1 > f(4)!$. Hence, $x_6 = x_5! > (f(4)!)! = f(6)$. Since $\mathcal{A} \subseteq B_6$, the statement Φ_6 and the inequality $x_6 > f(6)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_6) \in (\mathbb{N} \setminus \{0, 1\})^6$. According to Lemma 6, the equation $x_1! + 1 = x_4^2$ has infinitely many solutions in integers greater than 1. □

Corollary 1. *Assuming the statement* Φ_6*, a single query to an oracle for the halting problem decides whether or not the equation* $x! + 1 = y^2$ *has infinitely many solutions in integers greater than* 1.

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [7, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [6].

Theorem 2. *If the equation* $x_1! + 1 = x_4^2$ *has only finitely many solutions in positive integers, then the statement* Φ_6 *implies that each such solution* (x_1, x_4) *belongs to the set* $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. Assume that the antecedent holds. Assume that positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 6, the system $\mathcal{A}$ is solvable in integers x_2, x_3, x_5, x_6 greater than 1. Since $\mathcal{A} \subseteq B_6$, the statement Φ_6 implies that $x_6 = (x_1! + 1)! \leqslant f(6) = f(5)!$. Hence, $x_1! + 1 \leqslant f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24$. If $x_1 \in \{2, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

Similarly, we prove the following theorem.

Theorem 3. *If the equation* $x(x + 1) = y!$ *has only finitely many solutions in positive integers, then the statement* Φ_6 *implies that each such solution* (x, y) *belongs to the set* $\{(1, 2), (2, 3)\}$.

The question of solving the equation $x(x + 1) = y!$ was posed by P. Erdös, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers, see [4].

3. Are there infinitely many prime numbers of the form $n^2 + 1$?

Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

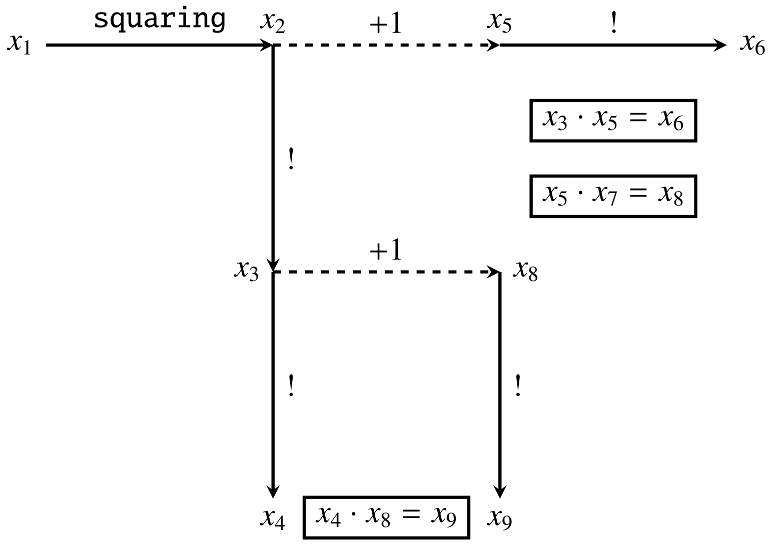

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 7. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than* 1 *if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. □

Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [5, pp. 37–38].

Theorem 4. *The statement Φ_9 proves the implication: if there exists an integer x_1 such that $x_1^2 + 1$ is prime and greater than $f(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that the antecedent holds. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement Φ_9 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. □

Corollary 2. *Assuming the statement Φ_9, a single query to an oracle for the halting problem decides Landau's problem.*

4. The twin prime conjecture

Let C denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{cases}$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system C.

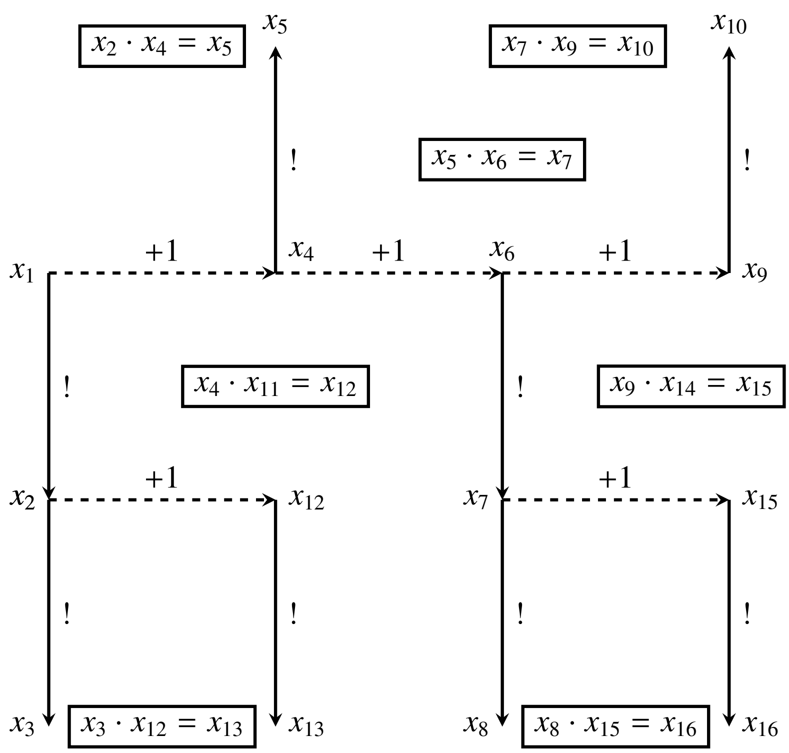

Fig. 4 Construction of the system C

Lemma 8. *If* $x_4 = 2$, *then the system* C *has no solutions in integers* $x_1, \ldots, x_{16}$ *greater than* 1.

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. □

Lemma 9. *If* $x_4 = 3$, *then the system* C *has no solutions in integers* $x_1, \ldots, x_{16}$ *greater than* 1.

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. □

Lemma 10. *For every $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0,1\}$, the system C is solvable in integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$
\begin{array}{rcl}
x_1 &=& x_4 - 1 \\
x_2 &=& (x_4 - 1)! \\
x_3 &=& ((x_4 - 1)!)! \\
x_5 &=& x_4! \\
x_6 &=& x_9 - 1 \\
x_7 &=& (x_9 - 1)! \\
x_8 &=& ((x_9 - 1)!)! \\
x_{10} &=& x_9! \\
x_{11} &=& \dfrac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &=& (x_4 - 1)! + 1 \\
x_{13} &=& ((x_4 - 1)! + 1)! \\
x_{14} &=& \dfrac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &=& (x_9 - 1)! + 1 \\
x_{16} &=& ((x_9 - 1)! + 1)!
\end{array}
$$

Proof. By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0,1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 10 follows from Lemma 5. □

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [5, p. 39].

Theorem 5. *The statement Φ_{16} proves the implication: if there exists a twin prime greater than $f(14)$, then there are infinitely many twin primes.*

Proof. Assume that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(14)$. Hence, $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$. By Lemma 10, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0,1\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Since $x_9 > f(14)$, we obtain that $x_9 - 1 \geqslant f(14)$. Therefore, $(x_9 - 1)! \geqslant f(14)! = f(15)$. Hence, $(x_9 - 1)! + 1 > f(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > f(15)! = f(16)$$

Since $C \subseteq B_{16}$, the statement Φ_{16} and the inequality $x_{16} > f(16)$ imply that the system C has infinitely many solutions in integers $x_1, \ldots, x_{16}$ greater than 1. According to Lemmas 8–10, there are infinitely many twin primes. □

Corollary 3. *Assuming the statement Φ_{16}, a single query to an oracle for the halting problem decides the twin prime problem.*

Corollary 3 conditionally solves the problem in [2].

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Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: `rttyszka@cyf-kr.edu.pl`