Open problems that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formally stated as they refer to current knowledge about $\mathcal{X}$

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Abstract

Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form n^2+1. Conditions (1)-(8) below concern sets $\mathcal{X} \subseteq \mathbb{N}$. (1) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite. (2) No known algorithm decides the finiteness of $\mathcal{X}$. (3) A known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. (4) An explicitly known integer n satisfies: $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$. (5) $\mathcal{X}$ is widely known in number theory. (6) We do not know any equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined simpler than $\mathcal{X}$. (7) For every finite set $\mathcal{F} \subseteq \mathbb{N}$, we do not know any definition of $\mathcal{X} \setminus \mathcal{F}$ simpler than the definition of $\mathcal{X}$. (8) For every set $\mathcal{Y} \subseteq \mathbb{N}$ that satisfies $\mathrm{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$, we do not know any definition of $\mathcal{Y}$ simpler than the definition of $\mathcal{X}$. Theorem. *For every explicitly known positive integer n, some simply defined set $\mathcal{X} \subseteq \mathbb{N}$ includes the set $(-\infty, n] \cap \mathbb{N}$ and satisfies conditions* (1)-(4)*. The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions* (1)-(3) *and* (5)-(8)*. The set $\mathcal{X} = \{k \in \mathbb{N} :$ the number of digits of k belongs to $\mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions* (1)-(3) *and* (6)-(8)*. Some hypothetical statement implies that these sets $\mathcal{X}$ satisfy condition* (4). We do not know any set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)-(4) and (5). The same is true, if condition (5) is replaced by condition (6) or (7) or (8).

Key words and phrases: arithmetical operations on huge integers cannot be practically performed; computable set $\mathcal{X} \subseteq \mathbb{N}$; explicitly known integer n; finiteness (infiniteness) of $\mathcal{X}$ remains conjectured; n bounds $\mathcal{X}$, if $\mathcal{X}$ is finite; no known algorithm decides the finiteness of $\mathcal{X}$.

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1 Introduction, basic definitions and lemmas

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [4, p. 199]. In this article, we present an argument against logicism: there are open problems that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formally stated as they refer to current knowledge about $\mathcal{X}$ and an intuitive concept of simplicity.

Definition 1. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Lemma 2. $((7!)!)! \approx 10^{10^{16477.87280582041}}$.

Proof. We ask Wolfram Alpha about $0.0 + ((7!)!)!$. □

Definition 2. *We say that an integer $m \geqslant -1$ is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than m, cf. [11] and [12].*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $m \geqslant -1$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $\{\max(\mathcal{X}), \max(\mathcal{X}) + 1, \max(\mathcal{X}) + 2, \ldots\}$.

Definition 3. *We say that a non-negative integer m is a weak threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if* $\mathrm{card}(\mathcal{X}) > m$.

Theorem 1. *For every $\mathcal{X} \subseteq \mathbb{N}$, if an integer $m \geqslant -1$ is a threshold number of $\mathcal{X}$, then $m + 1$ is a weak threshold number of $\mathcal{X}$.*

Proof. For every $\mathcal{X} \subseteq \mathbb{N}$, if $m \in [-1, \infty) \cap \mathbb{Z}$ and $\mathrm{card}(\mathcal{X}) > m + 1$, then $\mathcal{X} \cap [m + 1, \infty) \neq \emptyset$. □

Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form $n^2 + 1$ We do not know any weak threshold number of $\mathcal{P}_{n^2+1}$. The same is true for the sets

$$\left\{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\right\}$$

and

$$\{n \in \mathbb{N} : n! + 1 \text{ is a square}\}$$

Lemma 3. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. *(Wilson's theorem, [1, p. 89]). For every integer $x \geqslant 2$, x is prime if and only if x divides* $(x - 1)! + 1$.

Conditions (1)-(8) and (4•) below concern sets $\mathcal{X} \subseteq \mathbb{N}$.

(1) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.
(2) No known algorithm decides the finiteness of $\mathcal{X}$.
(3) A known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$.
(4) An explicitly known integer n satisfies: $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$.
(5) $\mathcal{X}$ is widely known in number theory.
(6) We do not know any equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined simpler than $\mathcal{X}$.
(7) For every finite set $\mathcal{F} \subseteq \mathbb{N}$, we do not know any definition of $\mathcal{X} \setminus \mathcal{F}$ simpler than the definition of $\mathcal{X}$.
(8) For every set $\mathcal{Y} \subseteq \mathbb{N}$ that satisfies $\mathrm{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$, we do not know any definition of $\mathcal{Y}$ simpler than the definition of $\mathcal{X}$.
(4•) An explicitly known integer n satisfies: $\mathrm{card}(\mathcal{X}) = \omega \Longleftrightarrow \mathrm{card}(\mathcal{X}) > n$.

2 Open Problems 1 and 2

The following two open problems cannot be formally stated as they refer to current knowledge about $\mathcal{X}$ and an intuitive concept of simplicity.

Open Problem 1. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1)-(3), (4•), *and* (5).

Open Problem 2. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1)-(5).

Theorem 2. *Open Problem 2 claims more than Open Problem 1.*

Proof. By Theorem 1, condition (4) implies condition (4•). □

Open Problems 1 and 2 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

3 Partial solutions to Open Problem 2

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ is infinite, see [5, pp. 37–38] and [8]. Let $\mathcal{M}$ denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$.

Theorem 3. *The set* $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{M}$ *satisfies conditions* `(1)-(4)`.

Proof. Condition `(1)` holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set $\mathcal{M}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

Let $[\cdot]$ denote the integer part function.

Lemma 5. *For every non-negative integer n,* $\left[\dfrac{3n-3\beta+3}{3n-3\beta+2}\right]$ *equals* 0 *or* 1. *The first case holds when* $n \leqslant \beta - 1$. *The second case holds when* $n \geqslant \beta$.

Lemma 6. *The function*

$$\mathbb{N} \cap [\beta, \infty) \ni n \xrightarrow{\theta} \beta + n - \left[\sqrt{n}\right]^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geqslant \beta$ *infinitely many times.*

Proof. Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \left\{(t+0)^2 + t,\ (t+1)^2 + t,\ (t+2)^2 + t, \ldots\right\} \cap [\beta, \infty)$$

□

Theorem 4. *The set*

$$\mathcal{X} = \left\{n \in \mathbb{N} : 2 + \left[\frac{3n-3\beta+3}{3n-3\beta+2}\right] \cdot \left(\left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 - 1\right) \text{ is prime}\right\}$$

satisfies conditions `(1)-(4)`.

Proof. Condition `(3)` holds trivially. By Lemma 5, $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$, where

$$\mathcal{H} = \left\{n \in \mathbb{N} \cap [\beta, \infty) : \left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 + 1 \text{ is prime}\right\}$$

By Lemma 6, the set $\mathcal{H}$ is empty or infinite. The second case holds when

$$\exists\, k \in \mathbb{N} \cap [\beta, \infty)\ \ k^2 + 1 \text{ is prime} \tag{G}$$

The equality $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of $\mathcal{X}$ and conditions `(1)` and `(4)` hold. Condition `(2)` holds as due to known physics we are not able to confirm the statement `(G)` by a direct computation. □

4 The statements Ψ_n, which seem to be true for every $n \in \{1, \ldots, 9\}$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

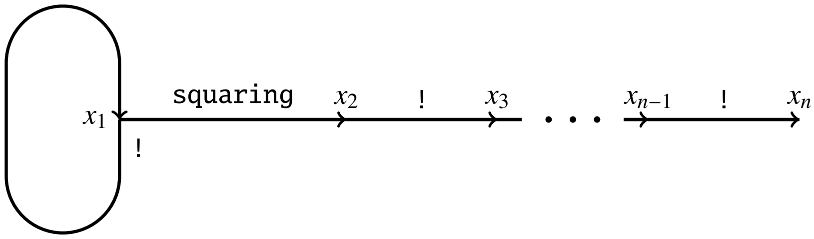

Fig. 1 Construction of the system $\mathcal{U}_n$

Lemma 7. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \dots, 1)$ *and* $\big(f(1), \dots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k : \; i, k \in \{1, \dots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : \; i, j, k \in \{1, \dots, n\}\Big\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The author's guess is that the statements $\Psi_1, \dots, \Psi_9$ are true.

Theorem 5. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

Theorem 6. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 7 because $\mathcal{U}_n \subseteq B_n$. □

5 The statement Ψ_9 solves Open Problem 2

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

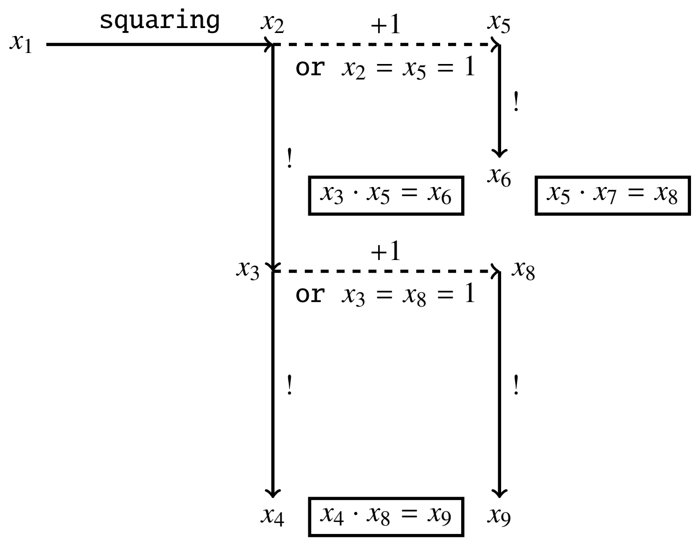

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 4. □

Lemma 9. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{A}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Theorem 7. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Since $\mathcal{A} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 8 and 9 the set $\mathcal{P}_{n^2+1}$ is infinite. □

Let $\mathcal{K} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$.

Lemma 10. $\operatorname{card}(\mathcal{K}) \geqslant 9 \cdot 10^{9 \cdot 4^{747}} \approx 10^{10^{450.6930560314272}}$.

Proof. The following PARI/GP ([7]) command

```
isprime(1+9*4^747,{flag=2})
```

returns %1 = 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([10, p. 226]). It rigorously shows that the number $\left(3 \cdot 2^{747}\right)^2 + 1$ is prime. Since $9 \cdot 10^{9 \cdot 4^{747}}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10\^{}(9 * 4\^{}747))$. □

Theorem 8. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions* (1)-(3) *and* (5)-(8)*. The set $\mathcal{X} = \mathcal{K}$ satisfies conditions* (1)-(3) *and* (6)-(8)*. The statement Ψ_9 implies that these sets $\mathcal{X}$ satisfy condition* (4)*.*

Proof. Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, Lemma 10 implies condition (1) for both sets $\mathcal{X}$. Conditions (3) and (6)-(8) hold trivially for both sets $\mathcal{X}$. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$. Thus condition (2) holds for both sets $\mathcal{X}$. Suppose that the statement Ψ_9 is true. By Theorem 7, $f(7)$ is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. By Theorem 7, $\underbrace{9 \ldots 9}_{f(7) \text{ digits}}$ is a threshold number of $\mathcal{X} = \mathcal{K}$. Thus condition (4) holds for both sets $\mathcal{X}$. □

6 Open Problems 3 and 4

Definition 4. *Let* (1⋄) *denote the following condition: there are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X} = \mathbb{N}$.*

Definition 5. *Let* (2⋄) *denote the following condition: no known algorithm decides the equality $\mathcal{X} = \mathbb{N}$.*

The following two open problems cannot be formally stated as they refer to current knowledge about $\mathcal{X}$ and an intuitive concept of simplicity.

Open Problem 3. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1⋄)-(2⋄)*,* (2)-(3)*,* (4•)*, and* (5)*.*

Open Problem 3 claims more than Open Problem 1 as condition (1⋄) implies condition (1).

Open Problem 4. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1⋄)-(2⋄) *and* (2)-(5)*.*

Open Problem 4 claims more than Open Problem 2 as condition (1⋄) implies condition (1).

Theorem 9. *Open Problem 4 claims more than Open Problem 3.*

Proof. By Theorem 1, condition (4) implies condition (4•). □

Open Problems 3 and 4 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

7 A partial solution to Open Problem 4

Let $\mathcal{V}$ denote the set of all positive multiples of elements of the set

$$\{n \in \{\beta + 1,\ \beta + 2,\ \beta + 3,\ \ldots\} : 2^{2^n} + 1 \text{ is composite}\}$$

Theorem 10. *The set $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{V}$ satisfies conditions* (1⋄)-(2⋄) *and* (2)-(4)*.*

Proof. The inequality card($\mathcal{X}$) > β holds trivially. Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [2, p. 23]. These two facts imply conditions `(1⋄)` and `(2⋄)`. Condition `(3)` holds trivially. Since the set $\mathcal{V}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. The question of finiteness of the set $\{n \in \mathbb{N} : 2^{2^n}+1 \text{ is composite}\}$ remains open, see [3, p. 159]. By this and Lemma 1, the question of emptiness of the set

$$\{n \in \{\beta+1,\ \beta+2,\ \beta+3,\ \ldots\} : 2^{2^n}+1 \text{ is composite}\}$$

remains open. Therefore, the question of finiteness of the set $\mathcal{V}$ remains open. Consequently, the question of finiteness of the set $\mathcal{X}$ remains open and condition `(2)` holds. □

8 Open Problems 5 and 6

Definition 6. *Let* `(1*)` *denote the following condition: there are a large number of elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is finite.*

The following two open problems cannot be formally stated as they refer to current knowledge about $\mathcal{X}$ and an intuitive concept of simplicity.

Open Problem 5. *Simply define a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1*)`, `(2)-(3)`, `(4•)`, *and* `(5)`.

Open Problem 6. *Simply define a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1*)` *and* `(2)-(5)`.

Theorem 11. *Open Problem 6 claims more than Open Problem 5.*

Proof. By Theorem 1, condition `(4)` implies condition `(4•)`. □

Open Problems 5 and 6 remain open, if condition `(5)` is replaced by condition `(6)` or `(7)` or `(8)`.

9 Partial solutions to Open Problem 6

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x!+1=y^2$, see [6].

Lemma 11. *([9, p. 297]). It is conjectured that* $x!+1$ *is a square only for* $x \in \{4,5,7\}$.

Let $\mathcal{W}$ denote the set of all integers x greater than β such that $x!+1$ is a square.

Theorem 12. *The set*

$$\mathcal{X} = \{0,\ldots,\beta\} \cup \{k \cdot x :\ (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

satisfies conditions `(1*)` *and* `(2)-(4)`.

Proof. Condition `(1*)` holds as card($\mathcal{X}$) > β and the set $\mathcal{W}$ is conjecturally empty by Lemma 11. Condition `(3)` holds trivially. We do not know any algorithm that decides the emptiness of $\mathcal{W}$ and the set

$$\mathcal{Y} = \{k \cdot x :\ (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

is empty or infinite. Thus condition `(2)` holds. Since the set $\mathcal{Y}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

Let C denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system C.

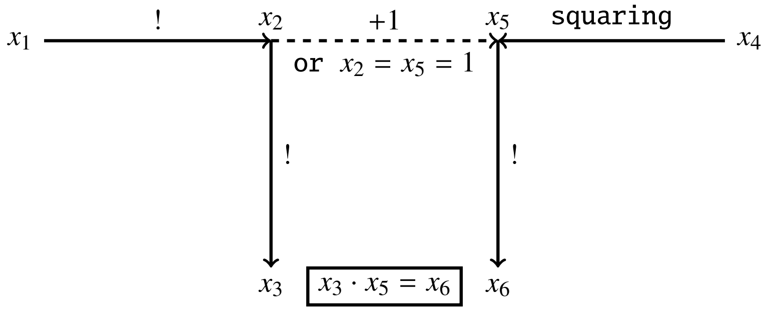

Fig. 3 Construction of the system C

Lemma 12. *For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 3. □

Theorem 13. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) satisfies $x_1 < 24!$.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 12, the system C is solvable in positive integers x_2, x_3, x_5, x_6. Since $C \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leqslant f(6) = f(5)!$. Hence, $x_1! + 1 \leqslant f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24!$. □

Theorem 14. *Let $\mathcal{X}$ denote the set of all non-negative integers n which have $((k!)!)!$ digits for some $k \in \{m \in \mathbb{N} : m! + 1 \text{ is a square}\}$. We claim that $\mathcal{X}$ satisfies conditions* `(1*)`*,* `(2)-(3)`*, and* `(6)-(8)`*. The statement Ψ_6 implies that $\mathcal{X}$ satisfies condition* `(4)`.

Proof. Let $d = ((7!)!)!$. Since $7! + 1 = 71^2$, we obtain that $\{10^{d-1}, \ldots, \underbrace{9 \ldots 9}_{d \text{ digits}}\} \subseteq \mathcal{X}$. Hence, $\text{card}(\mathcal{X}) \geqslant 9 \cdot 10^{d-1}$. By this and Lemmas 2 and 11, condition `(1*)` holds. Conditions `(2)-(3)` and `(6)-(8)` hold trivially. By Theorem 13, the statement Ψ_6 implies that $\underbrace{9 \ldots 9}_{\beta \text{ digits}}$ is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

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References

[1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory,* 2nd ed., CRC Press, Boca Raton, FL, 2016.

[2] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers,* American Mathematical Society, Providence, RI, 2012.

[3] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry,* Springer, New York, 2001.

[4] W. Marciszewski, *Logic, modern, history of,* in: *Dictionary of logic as applied in the study of language* (ed. W. Marciszewski), pp. 183–200, Springer, Dordrecht, 1981.

[5] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT,* Springer, London, 2012.

[6] M. Overholt, *The Diophantine equation* $n! + 1 = m^2$, Bull. London Math. Soc. 25 (1993), no. 2, p. 104.

[7] PARI/GP *online documentation,* http://pari.math.u-bordeaux.fr/dochtml/html/Arithmetic_functions.html.

[8] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002496, Primes of the form* $n^2 + 1$, http://oeis.org/A002496.

[9] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics,* 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.

[10] S. Y. Yan, *Number theory for computing,* 2nd ed., Springer, Berlin, 2002.

[11] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity,* Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm.

[12] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer,* in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

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