

Open problems that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formally stated as they refer to current knowledge about \mathcal{X}

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Abstract

Conditions (1)–(8) below concern sets $\mathcal{X} \subseteq \mathbb{N}$. (1) There are a large number of elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite. (2) No known algorithm decides the finiteness of \mathcal{X} . (3) A known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. (4) An explicitly known integer n satisfies: $\text{card}(\mathcal{X}) < \omega \implies \mathcal{X} \subseteq (-\infty, n]$. (5) \mathcal{X} is widely known in number theory. (6) There is no known equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where \mathcal{X}_1 and \mathcal{X}_2 are defined simpler than \mathcal{X} . (7) No known set \mathcal{Y} is defined simpler than \mathcal{X} and satisfies $(\mathcal{Y} \subseteq \mathcal{X}) \wedge (\text{card}(\mathcal{X} \setminus \mathcal{Y}) < \omega)$. (8) No known set \mathcal{Y} is defined simpler than \mathcal{X} and satisfies $\text{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$. We do not know any set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(4) and (5). The same is true, if condition (5) is replaced by condition (6) or (7) or (8). For every explicitly known integer n , some simply defined set $\mathcal{X} \subseteq \mathbb{N}$ includes the set $(-\infty, n] \cap \mathbb{N}$ and satisfies conditions (1)–(4). Let \mathcal{P}_{n^2+1} denote the set of primes of the form $n^2 + 1$. The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3) and (5)–(8). The set $\mathcal{X} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$ contains 10^{10450} consecutive integers and satisfies conditions (1)–(3) and (6)–(8). Some hypothetical statement implies that these sets \mathcal{X} satisfy condition (4).

Key words and phrases: arithmetical operations on huge integers cannot be practically performed; computable set $\mathcal{X} \subseteq \mathbb{N}$; explicitly known integer n ; finiteness (infiniteness) of \mathcal{X} remains conjectured; n bounds \mathcal{X} , if \mathcal{X} is finite; no known algorithm decides the finiteness of \mathcal{X} .

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1 Introduction, basic definitions and lemmas

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [4], p. 199]. In this article, we present an argument against logicism: there are open problems that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formally stated as they refer to current knowledge about \mathcal{X} and an intuitive concept of simplicity.

Definition 1. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{25.16114896940657}}}$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. □

Lemma 2. $((7!)!)! \approx 10^{10^{16477.87280582041}}$.

Proof. We ask Wolfram Alpha about $0.0 + ((7!)!)!$. □

Definition 2. We say that an integer $m \geq -1$ is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if \mathcal{X} is infinite if and only if \mathcal{X} contains an element greater than m , cf. [11] and [12].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $m \geq -1$ is a threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of \mathcal{X} form the set $\{\max(\mathcal{X}), \max(\mathcal{X}) + 1, \max(\mathcal{X}) + 2, \dots\}$.

Definition 3. We say that a non-negative integer m is a weak threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if \mathcal{X} is infinite if and only if $\text{card}(\mathcal{X}) > m$.

Theorem 1. For every $\mathcal{X} \subseteq \mathbb{N}$, if an integer $m \geq -1$ is a threshold number of \mathcal{X} , then $m + 1$ is a weak threshold number of \mathcal{X} .

Proof. For every $\mathcal{X} \subseteq \mathbb{N}$, if $m \in [-1, \infty) \cap \mathbb{Z}$ and $\text{card}(\mathcal{X}) > m + 1$, then $\mathcal{X} \cap [m + 1, \infty) \neq \emptyset$. □

Let \mathcal{P}_{n^2+1} denote the set of primes of the form $n^2 + 1$. We do not know any weak threshold number of \mathcal{P}_{n^2+1} . The same is true for the sets

$$\left\{ n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite} \right\}$$

and

$$\{ n \in \mathbb{N} : n! + 1 \text{ is a square} \}$$

Lemma 3. For every positive integers x and y , $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. (Wilson's theorem, [1] p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x - 1)! + 1$.

Conditions (1)–(8) and (4•) below concern sets $\mathcal{X} \subseteq \mathbb{N}$.

- (1) There are a large number of elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite.
- (2) No known algorithm decides the finiteness of \mathcal{X} .
- (3) A known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$.
- (4) An explicitly known integer n satisfies: $\text{card}(\mathcal{X}) < \omega \implies \mathcal{X} \subseteq (-\infty, n]$.
- (5) \mathcal{X} is widely known in number theory.
- (6) There is no known equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where \mathcal{X}_1 and \mathcal{X}_2 are defined simpler than \mathcal{X} .
- (7) No known set \mathcal{Y} is defined simpler than \mathcal{X} and satisfies $(\mathcal{Y} \subseteq \mathcal{X}) \wedge (\text{card}(\mathcal{X} \setminus \mathcal{Y}) < \omega)$.
- (8) No known set \mathcal{Y} is defined simpler than \mathcal{X} and satisfies $\text{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$.
- (4•) An explicitly known integer n satisfies: $\text{card}(\mathcal{X}) = \omega \iff \text{card}(\mathcal{X}) > n$.

2 Open Problems 1 and 2

The following two open problems cannot be formally stated as they refer to current knowledge about \mathcal{X} and an intuitive concept of simplicity.

Open Problem 1. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(3), (4•), and (5).*

Open Problem 2. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(5).*

Theorem 2. *Open Problem 2 claims more than Open Problem 1*

Proof. By Theorem 1, condition (4) implies condition (4•). □

Open Problems 1 and 2 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

3 Partial solutions to Open Problem 2

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} is infinite, see [5] pp. 37–38] and [8]. Let \mathcal{M} denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$.

Theorem 3. *The set $\mathcal{X} = \{0, \dots, \beta\} \cup \mathcal{M}$ satisfies conditions (1)–(4).*

Proof. Condition (1) holds as $\text{card}(\mathcal{X}) > \beta$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β . Thus condition (2) holds. Condition (3) holds trivially. Since the set \mathcal{M} is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. □

Let $[\cdot]$ denote the integer part function.

Lemma 5. *For every non-negative integer n , $\left\lfloor \frac{3n - 3\beta + 3}{3n - 3\beta + 2} \right\rfloor$ equals 0 or 1. The first case holds when $n \leq \beta - 1$. The second case holds when $n \geq \beta$.*

Lemma 6. *The function*

$$\mathbb{N} \cap [\beta, \infty) \ni n \xrightarrow{\theta} \beta + n - \left[\sqrt{n} \right]^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geq \beta$ infinitely many times.

Proof. Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \left\{ (t+0)^2 + t, (t+1)^2 + t, (t+2)^2 + t, \dots \right\} \cap [\beta, \infty)$$

□

Theorem 4. *The set*

$$\mathcal{X} = \left\{ n \in \mathbb{N} : 2 + \left\lfloor \frac{3n - 3\beta + 3}{3n - 3\beta + 2} \right\rfloor \cdot \left(\left(\beta + n - \left[\sqrt{n} \right]^2 \right)^2 - 1 \right) \text{ is prime} \right\}$$

satisfies conditions (1)–(4).

Proof. Condition (3) holds trivially. By Lemma 5, $\mathcal{X} = \{0, \dots, \beta - 1\} \cup \mathcal{H}$, where

$$\mathcal{H} = \left\{ n \in \mathbb{N} \cap [\beta, \infty) : \left(\beta + n - \left[\sqrt{n} \right]^2 \right)^2 + 1 \text{ is prime} \right\}$$

By Lemma 6, the set \mathcal{H} is empty or infinite. The second case holds when

$$\exists k \in \mathbb{N} \cap [\beta, \infty) \quad k^2 + 1 \text{ is prime} \tag{G}$$

The equality $\mathcal{X} = \{0, \dots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of \mathcal{X} and conditions (1) and (4) hold. Condition (2) holds as due to known physics we are not able to confirm the statement (G) by a direct computation. □

4 The statements Ψ_n , which seem to be true for every $n \in \{1, \dots, 9\}$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{U}_n .

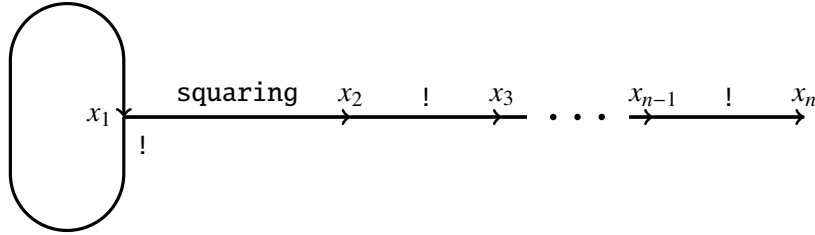


Fig. 1 Construction of the system \mathcal{U}_n

Lemma 7. For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.

Let

$$B_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Ψ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$. The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The author's guess is that the statements Ψ_1, \dots, Ψ_9 are true.

Theorem 5. Every statement Ψ_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

Theorem 6. For every statement Ψ_n , the bound $f(n)$ cannot be decreased.

Proof. It follows from Lemma 7 because $\mathcal{U}_n \subseteq B_n$. □

5 The statement Ψ_9 solves Open Problem 2

Let \mathcal{A} denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

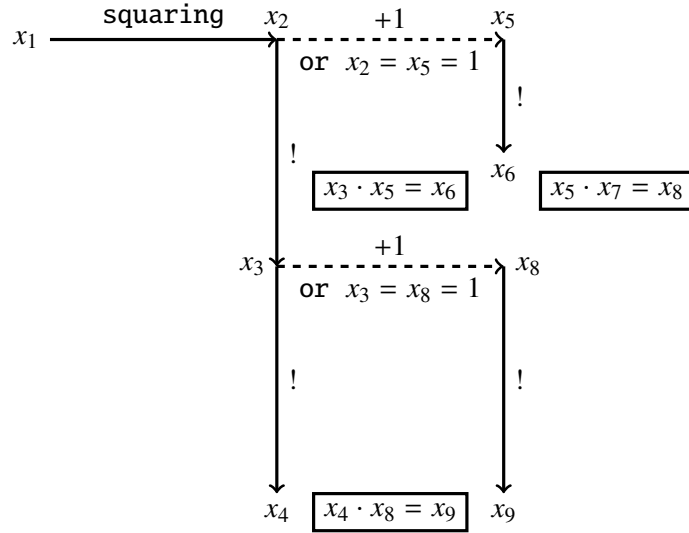


Fig. 2 Construction of the system \mathcal{A}

Lemma 8. For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

Proof. By Lemma 3, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 4. \square

Lemma 9. There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$.

Proof. If a tuple $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system \mathcal{A} and $x_1 = 1$, then $x_1, \dots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. \square

Theorem 7. The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.

Proof. Suppose that the antecedent holds. By Lemma 8, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Since $\mathcal{A} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 8 and 9 the set \mathcal{P}_{n^2+1} is infinite. \square

Let $\mathcal{K} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$.

Lemma 10. $\text{card}(\mathcal{K}) \geq 9 \cdot 10^9 \cdot 4^{747} \approx 10^{10^{450.6930560314272}}$.

Proof. The following PARI/GP ([7]) command

`isprime(1+9*4^747, {flag=2})`

returns %1 = 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([10], p. 226). It rigorously shows that the number $(3 \cdot 2^{747})^2 + 1$ is prime. Since $9 \cdot 10^9 \cdot 4^{747}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10^{(9 * 4^{747})})$. \square

Theorem 8. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)-(3) and (5)-(8). The set $\mathcal{X} = \mathcal{K}$ satisfies conditions (1)-(3) and (6)-(8). The statement Ψ_9 implies that these sets \mathcal{X} satisfy condition (4).*

Proof. Since the set \mathcal{P}_{n^2+1} is conjecturally infinite, Lemma [10] implies condition (1) for both sets \mathcal{X} . Conditions (3) and (6)-(8) hold trivially for both sets \mathcal{X} . By Lemma [1], due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than $f(7) = (((24!)!)!) = \beta$. Thus condition (2) holds for both sets \mathcal{X} . Suppose that the statement Ψ_9 is true. By Theorem [7], $f(7)$ is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. By Theorem [7], $\underbrace{9 \dots 9}_{f(7) \text{ digits}}$ is a threshold number of $\mathcal{X} = \mathcal{K}$. Thus condition (4) holds for both sets \mathcal{X} . \square

6 Open Problems [3] and [4]

Definition 4. *Let $(1\blacklozenge)$ denote the following condition: there are a large number of elements of \mathcal{X} and it is conjectured that $\mathcal{X} = \mathbb{N}$.*

Definition 5. *Let $(2\blacklozenge)$ denote the following condition: no known algorithm decides the equality $\mathcal{X} = \mathbb{N}$.*

The following two open problems cannot be formally stated as they refer to current knowledge about \mathcal{X} and an intuitive concept of simplicity.

Open Problem 3. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions $(1\blacklozenge)$ - $(2\blacklozenge)$, (2)-(3), $(4\bullet)$, and (5).*

Open Problem [3] claims more than Open Problem [1] as condition $(1\blacklozenge)$ implies condition (1).

Open Problem 4. *Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions $(1\blacklozenge)$ - $(2\blacklozenge)$ and (2)-(5).*

Open Problem [4] claims more than Open Problem [2] as condition $(1\blacklozenge)$ implies condition (1).

Theorem 9. *Open Problem [4] claims more than Open Problem [3]*

Proof. By Theorem [1], condition (4) implies condition $(4\bullet)$. \square

Open Problems [3] and [4] remain open, if condition (5) is replaced by condition (6) or (7) or (8).

7 A partial solution to Open Problem [4]

Let \mathcal{V} denote the set of all positive multiples of elements of the set

$$\{n \in \{\beta + 1, \beta + 2, \beta + 3, \dots\} : 2^{2^n} + 1 \text{ is composite}\}$$

Theorem 10. *The set $\mathcal{X} = \{0, \dots, \beta\} \cup \mathcal{V}$ satisfies conditions $(1\blacklozenge)$ - $(2\blacklozenge)$ and (2)-(4).*

Proof. The inequality $\text{card}(X) > \beta$ holds trivially. Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [2] p. 23]. These two facts imply conditions (1 \diamond) and (2 \diamond). Condition (3) holds trivially. Since the set \mathcal{V} is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. The question of finiteness of the set $\{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\}$ remains open, see [3] p. 159]. By this and Lemma [1], the question of emptiness of the set

$$\{n \in \{\beta + 1, \beta + 2, \beta + 3, \dots\} : 2^{2^n} + 1 \text{ is composite}\}$$

remains open. Therefore, the question of finiteness of the set \mathcal{V} remains open. Consequently, the question of finiteness of the set \mathcal{X} remains open and condition (2) holds. \square

8 Open Problems [5] and [6]

Definition 6. Let (1*) denote the following condition: there are a large number of elements of \mathcal{X} and it is conjectured that \mathcal{X} is finite.

The following two open problems cannot be formally stated as they refer to current knowledge about \mathcal{X} and an intuitive concept of simplicity.

Open Problem 5. Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1*), (2)–(3), (4 \bullet), and (5).

Open Problem 6. Simply define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1*) and (2)–(5).

Theorem 11. Open Problem [6] claims more than Open Problem [5]

Proof. By Theorem [1], condition (4) implies condition (4 \bullet). \square

Open Problems [5] and [6] remain open, if condition (5) is replaced by condition (6) or (7) or (8).

9 Partial solutions to Open Problem [6]

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [6].

Lemma 11. ([9] p. 297]). It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$.

Let \mathcal{W} denote the set of all integers x greater than β such that $x! + 1$ is a square.

Theorem 12. The set

$$\mathcal{X} = \{0, \dots, \beta\} \cup \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

satisfies conditions (1*) and (2)–(4).

Proof. Condition (1*) holds as $\text{card}(\mathcal{X}) > \beta$ and the set \mathcal{W} is conjecturally empty by Lemma [11]. Condition (3) holds trivially. We do not know any algorithm that decides the emptiness of \mathcal{W} and the set

$$\mathcal{Y} = \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

is empty or infinite. Thus condition (2) holds. Since the set \mathcal{Y} is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

Let C denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma [3] and the diagram in Figure 3 explain the construction of the system C .

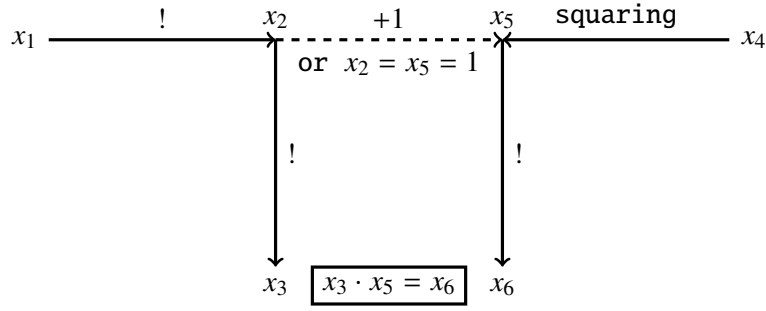


Fig. 3 Construction of the system C

Lemma 12. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 3. □

Theorem 13. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) satisfies $x_1 < 24!$.

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 12, the system C is solvable in positive integers x_2, x_3, x_5, x_6 . Since $C \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leq f(6) = f(5)!$. Hence, $x_1! + 1 \leq f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24!$. □

Theorem 14. Let X denote the set of all non-negative integers n which have $((k!)!)!$ digits for some $k \in \{m \in \mathbb{N} : m! + 1 \text{ is a square}\}$. We claim that X satisfies conditions (1*), (2)–(3), and (6)–(8). The statement Ψ_6 implies that X satisfies condition (4).

Proof. Let $d = ((7!)!)!$. Since $7! + 1 = 71^2$, we obtain that $\{10^{d-1}, \dots, \underbrace{9 \dots 9}_{d \text{ digits}}\} \subseteq X$. Hence, $\text{card}(X) \geq 9 \cdot 10^{d-1}$. By this and Lemmas 2 and 11, condition (1*) holds. Conditions (2)–(3) and (6)–(8) hold trivially. By Theorem 13, the statement Ψ_6 implies that $\underbrace{9 \dots 9}_{\beta \text{ digits}}$ is a threshold number of X . Thus condition (4) holds. □

References

- [1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [2] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers*, American Mathematical Society, Providence, RI, 2012.
- [3] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry*, Springer, New York, 2001.
- [4] W. Marciszewski, *Logic, modern, history of*, in: *Dictionary of logic as applied in the study of language* (ed. W. Marciszewski), pp. 183–200, Springer, Dordrecht, 1981.

- [5] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT*, Springer, London, 2012.
- [6] M. Overholt, *The Diophantine equation $n! + 1 = m^2$* , Bull. London Math. Soc. 25 (1993), no. 2, p. 104.
- [7] PARI/GP *online documentation*, http://pari.math.u-bordeaux.fr/dochtml/html/Arithmetic_functions.html.
- [8] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A002496, *Primes of the form $n^2 + 1$* , <http://oeis.org/A002496>.
- [9] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [10] S. Y. Yan, *Number theory for computing*, 2nd ed., Springer, Berlin, 2002.
- [11] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity*, Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, <http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm>.
- [12] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer*, in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

On *ZFC*-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite

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Abstract

Let $\Gamma(k)$ denote $(k-1)!$, and let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \dots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \dots, 16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ with Γ instead of Γ_n has only finitely many solutions in positive integers x_1, \dots, x_n , then every tuple $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system \mathcal{S} satisfies $x_1, \dots, x_n \leq 2^{2^{n-2}}$. Our hypothesis claims that the statements $\Sigma_3, \dots, \Sigma_{16}$ are true. The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(1, 2), (2, 3)\}$. The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$. The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$. The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geq 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$. The statement Σ_{14} implies the infinitude of twin primes. The statement Σ_{16} implies the infinitude of Sophie Germain primes.

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation $x! + 1 = y^2$, composite Fermat numbers, decidability in the limit, Erdős' equation $x(x+1) = y!$, finiteness of a set, infiniteness of a set, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, single query to an oracle for the halting problem, Sophie Germain primes, twin primes.

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1 Introduction and basic lemmas

The phrase “we know a non-negative integer n ” in the title means that we know an algorithm which returns n . The title of the article cannot be formalised in *ZFC* because the phrase “we know a non-negative integer n ” refers to currently known non-negative integers n with some property. A formally stated title may look like this: *On ZFC-formulae $\varphi(x)$ for which there exists a non-negative integer n such that ZFC proves that*

$$\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) < \infty \implies \{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer n such that *ZFC* proves the above implication.

Lemma 1. *For every non-negative integer n , $\text{card}(\{x \in \mathbb{N} : x \leq n - 1\}) = n$.*

Corollary 1. *The title altered to “On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) \leq n$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite” involves a weaker assumption on $\varphi(x)$.*

Lemma 2. *For every positive integers x and y , $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Let $\Gamma(k)$ denote $(k - 1)!$.

Lemma 3. For every positive integers x and y , $x \cdot \Gamma(x) = \Gamma(y)$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. For every non-negative integers b and c , $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.

Lemma 5. (Wilson's theorem, [8] p. 89). For every positive integer x , x divides $(x - 1)! + 1$ if and only if $x = 1$ or x is prime.

2 Subsets of \mathbb{N} and their threshold numbers

We say that a non-negative integer m is a threshold number of a set $X \subseteq \mathbb{N}$, if X is infinite if and only if X contains an element greater than m , cf. [24] and [25]. If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer m is a threshold number of X . If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $\{\max(X), \max(X) + 1, \max(X) + 2, \dots\}$.

It is conjectured that the set of prime numbers of the form $n^2 + 1$ is infinite, see [14], pp. 37–38]. It is conjectured that the set of prime numbers of the form $n! + 1$ is infinite, see [3] p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [14], p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^n} + 1$ is infinite, see [10], p. 23] and [11], pp. 158–159]. A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [22]. It is conjectured that the set of Sophie Germain primes is infinite, see [17], p. 330]. For each of these sets, we do not know any threshold number.

The following statement:

for every non-negative integer n there exist

$$\text{prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in [10^n, 10^n + 1] \quad (1)$$

is a Π_1 statement which strengthens the twin prime conjecture, see [4], p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger Π_1 statements, see [1]. Statement (1) is equivalent to the non-halting of a Turing machine. If a set $X \subseteq \mathbb{N}$ is computable and we know a threshold number of X , then the infinity of X is equivalent to the halting of a Turing machine.

The height of a rational number $\frac{p}{q}$ is denoted by $H\left(\frac{p}{q}\right)$ and equals $\max(|p|, |q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple (x_1, \dots, x_n) is denoted by $H(x_1, \dots, x_n)$ and equals $\max(H(x_1), \dots, H(x_n))$.

Lemma 6. The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [13] p. 212]. The known rational solutions are $(x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left(\frac{1}{4}, \frac{15}{32}\right), \left(\frac{1}{4}, \frac{17}{32}\right), \left(-\frac{15}{16}, -\frac{185}{1024}\right), \left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [18] pp. 223–224].

Corollary 2. The set $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$ is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of \mathcal{T} .

Let \mathcal{L} denote the following system of equations:

$$\begin{cases} x^2 + y^2 = s^2 \\ x^2 + z^2 = t^2 \\ y^2 + z^2 = u^2 \\ x^2 + y^2 + z^2 = v^2 \end{cases}$$

Let

$$\mathcal{F} = \left\{ n \in \mathbb{N} \setminus \{0\} : \left(\text{the system } \mathcal{L} \text{ has no solutions in } \{1, \dots, n\}^7 \right) \wedge \right. \\ \left. \left(\text{the system } \mathcal{L} \text{ has a solution in } \{1, \dots, n+1\}^7 \right) \right\}$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

Lemma 7. ([21]). *No perfect cuboids are known.*

Corollary 3. *We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{F}$. ZFC proves that $\text{card}(\mathcal{F}) \in \{0, 1\}$. We do not know any algorithm which returns $\text{card}(\mathcal{F})$. We do not know any algorithm which returns a threshold number of \mathcal{F} .*

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, & \text{if } \sin\left(999999\right) < 0 \\ \mathbb{N} \cap \left[0, \sin\left(999999\right) \cdot 999999 \right) & \text{otherwise} \end{cases}$$

We do not know whether or not the set \mathcal{H} is finite.

Proposition 1. *The number 999999 is a threshold number of \mathcal{H} . We know an algorithm which decides the equality $\mathcal{H} = \mathbb{N}$. If $\mathcal{H} \neq \mathbb{N}$, then the set \mathcal{H} consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{H}$.*

Let

$$\mathcal{K} = \begin{cases} \{n\}, & \text{if } (n \in \mathbb{N}) \wedge (2^{\aleph_0} = \aleph_{n+1}) \\ \{0\}, & \text{if } 2^{\aleph_0} \geq \aleph_{\omega} \end{cases}$$

Proposition 2. *ZFC proves that $\text{card}(\mathcal{K}) = 1$. If ZFC is consistent, then for every $n \in \mathbb{N}$ the sentences "n is a threshold number of \mathcal{K} " and "n is not a threshold number of \mathcal{K} " are not provable in ZFC.*

Proof. It suffices to observe that 2^{\aleph_0} can attain every value from the set $\{\aleph_1, \aleph_2, \aleph_3, \dots\}$, see [7] and [9] p. 232]. \square

3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matijasevich theorem imply the following theorem.

Theorem 1. ([5] p. 35]. *There exists a polynomial $D(x_1, \dots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation $D(x_1, \dots, x_m) = 0$ is solvable in non-negative integers" and "The equation $D(x_1, \dots, x_m) = 0$ is not solvable in non-negative integers" are not provable in ZFC.*

Let \mathcal{Y} denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has no solutions in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Theorem 1 implies the next theorem.

Theorem 2. For every $n \in \mathbb{N}$, ZFC proves that $n \in \mathcal{Y}$. If ZFC is arithmetically consistent, then the sentences “ \mathcal{Y} is finite” and “ \mathcal{Y} is infinite” are not provable in ZFC. If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences “ n is a threshold number of \mathcal{Y} ” and “ n is not a threshold number of \mathcal{Y} ” are not provable in ZFC.

Let \mathcal{E} denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has a solution in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{E}$. Theorem 1 implies the next theorem.

Theorem 3. The set \mathcal{E} is empty or infinite. In both cases, every non-negative integer n is a threshold number of \mathcal{E} . If ZFC is arithmetically consistent, then the sentences “ \mathcal{E} is empty”, “ \mathcal{E} is not empty”, “ \mathcal{E} is finite”, and “ \mathcal{E} is infinite” are not provable in ZFC.

Let

$$\mathcal{V} = \left\{ n \in \mathbb{N} : \left(\text{the polynomial } D(x_1, \dots, x_m) \text{ has no solutions in } \{0, \dots, n\}^m \right) \wedge \right. \\ \left. \left(\text{the polynomial } D(x_1, \dots, x_m) \text{ has a solution in } \{0, \dots, n+1\}^m \right) \right\}$$

Since the sets $\{0, \dots, n\}^m$ and $\{0, \dots, n+1\}^m$ are finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{V}$. Theorem 1 implies the next theorem.

Theorem 4. ZFC proves that $\text{card}(\mathcal{V}) \in \{0, 1\}$. For every $n \in \mathbb{N}$, ZFC proves that $n \notin \mathcal{V}$. ZFC does not prove the emptiness of \mathcal{V} , if ZFC is arithmetically consistent. For every $n \in \mathbb{N}$, the sentence “ n is a threshold number of \mathcal{V} ” is not provable in ZFC, if ZFC is arithmetically consistent.

4 Hypothetical statements Ψ_3, \dots, Ψ_{16}

For an integer $n \geq 3$, let \mathcal{U}_n denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{2\} \ x_i! = x_{i+1} \\ x_1 \cdot x_2 = x_3 \\ x_2 \cdot x_2 = x_3 \end{array} \right\}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{U}_n .

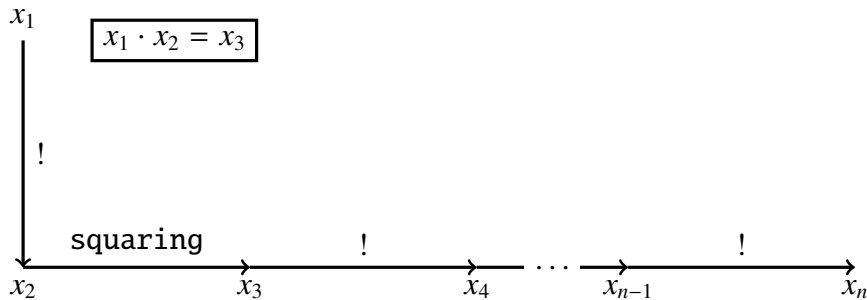


Fig. 1 Construction of the system \mathcal{U}_n

Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geq 3$.

Lemma 8. For every integer $n \geq 3$, the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(2, 2, g(3), \dots, g(n))$.

Let

$$B_n = \{x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer $n \geq 3$, let Ψ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq g(n)$. The statement Ψ_n says that for subsystems of B_n the largest known solution is indeed the largest possible.

Hypothesis 1. The statements Ψ_3, \dots, Ψ_{16} are true.

Proposition 3. Every statement Ψ_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

Proposition 4. For every statement Ψ_n , the bound $g(n)$ cannot be decreased.

Proof. It follows from Lemma 8 because $\mathcal{U}_n \subseteq B_n$. □

5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let \mathcal{A} denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

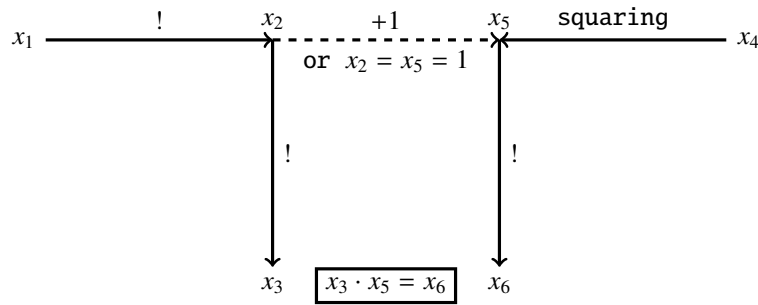


Fig. 2 Construction of the system \mathcal{A}

Lemma 9. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system \mathcal{A} is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [15].

Theorem 5. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 9, the system \mathcal{A} is solvable in positive integers x_2, x_3, x_5, x_6 . Since $\mathcal{A} \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$. Hence, $x_1! + 1 \leq g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \dots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

6 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [14, pp. 37–38]. Let \mathcal{B} denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system \mathcal{B} .

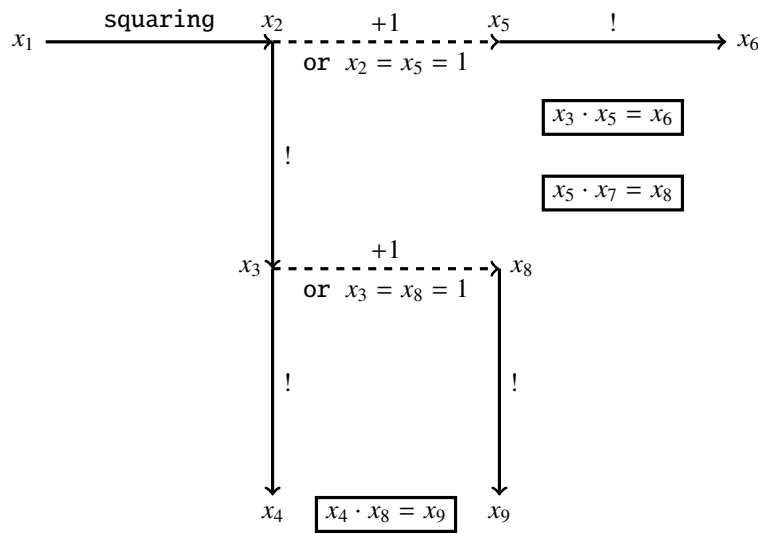


Fig. 3 Construction of the system \mathcal{B}

Lemma 10. For every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 2, for every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 10 follows from Lemma 5. \square

Lemma 11. There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system \mathcal{B} and satisfy $x_1 = 1$.

Proof. If a tuple $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system \mathcal{B} and $x_1 = 1$, then $x_1, \dots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. \square

Theorem 6. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Suppose that the antecedent holds. By Lemma [10](#), there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{B} . Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geq g(7)$. Hence, $(x_1^2)! \geq g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > g(9)$ imply that the system \mathcal{B} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas [10](#) and [11](#), there are infinitely many primes of the form $n^2 + 1$. \square

Corollary 4. *Let X_9 denote the set of primes of the form $n^2 + 1$. The statement Ψ_9 implies that we know an algorithm such that it returns a threshold number of X_9 , and this number equals $\max(X_9)$, if X_9 is finite. Assuming the statement Ψ_9 , a single query to an oracle for the halting problem decides the infinity of X_9 . Assuming the statement Ψ_9 , the infinity of X_9 is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(X_9 \cap [1, g(7)])$. \square

7 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [\[3\]](#) p. 443].

Theorem 7. *(cf. Theorem [11](#)). The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. We leave the analogous proof to the reader. \square

8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [\[14\]](#) p. 39]. Let C denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{array} \right.$$

Lemma [2](#) and the diagram in Figure 4 explain the construction of the system C .

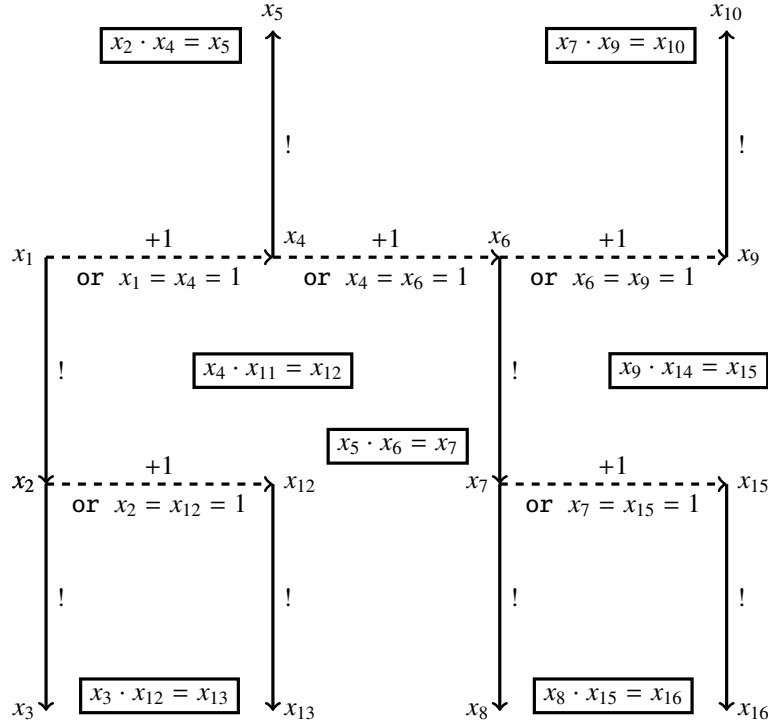


Fig. 4 Construction of the system C

Lemma 12. *For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$\begin{aligned}
 x_1 &= x_4 - 1 \\
 x_2 &= (x_4 - 1)! \\
 x_3 &= ((x_4 - 1)!)! \\
 x_5 &= x_4! \\
 x_6 &= x_9 - 1 \\
 x_7 &= (x_9 - 1)! \\
 x_8 &= ((x_9 - 1)!)! \\
 x_{10} &= x_9! \\
 x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
 x_{12} &= (x_4 - 1)! + 1 \\
 x_{13} &= ((x_4 - 1)! + 1)! \\
 x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
 x_{15} &= (x_9 - 1)! + 1 \\
 x_{16} &= ((x_9 - 1)! + 1)!
 \end{aligned}$$

Proof. By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | ((x_4 - 1)! + 1)) \wedge (x_9 | ((x_9 - 1)! + 1))$$

Hence, the claim of Lemma 12 follows from Lemma 5. \square

Lemma 13. *There are only finitely many tuples $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system C and satisfy $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$.*

Proof. If a tuple $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system C and $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$, then $x_1, \dots, x_{16} \leq 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. \square

Theorem 8. *The statement Ψ_{16} proves the following implication: if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.*

Proof. Suppose that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma [12](#), there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$ such that the tuple (x_1, \dots, x_{16}) solves the system C . Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geq g(14)$. Therefore, $(x_9 - 1)! \geq g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement Ψ_{16} and the inequality $x_{16} > g(16)$ imply that the system C has infinitely many solutions in positive integers x_1, \dots, x_{16} . According to Lemmas [12](#) and [13](#), there are infinitely many twin primes. \square

Corollary 5. (cf. [\[6\]](#)). *Let \mathcal{X}_{16} denote the set of twin primes. The statement Ψ_{16} implies that we know an algorithm such that it returns a threshold number of \mathcal{X}_{16} , and this number equals $\max(\mathcal{X}_{16})$, if \mathcal{X}_{16} is finite. Assuming the statement Ψ_{16} , a single query to an oracle for the halting problem decides the infinity of \mathcal{X}_{16} . Assuming the statement Ψ_{16} , the infinity of \mathcal{X}_{16} is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_{16} \cap [1, g(14)])$. \square

9 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n + 1) = \Gamma(\lambda(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let \mathcal{J}_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{3\} \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{cases}$$

Lemma [3](#) and the diagram in Figure 5 explain the construction of the system \mathcal{J}_n .

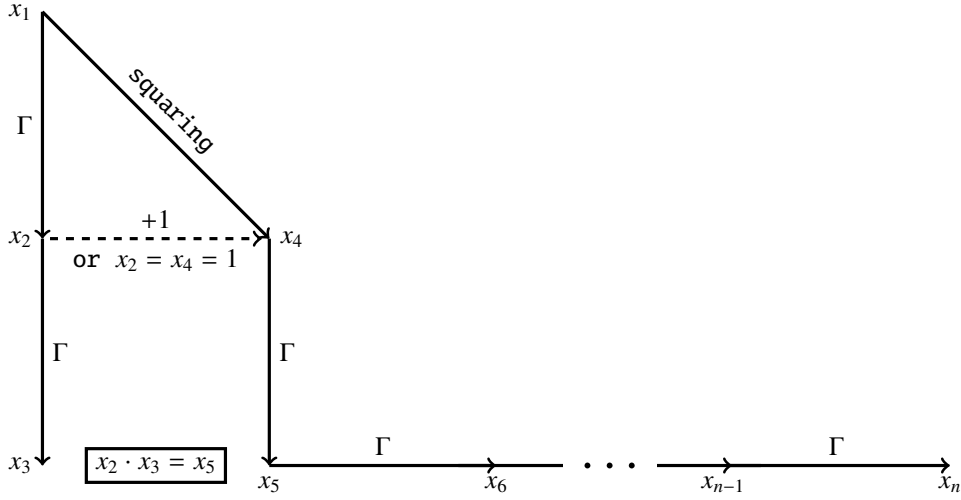


Fig. 5 Construction of the system \mathcal{J}_n

For every integer $n \geq 5$, the system \mathcal{J}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \dots, \lambda(n))$. For an integer $n \geq 5$, let Δ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq \lambda(n)$.*

Hypothesis 2. *The statements $\Delta_5, \dots, \Delta_{14}$ are true.*

Lemmas [3](#) and [5](#) imply that the statements Δ_n have similar consequences as the statements Ψ_n .

Theorem 9. *The statement Δ_6 implies that any prime number $p \geq 25$ proves the infinitude of primes.*

Proof. It follows from Lemmas [3](#) and [5](#). We leave the details to the reader. \square

10 Hypothetical statements $\Sigma_3, \dots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \dots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \dots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer $n \in \{3, \dots, 16\}$, let P_n denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \Gamma_n(x_2) = x_1 \\ \forall i \in \{2, \dots, n-1\} x_i \cdot x_i = x_{i+1} \end{cases}$$

Lemma 14. *For every integer $n \in \{3, \dots, 16\}$, $P_n \subseteq Q_n$ and the system P_n with Γ instead of Γ_n has exactly one solution in positive integers x_1, \dots, x_n , namely $(1, 2^{2^0}, 2^{2^1}, 2^{2^2}, \dots, 2^{2^{n-2}})$.*

For an integer $n \in \{3, \dots, 16\}$, let Σ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq Q_n$ with Γ instead of Γ_n has only finitely many solutions in positive integers x_1, \dots, x_n , then every tuple $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system \mathcal{S} satisfies $x_1, \dots, x_n \leq 2^{2^{n-2}}$.*

Hypothesis 3. *The statements $\Sigma_3, \dots, \Sigma_{16}$ are true.*

Lemma 15. (cf. Lemma [3](#)). *For every integer $n \in \{4, \dots, 16\}$ and for every positive integers x and y , $x \cdot \Gamma_n(x) = \Gamma_n(y)$ if and only if $(x+1 = y) \wedge (x \geq 2^{2^{n-3}} + 1)$.*

Let $\mathcal{Z}_9 \subseteq \mathcal{Q}_9$ be the system of equations in Figure 6.

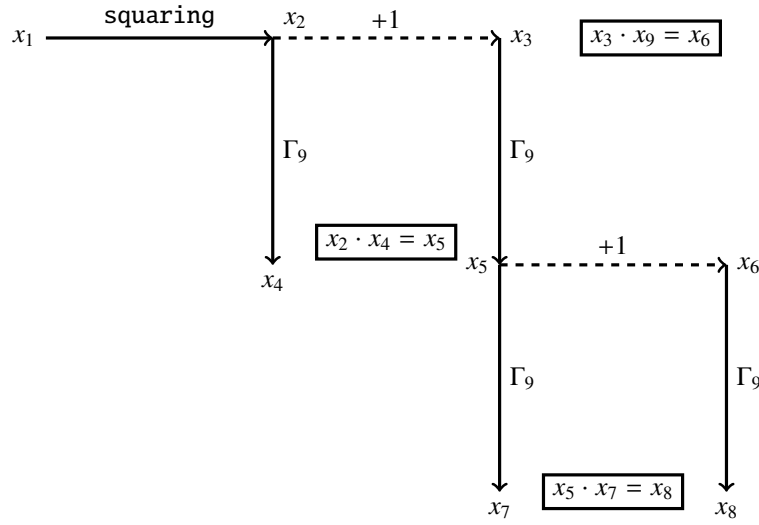


Fig. 6 Construction of the system \mathcal{Z}_9

Lemma 16. For every positive integer x_1 , the system \mathcal{Z}_9 is solvable in positive integers x_2, \dots, x_9 if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers x_2, \dots, x_9 are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with n and solve the system \mathcal{Z}_9 with Γ instead of Γ_9 .

Proof. It follows from Lemmas [3](#), [5](#), and [15](#). □

Lemma 17. ([\[19\]](#)). The number $(13!)^2 + 1 = 38775788043632640001$ is prime.

Lemma 18. $\left((13!)^2 \geq 2^{2^{9-3}} + 1 = 18446744073709551617 \right) \wedge \left(\Gamma_9((13!)^2) > 2^{2^{9-2}} \right)$.

Theorem 10. The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$.

Proof. It follows from Lemmas [16](#)-[18](#). □

Theorem 11. (cf. Theorem [7](#)). The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geq 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$.

Proof. We leave the proof to the reader. □

Corollary 6. Let \mathcal{Y}_9 denote the set of primes of the form $n! + 1$. The statement Σ_9 implies that we know an algorithm such that it returns a threshold number of \mathcal{Y}_9 , and this number equals $\max(\mathcal{Y}_9)$, if \mathcal{Y}_9 is finite. Assuming the statement Σ_9 , a single query to an oracle for the halting problem decides the infinity of \mathcal{Y}_9 . Assuming the statement Σ_9 , the infinity of \mathcal{Y}_9 is decidable in the limit.

Proof. We consider an algorithm which computes $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$. □

Let $\mathcal{Z}_{14} \subseteq \mathcal{Q}_{14}$ be the system of equations in Figure 7.

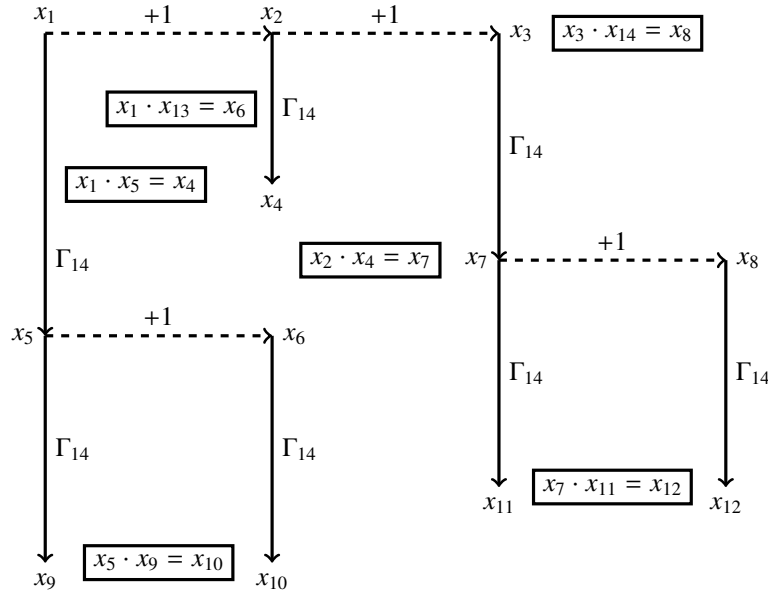


Fig. 7 Construction of the system \mathcal{Z}_{14}

Lemma 19. For every positive integer x_1 , the system \mathcal{Z}_{14} is solvable in positive integers x_2, \dots, x_{14} if and only if x_1 and $x_1 + 2$ are prime and $x_1 \geq 2^{2^{14-3}} + 1$. In this case, positive integers x_2, \dots, x_{14} are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with n and solve the system \mathcal{Z}_{14} with Γ instead of Γ_{14} .

Proof. It follows from Lemmas [3](#), [5](#), and [15](#). □

Lemma 20. ([\[23\]](#) p. 87]). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).

Lemma 21. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 12. The statement Σ_{14} implies the infinitude of twin primes.

Proof. It follows from Lemmas [19](#)-[21](#). □

A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [\[22\]](#). It is conjectured that there are infinitely many Sophie Germain primes, see [\[17\]](#) p. 330]. Let $\mathcal{Z}_{16} \subseteq \mathcal{Q}_{16}$ be the system of equations in Figure 8.

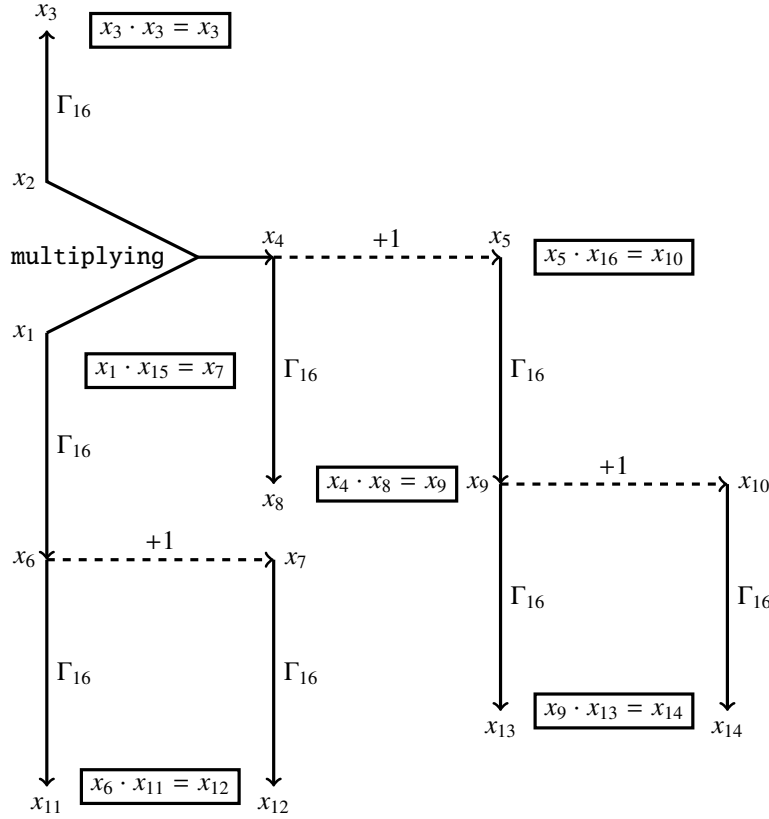


Fig. 8 Construction of the system \mathcal{Z}_{16}

Lemma 22. For every positive integer x_1 , the system \mathcal{Z}_{16} is solvable in positive integers x_2, \dots, x_{16} if and only if x_1 is a Sophie Germain prime and $x_1 \geq 2^{2^{16-3}} + 1$. In this case, positive integers x_2, \dots, x_{16} are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with n and solve the system \mathcal{Z}_{16} with Γ instead of Γ_{16} .

Proof. It follows from Lemmas [3](#), [5](#), and [15](#). □

Lemma 23. ([\[17\]](#) p. 330). $8069496435 \cdot 10^{5072} - 1$ is a Sophie Germain prime (Harvey Dubner).

Lemma 24. $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$.

Theorem 13. The statement Σ_{16} implies the infinitude of Sophie Germain primes.

Proof. It follows from Lemmas [22](#)-[24](#). □

Theorem 14. The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(1, 2), (2, 3)\}$.

Proof. We leave the proof to the reader. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdős, see [\[2\]](#). F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [\[12\]](#).

Theorem 15. The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. We leave the proof to the reader. □

11 Hypothetical statements $\Omega_3, \dots, \Omega_{16}$ and their consequences

For an integer $n \in \{3, \dots, 16\}$, let Ω_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has a solution in integers x_1, \dots, x_n greater than $2^{2^{n-2}}$, then \mathcal{S} has infinitely many solutions in positive integers x_1, \dots, x_n .* For every $n \in \{3, \dots, 16\}$, the statement Σ_n implies the statement Ω_n .

Lemma 25. *The number $(65!)^2 + 1$ is prime and $65! > 2^{2^{9-2}}$.*

Proof. The following PARI/GP ([16]) command

```
(04:04) gp > isprime((65!)^2+1,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([23] p. 226). It rigorously shows that the number $(65!)^2 + 1$ is prime. \square

Lemma 26. *If positive integers x_1, \dots, x_9 solve the system \mathcal{Z}_9 and $x_1 > 2^{2^{9-2}}$, then $x_1 = \min(x_1, \dots, x_9)$.*

Theorem 16. *The statement Ω_9 implies the infinitude of primes of the form $n^2 + 1$.*

Proof. It follows from Lemmas [16] and [25-26]. \square

Lemma 27. *If positive integers x_1, \dots, x_{14} solve the system \mathcal{Z}_{14} and $x_1 > 2^{2^{14-2}}$, then $x_1 = \min(x_1, \dots, x_{14})$.*

Theorem 17. *The statement Ω_{14} implies the infinitude of twin primes.*

Proof. It follows from Lemmas [19-21] and [27]. \square

12 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [11] p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [11] p. 1].

Open Problem. ([11] p. 159]. *Are there infinitely many composite numbers of the form $2^{2^n} + 1$? Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [10] p. 23]. Let*

$$H_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Let $h(1) = 1$, and let $h(n+1) = 2^{2^{h(n)}}$ for every positive integer n .

Lemma 28. *The following subsystem of H_n*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \dots, h(n))$.

For a positive integer n , let ξ_n denote the following statement: *if a system of equations $S \subseteq H_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq h(n)$* . The statement ξ_n says that for subsystems of H_n the largest known solution is indeed the largest possible.

Hypothesis 4. *The statements ξ_1, \dots, ξ_{13} are true.*

Proposition 5. *Every statement ξ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system H_n has a finite number of subsystems. □

Theorem 18. *The statement ξ_{13} proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers z .*

Proof. Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{2}$$

in positive integers. By Lemma 4, we can transform equation (2) into an equivalent system of equations \mathcal{G} which has 13 variables (x, y, z , and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 9.

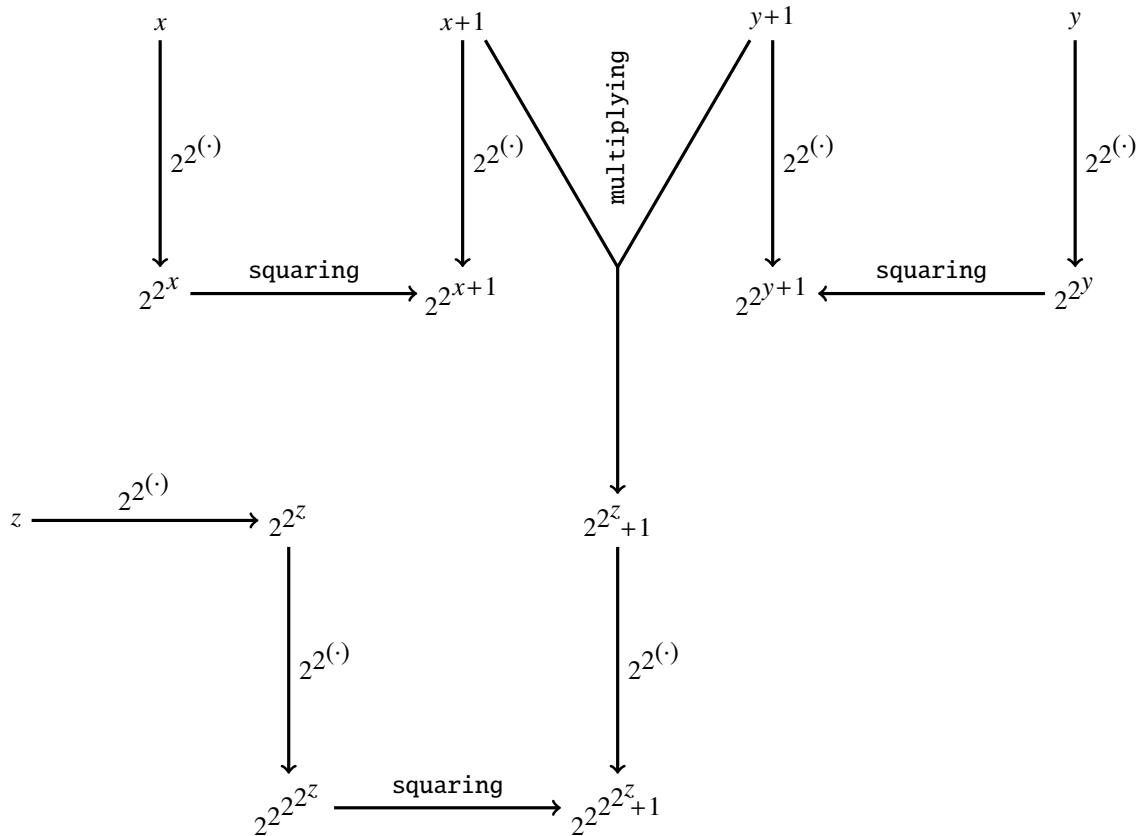


Fig. 9 Construction of the system \mathcal{G}

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^z} + 1} > h(13)$. By this, the statement ξ_{13} implies that the system \mathcal{G} has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

Corollary 7. Let \mathcal{W}_{13} denote the set of composite Fermat numbers. The statement ξ_{13} implies that we know an algorithm such that it returns a threshold number of \mathcal{W}_{13} , and this number equals $\max(\mathcal{W}_{13})$, if \mathcal{W}_{13} is finite. Assuming the statement ξ_{13} , a single query to an oracle for the halting problem decides the infinity of \mathcal{W}_{13} . Assuming the statement ξ_{13} , the infinity of \mathcal{W}_{13} is decidable in the limit.

Proof. We consider an algorithm which computes $\max(\mathcal{W}_{13} \cap [1, h(12)])$. □

References

- [1] C. H. Bennett, *Chaitin's Omega*, in: *Fractal music, hypercards, and more ...* (M. Gardner, ed.), W. H. Freeman, New York, 1992, 307–319.
- [2] D. Berend and J. E. Harmse, *On polynomial-factorial Diophantine equations*, *Trans. Amer. Math. Soc.* 358 (2006), no. 4, 1741–1779.
- [3] C. K. Caldwell and Y. Gallot, *On the primality of $n! \pm 1$ and $2 \times 3 \times 5 \times \dots \times p \pm 1$* , *Math. Comp.* 71 (2002), no. 237, 441–448, <http://doi.org/10.1090/S0025-5718-01-01315-1>.
- [4] C. S. Calude, H. Jürgensen, S. Legg, *Solving problems with finite test sets*, in: *Finite versus Infinite: Contributions to an Eternal Dilemma* (C. Calude and G. Păun, eds.), 39–52, Springer, London, 2000.
- [5] N. C. A. da Costa and F. A. Doria, *On the foundations of science (LIVRO): essays, first series*, E-papers Serviços Editoriais Ltda, Rio de Janeiro, 2013.
- [6] F. G. Dorais, *Can the twin prime problem be solved with a single use of a halting oracle?* July 23, 2011, <http://mathoverflow.net/questions/71050>.
- [7] W. B. Easton, *Powers of regular cardinals*, *Ann. Math. Logic* 1 (1970), 139–178.
- [8] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [9] T. Jech, *Set theory*, Springer, Berlin, 2003.
- [10] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers*, American Mathematical Society, Providence, RI, 2012.
- [11] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry*, Springer, New York, 2001.
- [12] F. Luca, *The Diophantine equation $P(x) = n!$ and a result of M. Overholt*, *Glas. Mat. Ser. III* 37 (57) (2002), no. 2, 269–273
- [13] M. Mignotte and A. Pethő, *On the Diophantine equation $x^p - x = y^q - y$* , *Publ. Mat.* 43 (1999), no. 1, 207–216.
- [14] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT*, Springer, London, 2012.
- [15] M. Overholt, *The Diophantine equation $n! + 1 = m^2$* , *Bull. London Math. Soc.* 25 (1993), no. 2, 104.
- [16] PARI/GP online documentation, http://pari.math.u-bordeaux.fr/dochtm/html/Arithmetic_functions.html.
- [17] P. Ribenboim, *The new book of prime number records*, Springer, New York, 1996, <http://doi.org/10.1007/978-1-4612-0759-7>.

- [18] S. Siksek, *Chabauty and the Mordell–Weil Sieve*, in: *Advances on Superelliptic Curves and Their Applications* (eds. L. Beshaj, T. Shaska, E. Zhupa), 194–224, IOS Press, Amsterdam, 2015, <http://dx.doi.org/10.3233/978-1-61499-520-3-194>.
- [19] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, *Smallest prime factor of $A020549(n) = (n!)^2 + 1$* , <http://oeis.org/A282706>.
- [20] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [21] Wolfram MathWorld, *Perfect Cuboid*, <http://mathworld.wolfram.com/PerfectCuboid.html>.
- [22] Wolfram MathWorld, *Sophie Germain prime*, <http://mathworld.wolfram.com/SophieGermainPrime.html>.
- [23] S. Y. Yan, *Number theory for computing*, 2nd ed., Springer, Berlin, 2002.
- [24] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity*, Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, <http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm>.
- [25] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer*, in: J. Cachro and K. Kijania-Placek (eds.), *Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science*, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

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