Statements that concern computable sets $X \subseteq \mathbb{N}$ and cannot be formalized in $ZFC$ because they refer to the currently known/unknown theorems about $X$

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Conditions (1)–(8) below concern sets $X \subseteq \mathbb{N}$. (1) There are a large number of elements of $X$ and it is conjectured that $X$ is infinite. (2) No known algorithm decides the finiteness of $X$. (3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in X$. (4) There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \implies X \subseteq (-\infty, n]$. (5) $X$ is widely known in number theory. (6) There is no known equality $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are defined simpler than $X$. (7) No known set $Y$ is defined simpler than $X$ and satisfies $(Y \subseteq X) \land (\text{card}(X \setminus Y) < \omega)$. (8) No known set $Y$ is defined simpler than $X$ and satisfies $\text{card}((X \setminus Y) \cup (Y \setminus X)) < \omega$. We do not know any set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(4) and (5). The same is true, if condition (5) is replaced by condition (6) or (7) or (8). For every explicitly known integer $n$, some simply defined set $X \subseteq \mathbb{N}$ includes the set $(-\infty, n] \cap \mathbb{N}$ and satisfies conditions (1)–(4). Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form $n^2 + 1$. The set $X = \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3) and (5)–(8). The set $X = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions (1)–(3) and (6)–(8). Some hypothetical statement implies that these sets $X$ satisfy condition (4).

2010 Mathematics Subject Classification: 03D20.

Key words and phrases: computable set $X \subseteq \mathbb{N}$, currently known/unknown theorems about $X$, explicitly known integer $n$, finiteness (infiniteness) of $X$ remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$, large number of elements of $X$, mathematical statement that cannot be formalized in $ZFC$, $n$ bounds $X$ if $X$ is finite, no known algorithm decides the finiteness of $X$.

1. Introduction, definitions, and basic lemmas

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [5, p. 199]. In this article, we present an argument against logicism: there are statements that concern computable sets $X \subseteq \mathbb{N}$ and cannot be formalized in $ZFC$ because they refer to the currently known/unknown theorems about $X$. 
Definition 1. Let $\beta = (((24!!))!!)$.

Lemma 1. $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

Proof. We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com). \qed

Lemma 2. $((7!!))! \approx 10^{10^{10^{10^{16477.87280582041}}}}$.

Proof. We ask Wolfram Alpha about $0 + ((7!!))!$. \qed

Definition 2. We say that an integer $n \geq -1$ is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \implies X \subseteq (-\infty, n]$, cf. [12] and [13].

Definition 3. We say that a non-negative integer $n$ is a weak threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \implies \text{card}(X) \leq n$.

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $n \geq -1$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $[\max(X), \infty) \cap \mathbb{N}$. If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer $n$ is a weak threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all weak threshold numbers of $X$ form the set $[\text{card}(X), \infty) \cap \mathbb{N}$.

Theorem 1. For every set $X \subseteq \mathbb{N}$, if an integer $n \geq -1$ is a threshold number of $X$, then $n + 1$ is a weak threshold number of $X$.

Proof. For every set $X \subseteq \mathbb{N}$ and for every integer $n \geq -1$, the inclusion $X \subseteq (-\infty, n]$ implies that $\text{card}(X) \leq n + 1$. \qed

Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form $n^2 + 1$. We do not know any weak threshold number of $\mathcal{P}_{n^2+1}$. The same is true for the sets

$$\left\{ n \in \mathbb{N} : 2^n + 1 \text{ is composite} \right\}$$

and

$$\left\{ n \in \mathbb{N} : n! + 1 \text{ is a square} \right\}$$

Lemma 3. For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if $(x + 1 = y) \lor (x = y = 1)$

Lemma 4. (Wilson’s theorem, [11] p. 89). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.

The conditions below concern sets $X \subseteq \mathbb{N}$.

1. There are a large number of elements of $X$ and it is conjectured that $X$ is infinite.
2. No known algorithm decides the finiteness of $X$.
3. There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in X$.
4. There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \implies X \subseteq (-\infty, n]$.
5. $X$ is widely known in number theory.
6. There is no known equality $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are defined simpler than $X$. 

Statements that concern computable sets $X \subseteq \mathbb{N}$

(7) No known set $Y$ is defined simpler than $X$ and satisfies 
$(Y \subseteq X) \land (\text{card}(X \setminus Y) < \omega)$.

(8) No known set $Y$ is defined simpler than $X$ and satisfies 
$\text{card}((X \setminus Y) \cup (Y \setminus X)) < \omega$.

(4•) There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \implies \text{card}(X) \leq n$.

(1•) There are a large number of elements of $X$ and it is conjectured that $X = \mathbb{N}$.

(2•) No known algorithm decides the equality $X = \mathbb{N}$.

(1*) There are a large number of elements of $X$ and it is conjectured that $X$ is finite.

2. Open Problems

The following two open problems cannot be formalized in $\text{ZFC}$ because they refer to current knowledge about $X$.

Open Problem 1. Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(3), (4•), and (5)?

Open Problem 2. Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)?

Theorem 2. Open Problem 2 claims more than Open Problem 1.

Proof. Condition (4) implies that $\text{card}(X) < \omega \implies X \subseteq (-\infty, |n|]$. Since $|n| \geq -1$, Theorem 1 guarantees that condition (4) implies condition (4•).

Open Problems 1 and 2 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

3. Partial solutions to Open Problem

Edmund Landau’s conjecture states that the set $P_{n^2+1}$ is infinite, see [6, pp. 37–38] and [9]. Let $M$ denote the set of all positive multiples of elements of the set $P_{n^2+1} \cap (\beta, \infty)$.

Statement 1. The set $X = \{0, \ldots, \beta\} \cup M$ satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\text{card}(X) > \beta$ and the set $P_{n^2+1}$ is conjecturally infinite. By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of $P_{n^2+1}$ is greater than $\beta$. Thus condition (2) holds. Condition (3) holds trivially. Since the set $M$ is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus condition (4) holds.

Statements 1–6 may be falsified because they refer to the currently known/unknown theorems about $X$. By the same reason, Statements 1–6 cannot be formalized in $\text{ZFC}$.

Let $\lfloor \cdot \rfloor$ denote the integer part function.

Lemma 5. For every non-negative integer $n$, $\lfloor \frac{3n - 3\beta + 3}{3n - 3\beta + 2} \rfloor$ equals 0 or 1. The first case holds when $n \leq \beta - 1$. The second case holds when $n \geq \beta$. 
4 Statements that concern computable sets $X \subseteq \mathbb{N}$

**Lemma 6.** The function

$$\mathbb{N} \cap [\beta, \infty) \ni n \mapsto \beta + n - \left\lfloor \sqrt{n} \right\rfloor^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geq \beta$ infinitely many times.

**Proof.** Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \left\{ (t+0)^2 + t, (t+1)^2 + t, (t+2)^2 + t, \ldots \right\} \cap [\beta, \infty)$$

\[\square\]

**Statement 2.** The set

$$X = \left\{ n \in \mathbb{N} : 2 + \left\lfloor \frac{3n - 3\beta + 3}{3n - 3\beta + 2} \right\rfloor \cdot \left( \beta + n - \left\lfloor \sqrt{n} \right\rfloor^2 \right)^2 - 1 \text{ is prime} \right\}$$

satisfies conditions (1)–(4).

**Proof.** Condition (3) holds trivially. By Lemma 5

$$X = \{0, \ldots, \beta - 1\} \cup \mathcal{H},$$

where

$$\mathcal{H} = \left\{ n \in \mathbb{N} \cap [\beta, \infty) : \left( \beta + n - \left\lfloor \sqrt{n} \right\rfloor^2 \right)^2 + 1 \text{ is prime} \right\}$$

By Lemma 6, the set $\mathcal{H}$ is empty or infinite. The second case holds when

$$\exists k \in \mathbb{N} \cap [\beta, \infty) \text{ such that } k^2 + 1 \text{ is prime} \quad \text{(G)}$$

The equality $X = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of $X$ and conditions (1) and (4) hold. Condition (2) holds as due to known physics we are not able to confirm the statement (G) by a direct computation. \[\square\]

4. **Number-theoretic statements $\Psi_n$**

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let $U_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $U_n$ denote the following system of equations:

$$\begin{align*}
    x_1! &= x_1 \\
    x_1 \cdot x_1 &= x_2 \\
    \forall i \in \{2, \ldots, n-1\} \quad x_i! &= x_{i+1}
\end{align*}$$

The diagram in Figure 1 illustrates the construction of the system $U_n$.  

![Fig. 1 Construction of the system $U_n$](image-url)
**Lemma 7.** For every positive integer \( n \), the system \( \mathcal{U}_n \) has exactly two solutions in positive integers, namely \((1, \ldots, 1)\) and \((f(1), \ldots, f(n))\).

Let
\[
B_n = \{x_i! = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}
\]

For a positive integer \( n \), let \( \Psi_n \) denote the following statement: if a system of equations \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then each such solution \((x_1, \ldots, x_n)\) satisfies \( x_1, \ldots, x_n \leq f(n) \). The statement \( \Psi_n \) says that for subsystems of \( B_n \) with a finite number of solutions, the largest known solution is indeed the largest possible. An elementary reasoning proves the statements \( \Psi_1 \) and \( \Psi_2 \).

**Theorem 3.** For every statement \( \Psi_n \), the bound \( f(n) \) cannot be decreased.

**Proof.** It follows from Lemma 7 because \( \mathcal{U}_n \subseteq B_n \).

**Theorem 4.** For every integer \( n \geq 2 \), the statement \( \Psi_{n+1} \) implies the statement \( \Psi_n \).

**Proof.** If a system \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then for every integer \( i \in \{1, \ldots, n\} \) the system \( S \cup \{x_i! = x_{n+1}\} \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_{n+1} \). The statement \( \Psi_{n+1} \) implies that \( x_i! = x_{n+1} \leq f(n + 1) = f(n)! \). Hence, \( x_i \leq f(n) \).

**Theorem 5.** Every statement \( \Psi_n \) is true with an unknown integer bound that depends on \( n \).

**Proof.** For every positive integer \( n \), the system \( B_n \) has a finite number of subsystems.

5. **A conjectural solution to Open Problem 2**

Let \( \mathcal{A} \) denote the following system of equations:
\[
\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}
\]

Lemma 8 and the diagram in Figure 2 explain the construction of the system \( \mathcal{A} \).
Lemma 8. For every integer \( x_1 \geq 2 \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) is prime. In this case, the integers \( x_2, \ldots, x_9 \) are uniquely determined by the following equalities:

\[
\begin{align*}
  x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{align*}
\]

Proof. By Lemma 3, for every integer \( x_1 \geq 2 \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) divides \((x_1^2)! + 1\). Hence, the claim of Lemma 8 follows from Lemma 4.

Lemma 9. There are only finitely many tuples \( (x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 \), which solve the system \( \mathcal{A} \) and satisfy \( x_1 = 1 \).

Proof. If a tuple \( (x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 \) solves the system \( \mathcal{A} \) and \( x_1 = 1 \), then \( x_1, \ldots, x_9 \leq 2 \). Indeed, \( x_1 = 1 \) implies that \( x_2 = x_1^2 = 1 \). Hence, for example, \( x_3 = x_2! = 1 \). Therefore, \( x_8 = x_3 + 1 = 2 \) or \( x_8 = 1 \). Consequently, \( x_9 = x_8! \leq 2 \).

Let \( \Phi_0 \) denote the statement \( \Psi_9 \) restricted to the system \( \mathcal{A} \). Apoloniusz Tyszka believes that the statement \( \Phi_0 \) is true.

Theorem 6. The statement \( \Phi_0 \) proves the following implication: if there exists an integer \( x_1 \geq 2 \) such that \( x_1^2 + 1 \) is prime and greater than \( f(7) \), then the set \( \mathcal{P}_{x_1^2 + 1} \) is infinite.
Statements that concern computable sets $X \subseteq \mathbb{N}$

**Proof.** Suppose that the antecedent holds. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently, $x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$

The statement $\Phi_9$ and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 8 and 9, the set $P_{n^2+1}$ is infinite.

Let $\mathcal{K} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } P_{n^2+1}\}$.

**Lemma 10.** $\text{card}(\mathcal{K}) \geq 9 \cdot 10^9 \cdot 4^{747} \approx 10^{10450.6930560314272}$.

**Proof.** The following PARI/GP (8) command

```plaintext
isprime(1+9*4^747,{flag=2})
```

returns %1 = 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([11, p. 226]). It rigorously shows that the number $(3 \cdot 2^{747})^2 + 1$ is prime. Since $9 \cdot 10^9 \cdot 4^{747}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 \ast (10^{(9 \ast 4^{747})})$.

**Statement 3.** The set $X = P_{n^2+1}$ satisfies conditions (1)–(3) and (5)–(8). The set $X = \mathcal{K}$ satisfies conditions (1)–(3) and (6)–(8). The statement $\Phi_9$ implies that these sets $X$ satisfy condition (4).

**Proof.** Since the set $P_{n^2+1}$ is conjecturally infinite, Lemma 10 implies condition (1) for both sets $X$. Conditions (3) and (6)–(8) hold trivially for both sets $X$. By Lemma 11, due to known physics we are not able to confirm by a direct computation that some element of $P_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$. Thus condition (2) holds for both sets $X$. Suppose that the statement $\Phi_9$ is true. By Theorem 6, $f(7)$ is a threshold number of $X = P_{n^2+1}$. By Theorem 6, $9 \ldots 9$ is a threshold number of $X = \mathcal{K}$. Thus condition (4) holds for both sets $X$.

**6. Open Problems 3 and 4**

The following two open problems cannot be formalized in ZFC because they refer to current knowledge about $X$.

**Open Problem 3.** Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1•)–(2•), (2)–(3), (4•), and (5)?

Open Problem 3 claims more than Open Problem 1 as condition (1•) implies condition (1).

**Open Problem 4.** Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1•)–(2•) and (2)–(5)?
Statements that concern computable sets \(X \subseteq \mathbb{N}\)

Open Problem 4 claims more than Open Problem 3 as condition (1\(\triangleleft\)) implies condition (1).

**Theorem 7.** Open Problem 4 claims more than Open Problem 3.

**Proof.** Condition (4) implies that \(\text{card}(X) < \omega \implies X \subseteq (-\infty, |n|].\) Since \(|n| \geq -1\), Theorem guarantees that condition (4) implies condition (4\(\circ\)). □

Open Problems 3 and 4 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

7. A partial solution to Open Problem 4

Let \(V\) denote the set of all positive multiples of elements of the set
\[
\left\{ n \in [\beta + 1, \infty) \cap \mathbb{N} : 2^{2^n} + 1 \text{ is composite} \right\}
\]

**Statement 4.** The set \(X = \{0, \ldots, \beta\} \cup V\) satisfies conditions (1\(\triangleleft\))–(2\(\triangleleft\)) and (2)–(4).

**Proof.** The inequality \(\text{card}(X) > \beta\) holds trivially. Most mathematicians believe that \(2^{2^n} + 1\) is composite for every integer \(n \geq 5\), see [3, p. 23]. These two facts imply conditions (1\(\triangleleft\)) and (2\(\triangleleft\)). Condition (3) holds trivially. Since the set \(V\) is empty or infinite, the integer \(\beta\) is a threshold number of \(X\). Thus condition (4) holds. The question of finiteness of the set
\[
\left\{ n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite} \right\}
\]
remains open, see [4, p. 159]. By this and Lemma the question of emptiness of the set
\[
\left\{ n \in [\beta + 1, \infty) \cap \mathbb{N} : 2^{2^n} + 1 \text{ is composite} \right\}
\]
remains open. Therefore, the question of finiteness of the set \(V\) remains open. Consequently, the question of finiteness of the set \(X\) remains open and condition (2) holds. □

8. Open Problems 5 and 6

The following two open problems cannot be formalized in \(ZFC\) because they refer to current knowledge about \(X\).

**Open Problem 5.** Is there a set \(X \subseteq \mathbb{N}\) that satisfies conditions (1\(*\)), (2)–(3), (4\(\bullet\)), and (5)?

**Open Problem 6.** Is there a set \(X \subseteq \mathbb{N}\) that satisfies conditions (1\(*\)) and (2)–(5)?

**Theorem 8.** Open Problem 6 claims more than Open Problem 5.

**Proof.** Condition (4) implies that \(\text{card}(X) < \omega \implies X \subseteq (-\infty, |n|]\). Since \(|n| \geq -1\), Theorem guarantees that condition (4) implies condition (4\(\bullet\)). □

Open Problems 5 and 6 remain open, if condition (5) is replaced by condition (6) or (7) or (8).
9. Partial solutions to Open Problem \[6\]

A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the equation \(x! + 1 = y^2\), see \[7\].

Lemma 11. ([10] p. 297]). \textit{It is conjectured that } \(x! + 1\) \textit{is a square only for } \(x \in \{4, 5, 7\}\).

Let \(W\) denote the set of all integers \(x\) greater than \(\beta\) such that \(x! + 1\) is a square.

Statement 5. \textit{The set} \(\mathcal{X} = \{0, \ldots, \beta\} \cup \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \land (x \in \mathcal{W})\}\) \textit{satisfies conditions (1\(^{*}\)) and (2)–(4)}.

Proof. Condition (1\(^{*}\)) holds as \(\text{card} (\mathcal{X}) > \beta\) and the set \(\mathcal{W}\) is conjecturally empty by Lemma 11. Condition (3) holds trivially. We do not know any algorithm that decides the emptiness of \(\mathcal{W}\) and the set \(\mathcal{Y} = \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \land (x \in \mathcal{W})\}\) is empty or infinite. Thus condition (2) holds. Since the set \(\mathcal{Y}\) is empty or infinite, the integer \(\beta\) is a threshold number of \(\mathcal{X}\). Thus condition (4) holds. \(\square\)

Let \(\mathcal{C}\) denote the following system of equations:

\[
\begin{align*}
  x_1! &= x_2 \\
  x_2! &= x_3 \\
  x_3! &= x_6 \\
  x_4 \cdot x_4 &= x_5 \\
  x_3 \cdot x_5 &= x_6 \\
\end{align*}
\]

Lemma 3 and the diagram in Figure 3 explain the construction of the system \(\mathcal{C}\).

\[
\begin{array}{cccc}
  & x_1! & x_2 & +1 & x_5 \\
  \text{squaring} & & & \text{or} & \text{or} \\
  & x_3 & x_4 & x_6 \\
  & x_3 \cdot x_5 = x_6 & \text{x3 \cdot x5 = x6} \\
\end{array}
\]

\textbf{Fig. 3} Construction of the system \(\mathcal{C}\)

Lemma 12. \textit{For every } \(x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}\), \textit{the system } \(\mathcal{C}\) \textit{is solvable in positive integers } \(x_2, x_3, x_5, x_6\) \textit{if and only if } \(x_1! + 1 = x_4^2\). \textit{In this case, the integers } \(x_2, x_3, x_5, x_6\) \textit{are uniquely determined by the following equalities:}

\[
\begin{align*}
  x_2 &= x_1! \\
  x_3 &= (x_1!)! \\
  x_5 &= x_1! + 1 \\
  x_6 &= (x_1! + 1)! \\
\end{align*}
\]
Proof. It follows from Lemma 3. □

Let \( \Phi_6 \) denote the statement \( \Psi_6 \) restricted to the system \( C \). Apoloniusz Tyszka believes that the statement \( \Phi_6 \) is true.

**Theorem 9.** If the equation \( x_1! + 1 = x_4^2 \) has only finitely many solutions in positive integers, then the statement \( \Phi_6 \) guarantees that each such solution \((x_1, x_4)\) satisfies \( x_1 < 24! \).

**Proof.** Suppose that the antecedent holds. Let positive integers \( x_1 \) and \( x_4 \) satisfy \( x_1! + 1 = x_4^2 \). Then, \( x_1, x_4 \in \mathbb{N} \setminus \{0, 1\} \). By Lemma 12, the system \( C \) is solvable in positive integers \( x_2, x_3, x_5, x_6 \). The statement \( \Phi_6 \) implies that \( x_6 = (x_1! + 1)! \leq f(6) = f(5)! \). Hence, \( x_1! + 1 \leq f(5) = f(4)! \). Consequently, \( x_1 < f(4) = 24! \). □

**Statement 6.** Let \( \mathcal{X} \) denote the set of all non-negative integers \( n \) which have \((k!)!)! \) digits for some \( k \in \{m \in \mathbb{N} : m! + 1 \text{ is a square}\} \). We claim that \( \mathcal{X} \) satisfies conditions (1*), (2)–(3), and (6)–(8). The statement \( \Phi_6 \) implies that \( \mathcal{X} \) satisfies condition (4).

**Proof.** Let \( d = ((7!)!)! \). Since \( 7! + 1 = 71^2 \), we obtain that \( \{10^{d-1}, \ldots, 9\ldots9\} \subseteq \mathcal{X} \).

Hence, \( \text{card}(\mathcal{X}) \geq 9 \cdot 10^{d-1} \). By this and Lemmas 2 and 11, condition (1*) holds. Conditions (2)–(3) and (6)–(8) hold trivially. By Theorem 9, the statement \( \Phi_6 \) implies that \( 9\ldots9 \) is a threshold number of \( \mathcal{X} \). Thus condition (4) holds. □

10. **Is it possible to strengthen the main results?**

Proving the statement \( \Phi_9 \) will strengthen Theorem 6 and Statement 5. Proving the statement \( \Phi_6 \) will strengthen Theorem 9 and Statement 6. The removal from conditions (6)–(8) of the word "known" will strengthen these conditions and will mean that it will be possible to formulate them in ZFC. Proving Statements 3 and 6 with such changed conditions (6)–(8) is the next goal.

Proving that the statement "The set \( \mathcal{P}_{n^2+1} \) is infinite" is independent of ZFC will falsify Statement 3 because then condition (1) will not be met. Such a statement is highly unlikely, cf. [2, p. 488]. Proving that the set \( \mathcal{P}_{n^2+1} \) is infinite will falsify Statement 5 because then condition (2) will not be fulfilled. Proving that the set \( \mathcal{P}_{n^2+1} \) is finite will falsify Statement 6 because then condition (2) will not be fulfilled. Such a statement is highly unlikely. When the above falsifications do not occur, we can try to prove that the consistency of ZFC implies the consistency of ZFC with the statement "The set \( \mathcal{P}_{n^2+1} \) is infinite". This will complement Statement 3 because it argues for the conjecture that the set \( \mathcal{P}_{n^2+1} \) is infinite, and this conjecture is part of condition (1).

**Acknowledgement.** Sławomir Kurpaska prepared three diagrams in TikZ. Apoloniusz Tyszka wrote the article.
References


Sławomir Kurpaska
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: rtkurpas@cyf-kr.edu.pl

Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl
On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer $n$ such that $\{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite

Apoloniusz Tyszka

Abstract

Let $\Gamma(k)$ denote $(k-1)!$, and let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in [2] \cup [2^{2n-3} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \ldots, 16\}$, let $\Sigma_n$ denote the following statement: if a system of equations $S \subseteq [\Gamma_n(x_i) = x_k : i, k \in \{1, \ldots, n\}] \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$ with $\Gamma$ instead of $\Gamma_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $S$ satisfies $x_1, \ldots, x_n \leq 2^{2n-2}$. Our hypothesis claims that the statements $\Sigma_3, \ldots, \Sigma_{16}$ are true. The statement $\Sigma_6$ proves the following implication: if the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(1, 2), (2, 3)\}$. The statement $\Sigma_8$ proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$. The statement $\Sigma_9$ implies the infinitude of primes of the form $n^2 + 1$. The statement $\Sigma_{10}$ implies that any prime of the form $n! + 1$ with $n \geq 2^{2n-3}$ proves the infinitude of primes of the form $n! + 1$. The statement $\Sigma_{11}$ implies the infinitude of twin primes. The statement $\Sigma_{12}$ implies the infinitude of Sophie Germain primes.

Key words and phrases: Brocard’s problem, Brocard-Ramanujan equation $x! + 1 = y^2$, composite Fermat numbers, decidability in the limit, Erdős’ equation $x(x + 1) = y!$, finiteness of a set, infiniteness of a set, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, single query to an oracle for the halting problem, Sophie Germain primes, twin primes.

2010 Mathematics Subject Classification: 03B30, 11A41.

1 Introduction and basic lemmas

The phrase “we know a non-negative integer $n$” in the title means that we know an algorithm which returns $n$. The title of the article cannot be formalised in ZFC because the phrase “we know a non-negative integer $n$” refers to currently known non-negative integers $n$ with some property. A formally stated title may look like this: On ZFC-formulae $\varphi(x)$ for which there exists a non-negative integer $n$ such that ZFC proves that

$$\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) < \infty \implies \{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer $n$ such that ZFC proves the above implication.

Lemma 1. For every non-negative integer $n$, $\text{card}(\{x \in \mathbb{N} : x \leq n - 1\}) = n$.

Corollary 1. The title altered to “On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer $n$ such that $\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) \leq n$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite” involves a weaker assumption on $\varphi(x)$.

Lemma 2. For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \lor (x = y = 1)$$
Let $\Gamma(k)$ denote $(k-1)!$.

**Lemma 3.** For every positive integers $x$ and $y$, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if

$$(x + 1 = y) \lor (x = y = 1)$$

**Lemma 4.** For every non-negative integers $b$ and $c$, $b + 1 = c$ if and only if $2^{2b} \cdot 2^{2b} = 2^{2c}$.

**Lemma 5.** (Wilson’s theorem, [8, p. 89]). For every positive integer $x$, $x$ divides $(x - 1)! + 1$ if and only if $x = 1$ or $x$ is prime.

## 2 Subsets of $\mathbb{N}$ and their threshold numbers

We say that a non-negative integer $m$ is a threshold number of a set $X \subseteq \mathbb{N}$, if $X$ is infinite if and only if $X$ contains an element greater than $m$, cf. [24] and [25]. If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer $m$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $(\max(X), \max(X) + 1, \max(X) + 2, \ldots)$.

It is conjectured that the set of prime numbers of the form $n^2 + 1$ is infinite, see [14, pp. 37–38]. It is conjectured that the set of prime numbers of the form $n! + 1$ is infinite, see [14, p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [14, p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^p} + 1$ is infinite, see [10, p. 23] and [11, pp. 158–159]. A prime $p$ is said to be a Sophie Germain prime if both $p$ and $2p + 1$ are prime, see [22]. It is conjectured that the set of Sophie Germain primes is infinite, see [17, p. 330]. For each of these sets, we do not know any threshold number.

The following statement:

for every non-negative integer $n$ there exist

$$\text{prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in \left[10^n, 10^n + 1\right]$$

is a $\Pi_1$ statement which strengthens the twin prime conjecture, see [4, p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger $\Pi_1$ statements, see [4]. Statement (1) is equivalent to the non-halting of a Turing machine. If a set $X \subseteq \mathbb{N}$ is computable and we know a threshold number of $X$, then the infinity of $X$ is equivalent to the halting of a Turing machine.

The height of a rational number $\frac{p}{q}$ is denoted by $H\left(\frac{p}{q}\right)$ and equals $\max(|p|, |q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple $(x_1, \ldots, x_n)$ is denoted by $H(x_1, \ldots, x_n)$ and equals $\max(H(x_1), \ldots, H(x_n))$.

**Lemma 6.** The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [12, p. 212]. The known rational solutions are $(x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left(\frac{1}{3}, \frac{15}{33}\right), \left(\frac{1}{3}, \frac{12}{33}\right), \left(-\frac{15}{16}, -\frac{185}{1024}\right), \left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [18, pp. 223–224].

**Corollary 2.** The set $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$ is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of $\mathcal{T}$.

Let $\mathcal{L}$ denote the following system of equations:

$$\left\{\begin{array}{c}
x^2 + y^2 = s^2 \\
x^2 + z^2 = t^2 \\
y^2 + z^2 = u^2 \\
x^2 + y^2 + z^2 = v^2
\end{array}\right.$$
Let \( F = \{ n \in \mathbb{N} \setminus \{ 0 \} : \text{(the system } L \text{ has no solutions in } \{1, \ldots, n\}^7) \land \) 
\( \text{(the system } L \text{ has a solution in } \{1, \ldots, n + 1\}^7) \} \)

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

**Lemma 7.** ([27]). No perfect cuboids are known.

**Corollary 3.** We know an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in F \). ZFC proves that \( \text{card}(F) \in \{0, 1\} \). We do not know any algorithm which returns \( \text{card}(F) \). We do not know any algorithm which returns a threshold number of \( F \).

Let \( H = \begin{cases} 
\mathbb{N}, & \text{if } \sin \left(99999\right) < 0 \\
\mathbb{N} \cap \left[0, \sin(99999) \cdot 99999\right], & \text{otherwise}
\end{cases} \)

We do not know whether or not the set \( H \) is finite.

**Proposition 1.** The number \( 99999 \) is a threshold number of \( H \). We know an algorithm which decides the equality \( H = \mathbb{N} \). If \( H \neq \mathbb{N} \), then the set \( H \) consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in H \).

Let \( K = \begin{cases} 
\{n\}, & \text{if } (n \in \mathbb{N}) \land \left(2^{N_0} = S_{n+1}\right) \\
\{0\}, & \text{if } 2^{N_0} \geq S_{\omega}
\end{cases} \)

**Proposition 2.** ZFC proves that \( \text{card}(K) = 1 \). If ZFC is consistent, then for every \( n \in \mathbb{N} \) the sentences "\( n \) is a threshold number of \( K \)" and "\( n \) is not a threshold number of \( K \)" are not provable in ZFC.

**Proof.** It suffices to observe that \( 2^{N_0} \) can attain every value from the set \( \{N_1, N_2, N_3, \ldots\} \), see [7] and [8, p. 232].

\( \square \)

### 3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel’s second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

**Theorem 1.** ([5, p. 35]). There exists a polynomial \( D(x_1, \ldots, x_m) \) with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation \( D(x_1, \ldots, x_m) = 0 \) is solvable in non-negative integers" and "The equation \( D(x_1, \ldots, x_m) = 0 \) is not solvable in non-negative integers" are not provable in ZFC.

Let \( Y \) denote the set of all non-negative integers \( k \) such that the equation \( D(x_1, \ldots, x_m) = 0 \) has no solutions in \( \{0, \ldots, k\}^m \). Since the set \( \{0, \ldots, k\}^m \) is finite, we know an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in Y \). Theorem \([5]\) implies the next theorem.
Theorem 2. For every \( n \in \mathbb{N} \), ZFC proves that \( n \in \mathcal{Y} \). If ZFC is arithmetically consistent, then the sentences “\( \mathcal{Y} \) is finite” and “\( \mathcal{Y} \) is infinite” are not provable in ZFC. If ZFC is arithmetically consistent, then for every \( n \in \mathbb{N} \) the sentences “\( n \) is a threshold number of \( \mathcal{Y} \)” and “\( n \) is not a threshold number of \( \mathcal{Y} \)” are not provable in ZFC.

Let \( \mathcal{E} \) denote the set of all non-negative integers \( k \) such that the equation \( D(x_1, \ldots, x_m) = 0 \) has a solution in \( \{0, \ldots, k\}^m \). Since the set \( \{0, \ldots, k\}^m \) is finite, we know an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in \mathcal{E} \). Theorem 2 implies the next theorem.

Theorem 3. The set \( \mathcal{E} \) is empty or infinite. In both cases, every non-negative integer \( n \) is a threshold number of \( \mathcal{E} \). If ZFC is arithmetically consistent, then the sentences “\( \mathcal{E} \) is empty”, “\( \mathcal{E} \) is not empty”, “\( \mathcal{E} \) is finite”, and “\( \mathcal{E} \) is infinite” are not provable in ZFC.

Let

\[
\mathcal{V} = \left\{ n \in \mathbb{N} : \begin{aligned}
&\text{(the polynomial } D(x_1, \ldots, x_m) \text{ has no solutions in } \{0, \ldots, n\}^m) \land \\
&\text{(the polynomial } D(x_1, \ldots, x_m) \text{ has a solution in } \{0, \ldots, n+1\}^m) \end{aligned} \right\}
\]

Since the sets \( \{0, \ldots, n\}^m \) and \( \{0, \ldots, n+1\}^m \) are finite, we know an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in \mathcal{V} \). Theorem 3 implies the next theorem.

Theorem 4. ZFC proves that \( \text{card} (\mathcal{V}) \in \{0, 1\} \). For every \( n \in \mathbb{N} \), ZFC proves that \( n \notin \mathcal{V} \). ZFC does not prove the emptiness of \( \mathcal{V} \), if ZFC is arithmetically consistent. For every \( n \in \mathbb{N} \), the sentence “\( n \) is a threshold number of \( \mathcal{V} \)” is not provable in ZFC, if ZFC is arithmetically consistent.

4 Hypothetical statements \( \Psi_3, \ldots, \Psi_{16} \)

For an integer \( n \geq 3 \), let \( \mathcal{U}_n \) denote the following system of equations:

\[
\begin{aligned}
\forall i \in \{1, \ldots, n-1\} \setminus \{2\} & x_i! = x_{i+1} \\
x_1 \cdot x_2 & = x_3 \\
x_2 \cdot x_2 & = x_3
\end{aligned}
\]

The diagram in Figure 1 illustrates the construction of the system \( \mathcal{U}_n \).

![Fig. 1 Construction of the system \( \mathcal{U}_n \)](image)

Let \( g(3) = 4 \), and let \( g(n+1) = g(n)! \) for every integer \( n \geq 3 \).

Lemma 8. For every integer \( n \geq 3 \), the system \( \mathcal{U}_n \) has exactly two solutions in positive integers, namely \((1,1,1,1,\ldots,1)\) and \((2,2,g(3),\ldots,g(n))\).

Let

\[
\mathcal{B}_n = \left\{ x_i! = x_k : (i, k \in \{1, \ldots, n\}) \land (i \neq k) \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\}
\]

For an integer \( n \geq 3 \), let \( \Psi_n \) denote the following statement: if a system of equations \( \mathcal{S} \subseteq \mathcal{B}_n \) has only finitely many solutions in positive integers \( x_1, x_2, \ldots, x_n \), then each such solution \((x_1, x_2, \ldots, x_n)\) satisfies \( x_1, \ldots, x_n \leq g(n) \). The statement \( \Psi_n \) says that for subsystems of \( \mathcal{B}_n \) the largest known solution is indeed the largest possible.
Hypothesis 1. The statements $\Psi_3, \ldots, \Psi_{16}$ are true.

Proposition 3. Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

Proof. For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

Proposition 4. For every statement $\Psi_n$, the bound $g(n)$ cannot be decreased.

Proof. It follows from Lemma 8 because $U_n \subseteq B_n$. □

5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let $\mathcal{A}$ denote the following system of equations:

\[
\begin{align*}
  x_1! &= x_2 \\
  x_2! &= x_3 \\
  x_5! &= x_6 \\
  x_4 \cdot x_4 &= x_5 \\
  x_3 \cdot x_5 &= x_6
\end{align*}
\]

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

Lemma 9. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system $\mathcal{A}$ is solvable in positive integers $x_2, x_3, x_5, x_6$ if and only if $x_1! + 1 = x_4^2$. In this case, the integers $x_2, x_3, x_5, x_6$ are uniquely determined by the following equalities:

\[
\begin{align*}
  x_2 &= x_1! \\
  x_3 &= (x_1!)! \\
  x_5 &= x_1! + 1 \\
  x_6 &= (x_1! + 1)!
\end{align*}
\]

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [20, p. 297]. A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [15].

Theorem 5. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement $\Psi_6$ guarantees that each such solution $(x_1, x_4)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. Suppose that the antecedent holds. Let positive integers $x_1$ and $x_4$ satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 8 the system $\mathcal{A}$ is solvable in positive integers $x_2, x_3, x_5, x_6$. Since $\mathcal{A} \subseteq B_6$, the statement $\Psi_6$ implies that $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$. Hence, $x_1! + 1 \leq g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □
6 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau’s conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [14, pp. 37–38]. Let $\mathcal{B}$ denote the following system of equations:

$$
\begin{align*}
    x_2! &= x_3 \\
    x_3! &= x_4 \\
    x_4! &= x_5 \\
    x_5! &= x_6 \\
    x_6! &= x_7 \\
    x_7! &= x_8 \\
    x_8! &= x_9 \\
    x_1 \cdot x_1 &= x_2 \\
    x_2 \cdot x_5 &= x_6 \\
    x_3 \cdot x_8 &= x_9 \\
    x_5 \cdot x_7 &= x_8
\end{align*}
$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

**Fig. 3** Construction of the system $\mathcal{B}$

**Lemma 10.** For every integer $x_1 \geq 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

$$
\begin{align*}
    x_2 &= x_1^2 \\
    x_3 &= (x_1^2)! \\
    x_4 &= ((x_1^2)!)! \\
    x_5 &= x_1^2 + 1 \\
    x_6 &= (x_1^2 + 1)! \\
    x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
    x_8 &= (x_1^2)! + 1 \\
    x_9 &= ((x_1^2)! + 1)!
\end{align*}
$$

**Proof.** By Lemma 2 for every integer $x_1 \geq 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 10 follows from Lemma 6.

**Lemma 11.** There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.

**Proof.** If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. 

□
Theorem 6. The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.

Proof. Suppose that the antecedent holds. By Lemma 10, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geq g(7)$. Hence, $(x_1^2)! \geq (g(7) + 1)! > (g(8)! = g(9)$.

Since $\mathcal{B} \subseteq \mathcal{B}_9$, the statement $\Psi_9$ and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 10 and 11 there are infinitely many primes of the form $n^2 + 1$. □

Corollary 4. Let $X_9$ denote the set of primes of the form $n^2 + 1$. The statement $\Psi_9$ implies that we know an algorithm such that it returns a threshold number of $X_9$, and this number equals $\max(X_9)$, if $X_9$ is finite. Assuming the statement $\Psi_9$, a single query to an oracle for the halting problem decides the infinity of $X_9$. Assuming the statement $\Psi_9$, the infinity of $X_9$ is decidable in the limit.

Proof. We consider an algorithm which computes $\max(X_9 \cap [1, g(7)])$. □

7 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [41, p. 443].

Theorem 7. (cf. Theorem 77). The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geq g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.

Proof. We leave the analogous proof to the reader. □

8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [41, p. 39]. Let $C$ denote the following system of equations:

\[
\begin{align*}
x_1! &= x_2 \\
x_2! &= x_3 \\
x_4! &= x_5 \\
x_6! &= x_7 \\
x_7! &= x_8 \\
x_9! &= x_{10} \\
x_{12}! &= x_{13} \\
x_{15}! &= x_{16} \\
x_2 \cdot x_4 &= x_5 \\
x_5 \cdot x_6 &= x_7 \\
x_7 \cdot x_9 &= x_{10} \\
x_4 \cdot x_{11} &= x_{12} \\
x_3 \cdot x_{12} &= x_{13} \\
x_9 \cdot x_{14} &= x_{15} \\
x_8 \cdot x_{15} &= x_{16}
\end{align*}
\]

Lemma 2 and the diagram in Figure 4 explain the construction of the system $C$. 

7
Lemma 12. For every \( x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\} \), the system \( C \) is solvable in positive integers \( x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \) if and only if \( x_4 \) and \( x_9 \) are prime and \( x_4 + 2 = x_9 \). In this case, the integers \( x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \) are uniquely determined by the following equalities:

\[
\begin{align*}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1))! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1))! \\
x_{10} &= x_9! \\
x_{11} &= (x_9 - 1)! + 1 \\
x_{12} &= (x_9 - 1)! + 1 \\
x_{13} &= ((x_9 - 1)! + 1)! \\
x_{14} &= (x_9 - 1)! + 1 \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)! 
\end{align*}
\]

Proof. By Lemma 5 for every \( x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\} \), the system \( C \) is solvable in positive integers \( x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \) if and only if

\[
\left( x_4 + 2 = x_9 \right) \land \left( x_4 | (x_4 - 1)! + 1 \right) \land \left( x_9 | (x_9 - 1)! + 1 \right)
\]

Hence, the claim of Lemma 12 follows from Lemma 5.

\[\square\]

Lemma 13. There are only finitely many tuples \( (x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16} \) which solve the system \( C \) and satisfy \( (x_4 \in \{1, 2\}) \lor (x_9 \in \{1, 2\}) \).
Proof. If a tuple \((x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}\) solves the system \(C\) and \((x_4 \in \{1, 2\}) \lor (x_9 \in \{1, 2\})\), then \(x_1, \ldots, x_{16} \leq 7!\). Indeed, for example, if \(x_4 = 2\) then \(x_6 = x_4 + 1 = 3\). Hence, \(x_7 = x_6! = 6\). Therefore, \(x_{15} = x_7 + 1 = 7\). Consequently, \(x_{16} = x_{15}! = 7!\). □

**Theorem 8.** The statement \(Ψ_{16}\) proves the following implication: if there exists a twin prime greater than \(g(14)\), then there are infinitely many twin primes.

Proof. Suppose that the antecedent holds. Then, there exist prime numbers \(x_4\) and \(x_9\) such that \(x_9 = x_4 + 2 > g(14)\). Hence, \(x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}\). By Lemma 12, there exists a unique tuple \((x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}\) such that the tuple \((x_1, \ldots, x_{16})\) solves the system \(C\). Since \(x_9 > g(14)\), we obtain that \(x_9 - 1 > g(14)\). Therefore, \((x_9 - 1)! > g(14)! = g(15)\). Hence, \((x_9 - 1)! + 1 > g(15)\). Consequently,

\[
x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)
\]

Since \(C \subseteq B_{16}\), the statement \(Ψ_{16}\) and the inequality \(x_{16} > g(16)\) imply that the system \(C\) has infinitely many solutions in positive integers \(x_1, \ldots, x_{16}\). According to Lemmas 12 and 13, there are infinitely many twin primes. □

**Corollary 5.** (cf. [6]). Let \(X_{16}\) denote the set of twin primes. The statement \(Ψ_{16}\) implies that we know an algorithm such that it returns a threshold number of \(X_{16}\), and this number equals \(\max(X_{16})\), if \(X_{16}\) is finite. Assuming the statement \(Ψ_{16}\), a single query to an oracle for the halting problem decides the infinity of \(X_{16}\). Assuming the statement \(Ψ_{16}\), the infinity of \(X_{16}\) is decidable in the limit.

Proof. We consider an algorithm which computes \(\max(X_{16} \cap \{1, g(14)\})\). □

9 Hypothetical statements \(Δ_5, \ldots, Δ_{14}\) and their consequences

Let \(\lambda(5) = \Gamma(25)\), and let \(\lambda(n + 1) = \Gamma(\lambda(n))\) for every integer \(n \geq 5\). For an integer \(n \geq 5\), let \(J_n\) denote the following system of equations:

\[
\begin{align*}
\forall i \in \{1, \ldots, n - 1\} \setminus \{3\} & : \Gamma(x_i) = x_{i+1} \\
x_1 \cdot x_1 & = x_4 \\
x_2 \cdot x_3 & = x_5
\end{align*}
\]

Lemma 8 and the diagram in Figure 5 explain the construction of the system \(J_n\).
For every integer $n \geq 5$, the system $\mathcal{F}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$. For an integer $n \geq 5$, let $\Delta_n$ denote the following statement: if a system of equations $S \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq \lambda(n)$.

**Hypothesis 2.** The statements $\Delta_5, \ldots, \Delta_{14}$ are true.

Lemmas 3 and 5 imply that the statements $\Delta_n$ have similar consequences as the statements $\Psi_n$.

**Theorem 9.** The statement $\Delta_6$ implies that any prime number $p \geq 25$ proves the infinitude of primes.

**Proof.** It follows from Lemmas 3 and 5. We leave the details to the reader. \hfill \Box

## 10 Hypothetical statements $\Sigma_3, \ldots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k - 1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup [2^{2n-3} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \ldots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

For an integer $n \in \{3, \ldots, 16\}$, let $P_n$ denote the following system of equations:

$$\begin{align*}
x_1 \cdot x_1 &= x_1 \\
\Gamma_n(x_2) &= x_1 \\
\forall i \in \{2, \ldots, n-1\} \ x_i \cdot x_i &= x_{i+1}
\end{align*}$$

**Lemma 14.** For every integer $n \in \{3, \ldots, 16\}$, $P_n \subseteq Q_n$ and the system $P_n$ with $\Gamma$ instead of $\Gamma_n$ has exactly one solution in positive integers $x_1, \ldots, x_n$, namely $(1, 2^{20}, 2^{21}, 2^{22}, \ldots, 2^{2n-2})$.

For an integer $n \in \{3, \ldots, 16\}$, let $\Sigma_n$ denote the following statement: if a system of equations $S \subseteq Q_n$ with $\Gamma$ instead of $\Gamma_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $S$ satisfies $x_1, \ldots, x_n \leq 2^{2n-2}$.

**Hypothesis 3.** The statements $\Sigma_3, \ldots, \Sigma_{16}$ are true.

**Lemma 15.** (cf. Lemma 3). For every integer $n \in \{4, \ldots, 16\}$ and for every positive integers $x$ and $y$, $x \cdot \Gamma_n(x) = \Gamma_n(y)$ if and only if $(x + 1 = y) \land \left( x \geq 2^{2n-3} + 1 \right)$. 


Let $Z_9 \subseteq Q_9$ be the system of equations in Figure 6.

**Fig. 6** Construction of the system $Z_9$

**Lemma 16.** For every positive integer $x_1$, the system $Z_9$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1 > 2^{9^9-4}$ and $x_1^2 + 1$ is prime. In this case, positive integers $x_2, \ldots, x_9$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with $n$ and solve the system $Z_9$ with $\Gamma$ instead of $\Gamma_9$.

**Proof.** It follows from Lemmas 3, 5, and 15. □

**Lemma 17.** (13!)^2 + 1 = 387757788043632640001 is prime.

**Lemma 18.** \((13!)^2 \geq 2^{2^{9-3}} + 1 = 18446744073709551617 \land (\Gamma_9((13!)^2)) > 2^{2^{9-2}}\).

**Theorem 10.** The statement $\Sigma_9$ implies the infinitude of primes of the form $n^2 + 1$.

**Proof.** It follows from Lemmas 16–18. □

**Theorem 11.** (cf. Theorem 7). The statement $\Sigma_9$ implies that any prime of the form $n! + 1$ with $n \geq 2^{9-3}$ proves the infinitude of primes of the form $n! + 1$.

**Proof.** We leave the proof to the reader. □

**Corollary 6.** Let $Y_9$ denote the set of primes of the form $n! + 1$. The statement $\Sigma_9$ implies that we know an algorithm such that it returns a threshold number of $Y_9$, and this number equals $\max(Y_9)$, if $Y_9$ is finite. Assuming the statement $\Sigma_9$, a single query to an oracle for the halting problem decides the infinity of $Y_9$. Assuming the statement $\Sigma_9$, the infinity of $Y_9$ is decidable in the limit.

**Proof.** We consider an algorithm which computes $\max(Y_9 \cap [1, (2^{9-3} - 1)! + 1])$. □
Let $\mathcal{Z}_{14} \subseteq \mathbb{Q}_{14}$ be the system of equations in Figure 7.

**Fig. 7** Construction of the system $\mathcal{Z}_{14}$

**Lemma 19.** For every positive integer $x_1$, the system $\mathcal{Z}_{14}$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if $x_1$ and $x_1 + 2$ are prime and $x_1 \geq 2^{14-3} + 1$. In this case, positive integers $x_2, \ldots, x_{14}$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with $n$ and solve the system $\mathcal{Z}_{14}$ with $\Gamma$ instead of $\Gamma_{14}$.

**Proof.** It follows from Lemmas 3, 5, and 15.

**Lemma 20.** ([23], p. 87). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).

**Lemma 21.** $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

**Theorem 12.** The statement $\Sigma_{14}$ implies the infinitude of twin primes.

**Proof.** It follows from Lemmas 19, 21.

A prime $p$ is said to be a Sophie Germain prime if both $p$ and $2p + 1$ are prime, see [22]. It is conjectured that there are infinitely many Sophie Germain primes, see [17], p. 330. Let $\mathcal{Z}_{16} \subseteq \mathbb{Q}_{16}$ be the system of equations in Figure 8.
Lemma 22. For every positive integer $x_1$, the system $\mathcal{Z}_{16}$ is solvable in positive integers $x_2, \ldots, x_{16}$ if and only if $x_1$ is a Sophie Germain prime and $x_1 \geq 2^{16-3} + 1$. In this case, positive integers $x_2, \ldots, x_{16}$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with $n$ and solve the system $\mathcal{Z}_{16}$ with $\Gamma$ instead of $\Gamma_{16}$.

Proof. It follows from Lemmas 3, 5, and 15.

Lemma 23. ([17, p. 330]). 8069496435 · $10^{5072} - 1$ is a Sophie Germain prime (Harvey Dubner).

Lemma 24. 8069496435 · $10^{5072} - 1 > 2^{2^{16-2}}$.

Theorem 13. The statement $\Sigma_{16}$ implies the infinitude of Sophie Germain primes.

Proof. It follows from Lemmas 22 and 24.

Theorem 14. The statement $\Sigma_6$ proves the following implication: if the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(1, 2), (2, 3)\}$.

Proof. We leave the proof to the reader.

The question of solving the equation $x(x + 1) = y!$ was posed by P. Erdős, see [2]. F. Luca proved that the abc conjecture implies that the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers, see [12].

Theorem 15. The statement $\Sigma_6$ proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. We leave the proof to the reader.
11 Hypothetical statements $\Omega_3, \ldots, \Omega_{16}$ and their consequences

For an integer $n \in \{3, \ldots, 16\}$, let $\Omega_n$ denote the following statement: if a system of equations $S \subseteq \{ \Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \}$ has a solution in integers $x_1, \ldots, x_n$ greater than $2^{2n-2}$, then $S$ has infinitely many solutions in positive integers $x_1, \ldots, x_n$. For every $n \in \{3, \ldots, 16\}$, the statement $\Sigma_n$ implies the statement $\Omega_n$.

Lemma 25. The number $(65!)^2 + 1$ is prime and $65! > 2^{2^9 - 2}$.

Proof. The following PARI/GP ([16]) command

(04:04) gp > isprime((65!)^2+1,{flag=2})

%1 = 1

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([23, p. 226]). It rigorously shows that the number $(65!)^2 + 1$ is prime. □

Lemma 26. If positive integers $x_1, \ldots, x_9$ solve the system $Z_9$ and $x_1 > 2^{2^9 - 2}$, then $x_1 = \min(x_1, \ldots, x_9)$.

Theorem 16. The statement $\Omega_9$ implies the infinitude of primes of the form $n^2 + 1$.


Lemma 27. If positive integers $x_1, \ldots, x_{14}$ solve the system $Z_{14}$ and $x_1 > 2^{2^{14} - 2}$, then $x_1 = \min(x_1, \ldots, x_{14})$.

Theorem 17. The statement $\Omega_{14}$ implies the infinitude of twin primes.

Proof. It follows from Lemmas 19–21 and 27. □

12 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [11, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3, 2^{2^1} + 1 = 5, 2^{2^2} + 1 = 17, 2^{2^3} + 1 = 257, \text{ and } 2^{2^4} + 1 = 65537$ are all prime, see [11, p. 1].

Open Problem. ([11, p. 159]). Are there infinitely many composite numbers of the form $2^{2^n} + 1$?

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [10, p. 23]. Let

$$H_n = \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \} \cup \{ 2^{2^{x_i}} = x_k : i, k \in \{1, \ldots, n\} \}$$

Let $h(1) = 1$, and let $h(n+1) = 2^{h(n)}$ for every positive integer $n$.

Lemma 28. The following subsystem of $H_n$

$$\begin{align*}
x_1 \cdot x_1 &= x_1 \\
\forall i \in \{1, \ldots, n-1\} \quad 2^{2^{x_i}} &= x_{i+1}
\end{align*}$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \ldots, h(n))$. 
For a positive integer $n$, let $\xi_n$ denote the following statement: if a system of equations $S \subseteq H_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq h(n)$. The statement $\xi_n$ says that for subsystems of $H_n$ the largest known solution is indeed the largest possible.

**Hypothesis 4.** The statements $\xi_1, \ldots, \xi_{13}$ are true.

**Proposition 5.** Every statement $\xi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $H_n$ has a finite number of subsystems. □

**Theorem 18.** The statement $\xi_{13}$ proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers $z$.

**Proof.** Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \quad (2)$$

in positive integers. By Lemma [4], we can transform equation (2) into an equivalent system of equations $G$ which has 13 variables ($x$, $y$, $z$, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 9.

![Fig. 9 Construction of the system $G$](image)

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^z} + 1} > h(13)$. By this, the statement $\xi_{13}$ implies that the system $G$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □
Corollary 7. Let $W_{13}$ denote the set of composite Fermat numbers. The statement $\xi_{13}$ implies that we know an algorithm such that it returns a threshold number of $W_{13}$, and this number equals $\max(W_{13})$, if $W_{13}$ is finite. Assuming the statement $\xi_{13}$, a single query to an oracle for the halting problem decides the infinity of $W_{13}$. Assuming the statement $\xi_{13}$, the infinity of $W_{13}$ is decidable in the limit.

Proof. We consider an algorithm which computes $\max(W_{13} \cap [1, h(12)])$. □

References


Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl