Statements that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in ZFC because they refer to the currently known/unknown theorems about $\mathcal{X}$

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Conditions (1)-(8) below concern sets $\mathcal{X} \subseteq \mathbb{N}$. (1) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite. (2) No known algorithm decides the finiteness of $\mathcal{X}$. (3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. (4) There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$. (5) $\mathcal{X}$ is widely known in number theory. (6) There is no known equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined simpler than $\mathcal{X}$. (7) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies $(\mathcal{Y} \subseteq \mathcal{X}) \wedge (\mathrm{card}(\mathcal{X} \setminus \mathcal{Y}) < \omega)$. (8) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies $\mathrm{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$. We do not know any set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)-(4) and (5). The same is true, if condition (5) is replaced by condition (6) or (7) or (8). For every explicitly known integer n, some simply defined set $\mathcal{X} \subseteq \mathbb{N}$ includes the set $(-\infty, n] \cap \mathbb{N}$ and satisfies conditions (1)-(4). Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form n^2+1. The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)-(3) and (5)-(8). The set $\mathcal{X} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions (1)-(3) and (6)-(8). Some hypothetical statement implies that these sets $\mathcal{X}$ satisfy condition (4).

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1. Introduction, definitions, and basic lemmas

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [5, p. 199]. In this article, we present an argument against logicism: there are statements that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in ZFC because they refer to the currently known/unknown theorems about $\mathcal{X}$.

Definition 1. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

Proof. We ask Wolfram Alpha at http://wolframalpha.com. □

Lemma 2. $((7!)!)! \approx 10^{10^{16477.87280582041}}$.

Proof. We ask Wolfram Alpha about $0.0 + ((7!)!)!$. □

Definition 2. *We say that an integer* $n \geqslant -1$ *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$, *if* $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$, *cf.* [12] *and* [13].

Definition 3. *We say that a non-negative integer n is a weak threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$, *if* $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathrm{card}(\mathcal{X}) \leqslant n$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $n \geqslant -1$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer n is a weak threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all weak threshold numbers of $\mathcal{X}$ form the set $[\mathrm{card}(\mathcal{X}), \infty) \cap \mathbb{N}$.

Theorem 1. *For every set* $\mathcal{X} \subseteq \mathbb{N}$, *if an integer* $n \geqslant -1$ *is a threshold number of* $\mathcal{X}$, *then* $n + 1$ *is a weak threshold number of* $\mathcal{X}$.

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$ and for every integer $n \geqslant -1$, the inclusion $\mathcal{X} \subseteq (-\infty, n]$ implies that $\mathrm{card}(\mathcal{X}) \leqslant n + 1$. □

Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form $n^2 + 1$. We do not know any weak threshold number of $\mathcal{P}_{n^2+1}$. The same is true for the sets

$$\left\{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\right\}$$

and

$$\{n \in \mathbb{N} : n! + 1 \text{ is a square}\}$$

Lemma 3. *For every positive integers x and y,* $x! \cdot y = y!$ *if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. *(Wilson's theorem,* [1, p. 89]*). For every integer* $x \geqslant 2$, *x is prime if and only if x divides* $(x - 1)! + 1$.

The conditions below concern sets $\mathcal{X} \subseteq \mathbb{N}$.

(1) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.

(2) No known algorithm decides the finiteness of $\mathcal{X}$.

(3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$.

(4) There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) $\mathcal{X}$ is widely known in number theory.

(6) There is no known equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined simpler than $\mathcal{X}$.

(7) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies $(\mathcal{Y} \subseteq \mathcal{X}) \wedge (\mathrm{card}(\mathcal{X} \setminus \mathcal{Y}) < \omega)$.

(8) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies $\mathrm{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$.

(4•) There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathrm{card}(\mathcal{X}) \leqslant n$.

(1⋄) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X} = \mathbb{N}$.

(2⋄) No known algorithm decides the equality $\mathcal{X} = \mathbb{N}$.

(1*) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is finite.

2. Open Problems 1 and 2

The following two open problems cannot be formalized in *ZFC* because they refer to current knowledge about $\mathcal{X}$.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1)-(3), (4•), *and* (5)*?*

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1)-(5)*?*

Theorem 2. *Open Problem 2 claims more than Open Problem 1.*

Proof. Condition (4) implies that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, |n|]$. Since $|n| \geqslant -1$, Theorem 1 guarantees that condition (4) implies condition (4•). □

Open Problems 1 and 2 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

3. Partial solutions to Open Problem 2

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ is infinite, see [6, pp. 37–38] and [9]. Let $\mathcal{M}$ denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$.

Statement 1. *The set $\mathcal{X} = \{0, \dots, \beta\} \cup \mathcal{M}$ satisfies conditions* (1)-(4).

Proof. Condition (1) holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β. Thus condition (2) holds. Condition (3) holds trivially. Since the set $\mathcal{M}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition (4) holds. □

Statements 1–6 may be falsified because they refer to the currently known/unknown theorems about $\mathcal{X}$. By the same reason, Statements 1–6 cannot be formalized in *ZFC*.

Let $[\cdot]$ denote the integer part function.

Lemma 5. *For every non-negative integer n,* $\left[\dfrac{3n-3\beta+3}{3n-3\beta+2}\right]$ *equals* 0 *or* 1. *The first case holds when $n \leqslant \beta - 1$. The second case holds when $n \geqslant \beta$.*

Lemma 6. *The function*

$$\mathbb{N} \cap [\beta, \infty) \ni n \xrightarrow{\theta} \beta + n - \left[\sqrt{n}\right]^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geqslant \beta$ infinitely many times.

Proof. Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \left\{(t+0)^2 + t,\ (t+1)^2 + t,\ (t+2)^2 + t,\ \ldots\right\} \cap [\beta, \infty)$$

□

Statement 2. *The set*

$$\mathcal{X} = \left\{n \in \mathbb{N} : 2 + \left[\frac{3n - 3\beta + 3}{3n - 3\beta + 2}\right] \cdot \left(\left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 - 1\right) \text{ is prime}\right\}$$

satisfies conditions `(1)-(4)`.

Proof. Condition `(3)` holds trivially. By Lemma 5, $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$, where

$$\mathcal{H} = \left\{n \in \mathbb{N} \cap [\beta, \infty) : \left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 + 1 \text{ is prime}\right\}$$

By Lemma 6, the set $\mathcal{H}$ is empty or infinite. The second case holds when

$$\exists\, k \in \mathbb{N} \cap [\beta, \infty)\ k^2 + 1 \text{ is prime} \tag{G}$$

The equality $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of $\mathcal{X}$ and conditions `(1)` and `(4)` hold. Condition `(2)` holds as due to known physics we are not able to confirm the statement `(G)` by a direct computation. □

4. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

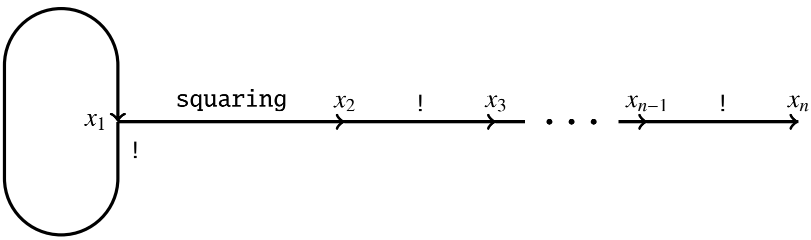

Fig. 1 Construction of the system $\mathcal{U}_n$

Lemma 7. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \dots, 1)$ *and* $\big(f(1), \dots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k : \ i, k \in \{1, \dots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \dots, n\}\Big\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. An elementary reasoning proves the statements Ψ_1 and Ψ_2.

Theorem 3. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 7 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 4. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \dots, x_n$, then for every integer $i \in \{1, \dots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \dots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 5. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

5. A conjectural solution to Open Problem 2

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

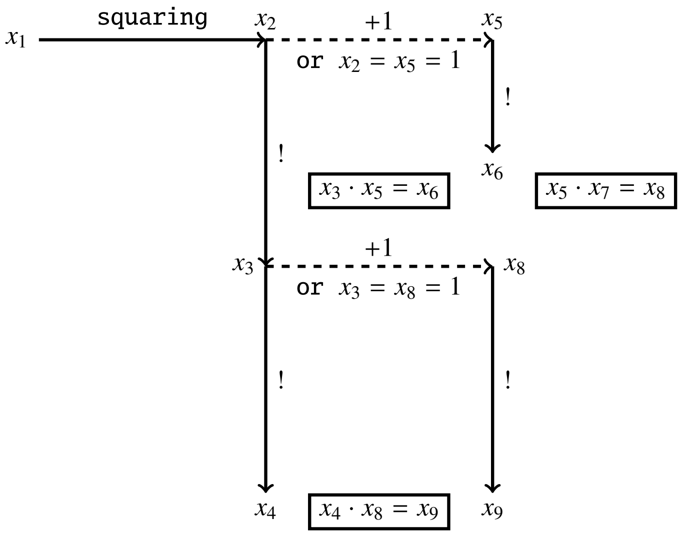

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{array}{rcl} x_2 & = & x_1^2 \\ x_3 & = & (x_1^2)! \\ x_4 & = & ((x_1^2)!)! \\ x_5 & = & x_1^2 + 1 \\ x_6 & = & (x_1^2 + 1)! \\ x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 & = & (x_1^2)! + 1 \\ x_9 & = & ((x_1^2)! + 1)! \end{array}$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 4. □

Lemma 9. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{A}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Let Φ_9 denote the statement Ψ_9 restricted to the system $\mathcal{A}$. Apoloniusz Tyszka believes that the statement Φ_9 is true.

Theorem 6. *The statement Φ_9 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N}\setminus\{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

The statement Φ_9 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 8 and 9, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Let $\mathcal{K} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$.

Lemma 10. $\mathrm{card}(\mathcal{K}) \geqslant 9 \cdot 10^{9 \cdot 4^{747}} \approx 10^{10^{450.6930560314272}}$.

Proof. The following `PARI/GP` ([8]) command

```
isprime(1+9*4^747,{flag=2})
```

returns $\%1 = 1$. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([11, p. 226]). It rigorously shows that the number $\left(3 \cdot 2^{747}\right)^2 + 1$ is prime. Since $9 \cdot 10^{9 \cdot 4^{747}}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10\hat{}(9 * 4\hat{}747))$. □

Statement 3. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions* `(1)`-`(3)` *and* `(5)`-`(8)`*. The set $\mathcal{X} = \mathcal{K}$ satisfies conditions* `(1)`-`(3)` *and* `(6)`-`(8)`*. The statement Φ_9 implies that these sets $\mathcal{X}$ satisfy condition* `(4)`*.*

Proof. Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, Lemma 10 implies condition `(1)` for both sets $\mathcal{X}$. Conditions `(3)` and `(6)`-`(8)` hold trivially for both sets $\mathcal{X}$. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$. Thus condition `(2)` holds for both sets $\mathcal{X}$. Suppose that the statement Φ_9 is true. By Theorem 6, $f(7)$ is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. By Theorem 6, $\underbrace{9\ldots9}_{f(7)\text{ digits}}$ is a threshold number of $\mathcal{X} = \mathcal{K}$. Thus condition `(4)` holds for both sets $\mathcal{X}$. □

6. Open Problems 3 and 4

The following two open problems cannot be formalized in *ZFC* because they refer to current knowledge about $\mathcal{X}$.

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1⋄)`-`(2⋄)`, `(2)`-`(3)`, `(4•)`, *and* `(5)`*?*

Open Problem 3 claims more than Open Problem 1 as condition `(1⋄)` implies condition `(1)`.

Open Problem 4. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1⋄)`-`(2⋄)` *and* `(2)`-`(5)`*?*

Open Problem 4 claims more than Open Problem 2 as condition `(1⋄)` implies condition `(1)`.

Theorem 7. *Open Problem 4 claims more than Open Problem 3.*

Proof. Condition `(4)` implies that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, |n|]$. Since $|n| \geqslant -1$, Theorem 1 guarantees that condition `(4)` implies condition `(4•)`. □

Open Problems 3 and 4 remain open, if condition `(5)` is replaced by condition `(6)` or `(7)` or `(8)`.

7. A partial solution to Open Problem 4

Let $\mathcal{V}$ denote the set of all positive multiples of elements of the set

$$\left\{n \in [\beta+1, \infty) \cap \mathbb{N} : 2^{2^n}+1 \text{ is composite}\right\}$$

Statement 4. *The set* $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{V}$ *satisfies conditions* `(1⋄)-(2⋄)` *and* `(2)-(4)`.

Proof. The inequality $\mathrm{card}(\mathcal{X}) > \beta$ holds trivially. Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [3, p. 23]. These two facts imply conditions `(1⋄)` and `(2⋄)`. Condition `(3)` holds trivially. Since the set $\mathcal{V}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. The question of finiteness of the set

$$\left\{n \in \mathbb{N} : 2^{2^n}+1 \text{ is composite}\right\}$$

remains open, see [4, p. 159]. By this and Lemma 1, the question of emptiness of the set

$$\left\{n \in [\beta+1, \infty) \cap \mathbb{N} : 2^{2^n}+1 \text{ is composite}\right\}$$

remains open. Therefore, the question of finiteness of the set $\mathcal{V}$ remains open. Consequently, the question of finiteness of the set $\mathcal{X}$ remains open and condition `(2)` holds. □

8. Open Problems 5 and 6

The following two open problems cannot be formalized in *ZFC* because they refer to current knowledge about $\mathcal{X}$.

Open Problem 5. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1*)`, `(2)-(3)`, `(4•)`, *and* `(5)`*?*

Open Problem 6. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1*)` *and* `(2)-(5)`*?*

Theorem 8. *Open Problem 6 claims more than Open Problem 5.*

Proof. Condition `(4)` implies that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, |n|]$. Since $|n| \geqslant -1$, Theorem 1 guarantees that condition `(4)` implies condition `(4•)`. □

Open Problems 5 and 6 remain open, if condition `(5)` is replaced by condition `(6)` or `(7)` or `(8)`.

9. Partial solutions to Open Problem 6

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [7].

Lemma 11. ([10, p. 297]). *It is conjectured that* $x! + 1$ *is a square only for* $x \in \{4, 5, 7\}$.

Let $\mathcal{W}$ denote the set of all integers x greater than β such that $x! + 1$ is a square.

Statement 5. *The set*

$$\mathcal{X} = \{0, \ldots, \beta\} \cup \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

satisfies conditions `(1*)` *and* `(2)-(4)`.

Proof. Condition `(1*)` holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{W}$ is conjecturally empty by Lemma 11. Condition `(3)` holds trivially. We do not know any algorithm that decides the emptiness of $\mathcal{W}$ and the set

$$\mathcal{Y} = \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

is empty or infinite. Thus condition `(2)` holds. Since the set $\mathcal{Y}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

Let C denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system C.

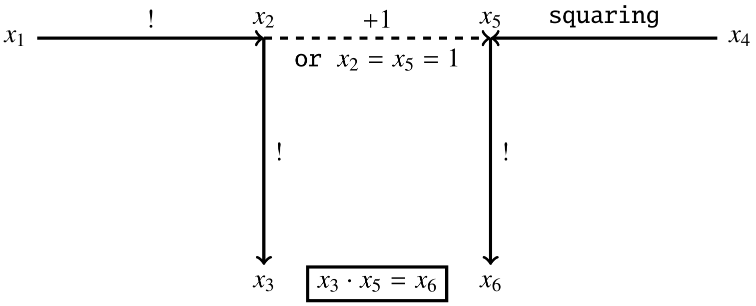

Fig. 3 Construction of the system C

Lemma 12. *For every* $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, *the system* C *is solvable in positive integers* x_2, x_3, x_5, x_6 *if and only if* $x_1! + 1 = x_4^2$. *In this case, the integers* x_2, x_3, x_5, x_6 *are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 3. □

Let Φ_6 denote the statement Ψ_6 restricted to the system C. Apoloniusz Tyszka believes that the statement Φ_6 is true.

Theorem 9. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Φ_6 guarantees that each such solution (x_1, x_4) satisfies $x_1 < 24!$.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 12, the system C is solvable in positive integers x_2, x_3, x_5, x_6. The statement Φ_6 implies that $x_6 = (x_1! + 1)! \leqslant f(6) = f(5)!$. Hence, $x_1! + 1 \leqslant f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24!$. □

Statement 6. *Let $\mathcal{X}$ denote the set of all non-negative integers n which have $((k!)!)!$ digits for some $k \in \{m \in \mathbb{N} : m! + 1 \text{ is a square}\}$. We claim that $\mathcal{X}$ satisfies conditions* `(1*)`, `(2)-(3)`, *and* `(6)-(8)`. *The statement Φ_6 implies that $\mathcal{X}$ satisfies condition* `(4)`.

Proof. Let $d = ((7!)!)!$. Since $7! + 1 = 71^2$, we obtain that $\{10^{d-1}, \ldots, \underbrace{9 \ldots 9}_{d \text{ digits}}\} \subseteq \mathcal{X}$. Hence, $\mathrm{card}(\mathcal{X}) \geqslant 9 \cdot 10^{d-1}$. By this and Lemmas 2 and 11, condition `(1*)` holds. Conditions `(2)-(3)` and `(6)-(8)` hold trivially. By Theorem 9, the statement Φ_6 implies that $\underbrace{9 \ldots 9}_{\beta \text{ digits}}$ is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

10. Is it possible to strengthen the main results?

Proving the statement Φ_9 will strengthen Theorem 6 and Statement 3. Proving the statement Φ_6 will strengthen Theorem 9 and Statement 6. The removal from conditions `(6)-(8)` of the word *"known"* will strengthen these conditions and will mean that it will be possibile to formulate them in ZFC. Proving Statements 3 and 6 with such changed conditions `(6)-(8)` is the next goal.

Proving that the statement *"The set $\mathcal{P}_{n^2+1}$ is infinite"* is independent of ZFC will falsify Statement 3, because then condition `(1)` will not be met. Such a statement is highly unlikely, cf. [2, p. 488]. Proving that the set $\mathcal{P}_{n^2+1}$ is infinite will falsify Statement 3, because then condition `(2)` will not be fulfilled. Proving that the set $\mathcal{P}_{n^2+1}$ is finite will falsify Statement 3, because then condition `(2)` will not be fulfilled. Such a statement is highly unlikely. When the above falsifications do not occur, we can try to prove that the consistency of ZFC implies the consistency of ZFC with the statement *"The set $\mathcal{P}_{n^2+1}$ is infinite"*. This will complement Statement 3, because it argues for the conjecture that the set $\mathcal{P}_{n^2+1}$ is infinite, and this conjecture is part of condition `(1)`.

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Sławomir Kurpaska
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: rtkurpas@cyf-kr.edu.pl

Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl

On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\{x \in \mathbb{N}\colon \varphi(x)\} \subseteq \{x \in \mathbb{N}\colon x \leqslant n-1\}$ if the set $\{x \in \mathbb{N}\colon \varphi(x)\}$ is finite

Apoloniusz Tyszka

Abstract

Let $\Gamma(k)$ denote $(k-1)!$, and let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3,\ldots,16\}$ and $k \in \{2\} \cup [2^{2^{n-3}}+1,\infty) \cap \mathbb{N}$. For an integer $n \in \{3,\ldots,16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq \{\Gamma_n(x_i)=x_k:\ i,k \in \{1,\ldots,n\}\} \cup \{x_i \cdot x_j = x_k:\ i,j,k \in \{1,\ldots,n\}\}$ with Γ instead of Γ_n has only finitely many solutions in positive integers $x_1,\ldots,x_n$, then every tuple $(x_1,\ldots,x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $\mathcal{S}$ satisfies $x_1,\ldots,x_n \leqslant 2^{2^{n-2}}$. Our hypothesis claims that the statements $\Sigma_3,\ldots,\Sigma_{16}$ are true. The statement Σ_6 proves the following implication: if the equation $x(x+1)=y!$ has only finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(1,2),(2,3)\}$. The statement Σ_6 proves the following implication: if the equation $x!+1=y^2$ has only finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(4,5),(5,11),(7,71)\}$. The statement Σ_9 implies the infinitude of primes of the form n^2+1. The statement Σ_9 implies that any prime of the form $n!+1$ with $n \geqslant 2^{2^{9-3}}$ proves the infinitude of primes of the form $n!+1$. The statement Σ_{14} implies the infinitude of twin primes. The statement Σ_{16} implies the infinitude of Sophie Germain primes.

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation $x!+1=y^2$, composite Fermat numbers, decidability in the limit, Erdös' equation $x(x+1)=y!$, finiteness of a set, infiniteness of a set, prime numbers of the form n^2+1, prime numbers of the form $n!+1$, single query to an oracle for the halting problem, Sophie Germain primes, twin primes.

2010 Mathematics Subject Classification: 03B30, 11A41.

1 Introduction and basic lemmas

The phrase "*we know a non-negative integer n*" in the title means that we know an algorithm which returns n. The title of the article cannot be formalised in ZFC because the phrase "*we know a non-negative integer n*" refers to currently known non-negative integers n with some property. A formally stated title may look like this: *On ZFC-formulae $\varphi(x)$ for which there exists a non-negative integer n such that ZFC proves that*

$$\mathrm{card}(\{x \in \mathbb{N}\colon \varphi(x)\}) < \infty \Longrightarrow \{x \in \mathbb{N}\colon \varphi(x)\} \subseteq \{x \in \mathbb{N}\colon x \leqslant n-1\}$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer n such that ZFC proves the above implication.

Lemma 1. *For every non-negative integer n,* $\mathrm{card}(\{x \in \mathbb{N}\colon x \leqslant n-1\}) = n$.

Corollary 1. *The title altered to* "On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\mathrm{card}(\{x \in \mathbb{N}\colon \varphi(x)\}) \leqslant n$ if the set $\{x \in \mathbb{N}\colon \varphi(x)\}$ is finite" *involves a weaker assumption on $\varphi(x)$.*

Lemma 2. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Let $\Gamma(k)$ denote $(k-1)!$.

Lemma 3. *For every positive integers x and y, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 4. *For every non-negative integers b and c, $b+1=c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Lemma 5. *(Wilson's theorem, [8, p. 89]). For every positive integer x, x divides $(x-1)!+1$ if and only if $x=1$ or x is prime.*

2 Subsets of $\mathbb{N}$ and their threshold numbers

We say that a non-negative integer m is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than m, cf. [24] and [25]. If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer m is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $\{\max(\mathcal{X}), \max(\mathcal{X})+1, \max(\mathcal{X})+2, \ldots\}$.

It is conjectured that the set of prime numbers of the form n^2+1 is infinite, see [14, pp. 37–38]. It is conjectured that the set of prime numbers of the form $n!+1$ is infinite, see [3, p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [14, p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^n}+1$ is infinite, see [10, p. 23] and [11, pp. 158–159]. A prime p is said to be a Sophie Germain prime if both p and $2p+1$ are prime, see [22]. It is conjectured that the set of Sophie Germain primes is infinite, see [17, p. 330]. For each of these sets, we do not know any threshold number.

The following statement:

$$\begin{gathered}\text{for every non-negative integer } n \text{ there exist} \\ \text{prime numbers } p \text{ and } q \text{ such that } p+2=q \text{ and } p \in \left[10^n, 10^{n+1}\right]\end{gathered} \tag{1}$$

is a Π_1 statement which strengthens the twin prime conjecture, see [4, p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger Π_1 statements, see [1]. Statement (1) is equivalent to the non-halting of a Turing machine. If a set $\mathcal{X} \subseteq \mathbb{N}$ is computable and we know a threshold number of $\mathcal{X}$, then the infinity of $\mathcal{X}$ is equivalent to the halting of a Turing machine.

The height of a rational number $\frac{p}{q}$ is denoted by $H\left(\frac{p}{q}\right)$ and equals $\max(|p|,|q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple $(x_1, \ldots, x_n)$ is denoted by $H(x_1, \ldots, x_n)$ and equals $\max(H(x_1), \ldots, H(x_n))$.

Lemma 6. *The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [13, p. 212]. The known rational solutions are $(x,y) = (-1,0)$, $(-1,1)$, $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(2,-5)$, $(2,6)$, $(3,-15)$, $(3,16)$, $(30,-4929)$, $(30,4930)$, $\left(\frac{1}{4}, \frac{15}{32}\right)$, $\left(\frac{1}{4}, \frac{17}{32}\right)$, $\left(-\frac{15}{16}, -\frac{185}{1024}\right)$, $\left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [18, pp. 223–224].*

Corollary 2. *The set $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$ is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of $\mathcal{T}$.*

Let $\mathcal{L}$ denote the following system of equations:

$$\begin{cases} x^2 + y^2 &= s^2 \\ x^2 + z^2 &= t^2 \\ y^2 + z^2 &= u^2 \\ x^2 + y^2 + z^2 &= v^2 \end{cases}$$

Let

$$\mathcal{F} = \Big\{n \in \mathbb{N} \setminus \{0\} : \Big(\text{the system } \mathcal{L} \text{ has no solutions in } \{1,\ldots,n\}^7\Big) \wedge$$
$$\Big(\text{the system } \mathcal{L} \text{ has a solution in } \{1,\ldots,n+1\}^7\Big)\Big\}$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

Lemma 7. *([21]). No perfect cuboids are known.*

Corollary 3. *We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{F}$. ZFC proves that* $\mathrm{card}(\mathcal{F}) \in \{0,1\}$. *We do not know any algorithm which returns* $\mathrm{card}(\mathcal{F})$. *We do not know any algorithm which returns a threshold number of $\mathcal{F}$.*

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, \text{ if } \sin\left(9^{9^{9^{9^{9}}}}\right) < 0 \\ \mathbb{N} \cap \left[0,\ \sin\left(9^{9^{9^{9^{9}}}}\right) \cdot 9^{9^{9^{9^{9}}}}\right) \text{ otherwise} \end{cases}$$

We do not know whether or not the set $\mathcal{H}$ is finite.

Proposition 1. *The number $9^{9^{9^{9^{9}}}}$ is a threshold number of $\mathcal{H}$. We know an algorithm which decides the equality $\mathcal{H} = \mathbb{N}$. If $\mathcal{H} \neq \mathbb{N}$, then the set $\mathcal{H}$ consists of all integers from* 0 *to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{H}$.*

Let

$$\mathcal{K} = \begin{cases} \{n\}, \text{ if } (n \in \mathbb{N}) \wedge \left(2^{\aleph_0} = \aleph_{n+1}\right) \\ \{0\}, \text{ if } 2^{\aleph_0} \geqslant \aleph_\omega \end{cases}$$

Proposition 2. *ZFC proves that* $\mathrm{card}(\mathcal{K}) = 1$. *If ZFC is consistent, then for every $n \in \mathbb{N}$ the sentences* "n is a threshold number of $\mathcal{K}$" *and* "n is not a threshold number of $\mathcal{K}$" *are not provable in ZFC.*

Proof. It suffices to observe that $2^{\aleph_0}$ can attain every value from the set $\{\aleph_1, \aleph_2, \aleph_3, \ldots\}$, see [7] and [9, p. 232]. □

3 A Diophantine equation whose non-solvability expresses the consistency of *ZFC*

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

Theorem 1. *([5, p. 35]). There exists a polynomial $D(x_1,\ldots,x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences* "The equation $D(x_1,\ldots,x_m) = 0$ is solvable in non-negative integers" *and* "The equation $D(x_1,\ldots,x_m) = 0$ is not solvable in non-negative integers" *are not provable in ZFC.*

Let $\mathcal{Y}$ denote the set of all non-negative integers k such that the equation $D(x_1,\ldots,x_m) = 0$ has no solutions in $\{0,\ldots,k\}^m$. Since the set $\{0,\ldots,k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Theorem 1 implies the next theorem.

Theorem 2. *For every $n \in \mathbb{N}$, ZFC proves that $n \in \mathcal{Y}$. If ZFC is arithmetically consistent, then the sentences* "$\mathcal{Y}$ is finite" *and* "$\mathcal{Y}$ is infinite" *are not provable in ZFC. If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences* "n is a threshold number of $\mathcal{Y}$" *and* "n is not a threshold number of $\mathcal{Y}$" *are not provable in ZFC.*

Let $\mathcal{E}$ denote the set of all non-negative integers k such that the equation $D(x_1, \ldots, x_m) = 0$ has a solution in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{E}$. Theorem 1 implies the next theorem.

Theorem 3. *The set $\mathcal{E}$ is empty or infinite. In both cases, every non-negative integer n is a threshold number of $\mathcal{E}$. If ZFC is arithmetically consistent, then the sentences* "$\mathcal{E}$ is empty", "$\mathcal{E}$ is not empty", "$\mathcal{E}$ is finite", *and* "$\mathcal{E}$ is infinite" *are not provable in ZFC.*

Let

$$\mathcal{V} = \Big\{n \in \mathbb{N} : \Big(\text{the polynomial } D(x_1, \ldots, x_m) \text{ has no solutions in } \{0, \ldots, n\}^m\Big) \wedge$$
$$\Big(\text{the polynomial } D(x_1, \ldots, x_m) \text{ has a solution in } \{0, \ldots, n+1\}^m\Big)\Big\}$$

Since the sets $\{0, \ldots, n\}^m$ and $\{0, \ldots, n+1\}^m$ are finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{V}$. Theorem 1 implies the next theorem.

Theorem 4. *ZFC proves that* $\mathrm{card}(\mathcal{V}) \in \{0, 1\}$. *For every $n \in \mathbb{N}$, ZFC proves that $n \notin \mathcal{V}$. ZFC does not prove the emptiness of $\mathcal{V}$, if ZFC is arithmetically consistent. For every $n \in \mathbb{N}$, the sentence* "n is a threshold number of $\mathcal{V}$" *is not provable in ZFC, if ZFC is arithmetically consistent.*

4 Hypothetical statements $\Psi_3, \ldots, \Psi_{16}$

For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{2\} \; x_i! & = & x_{i+1} \\ x_1 \cdot x_2 & = & x_3 \\ x_2 \cdot x_2 & = & x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

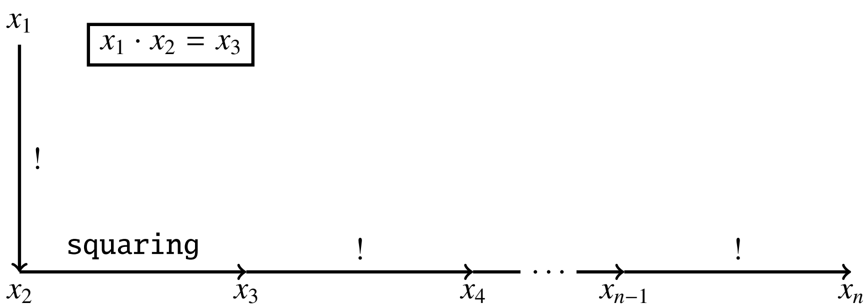

Fig. 1 Construction of the system $\mathcal{U}_n$

Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geqslant 3$.

Lemma 8. *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\big(2, 2, g(3), \ldots, g(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k : \; (i, k \in \{1, \ldots, n\}) \wedge (i \neq k)\Big\} \cup \Big\{x_i \cdot x_j = x_k : \; i, j, k \in \{1, \ldots, n\}\Big\}$$

For an integer $n \geqslant 3$, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant g(n)$.* The statement Ψ_n says that for subsystems of B_n the largest known solution is indeed the largest possible.

Hypothesis 1. *The statements* $\Psi_3, \ldots, \Psi_{16}$ *are true.*

Proposition 3. *Every statement* Ψ_n *is true with an unknown integer bound that depends on* n*.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

Proposition 4. *For every statement* Ψ_n*, the bound* $g(n)$ *cannot be decreased.*

Proof. It follows from Lemma 8 because $\mathcal{U}_n \subseteq B_n$. □

5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

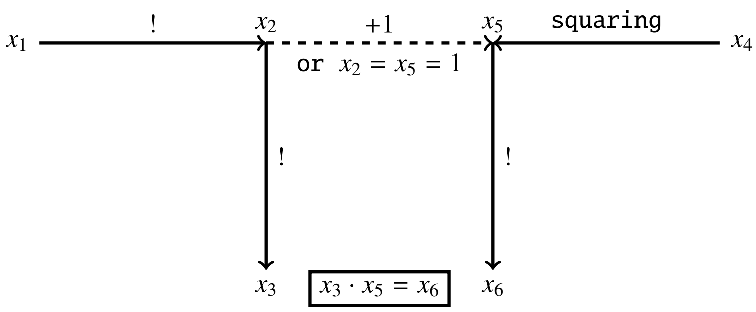

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 9. *For every* $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$*, the system* $\mathcal{A}$ *is solvable in positive integers* x_2*,* x_3*,* x_5*,* x_6 *if and only if* $x_1! + 1 = x_4^2$*. In this case, the integers* x_2, x_3, x_5, x_6 *are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [15].

Theorem 5. *If the equation* $x_1! + 1 = x_4^2$ *has only finitely many solutions in positive integers, then the statement* Ψ_6 *guarantees that each such solution* (x_1, x_4) *belongs to the set* $\{(4, 5), (5, 11), (7, 71)\}$*.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 9, the system $\mathcal{A}$ is solvable in positive integers x_2, x_3, x_5, x_6. Since $\mathcal{A} \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leqslant g(6) = g(5)!$. Hence, $x_1! + 1 \leqslant g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

6 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [14, pp. 37–38]. Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

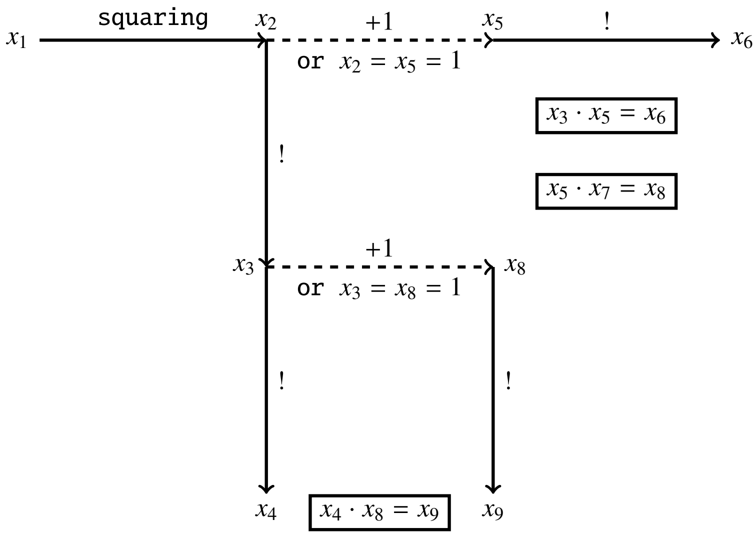

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 10. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 2, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 10 follows from Lemma 5. □

Lemma 11. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Theorem 6. *The statement* Ψ_9 *proves the following implication: if there exists an integer* $x_1 \geqslant 2$ *such that* $x_1^2 + 1$ *is prime and greater than* $g(7)$*, then there are infinitely many primes of the form* $n^2 + 1$.

Proof. Suppose that the antecedent holds. By Lemma 10, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geqslant g(7)$. Hence, $(x_1^2)! \geqslant g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 10 and 11, there are infinitely many primes of the form $n^2 + 1$. □

Corollary 4. *Let* $\mathcal{X}_9$ *denote the set of primes of the form* $n^2 + 1$*. The statement* Ψ_9 *implies that we know an algorithm such that it returns a threshold number of* $\mathcal{X}_9$*, and this number equals* $\max(\mathcal{X}_9)$*, if* $\mathcal{X}_9$ *is finite. Assuming the statement* Ψ_9*, a single query to an oracle for the halting problem decides the infinity of* $\mathcal{X}_9$*. Assuming the statement* Ψ_9*, the infinity of* $\mathcal{X}_9$ *is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_9 \cap [1, g(7)])$. □

7 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [3, p. 443].

Theorem 7. *(cf. Theorem 11). The statement* Ψ_9 *proves the following implication: if there exists an integer* $x_1 \geqslant g(6)$ *such that* $x_1! + 1$ *is prime, then there are infinitely many primes of the form* $n! + 1$.

Proof. We leave the analogous proof to the reader. □

8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [14, p. 39]. Let C denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{cases}$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system C.

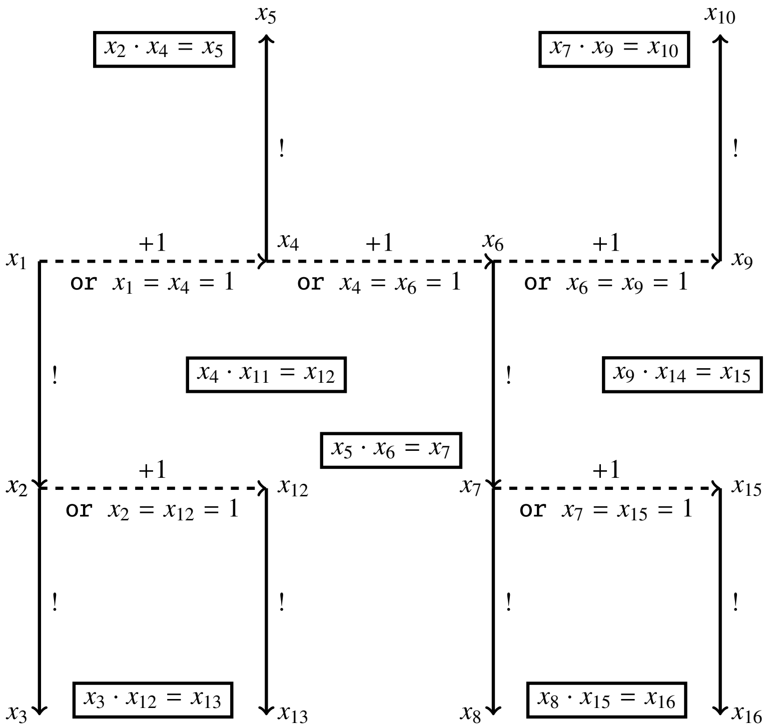

Fig. 4 Construction of the system C

Lemma 12. *For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$\begin{array}{rcl}
x_1 & = & x_4 - 1 \\
x_2 & = & (x_4 - 1)! \\
x_3 & = & ((x_4 - 1)!)! \\
x_5 & = & x_4! \\
x_6 & = & x_9 - 1 \\
x_7 & = & (x_9 - 1)! \\
x_8 & = & ((x_9 - 1)!)! \\
x_{10} & = & x_9! \\
x_{11} & = & \dfrac{(x_4 - 1)! + 1}{x_4} \\
x_{12} & = & (x_4 - 1)! + 1 \\
x_{13} & = & ((x_4 - 1)! + 1)! \\
x_{14} & = & \dfrac{(x_9 - 1)! + 1}{x_9} \\
x_{15} & = & (x_9 - 1)! + 1 \\
x_{16} & = & ((x_9 - 1)! + 1)!
\end{array}$$

Proof. By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 12 follows from Lemma 5. □

Lemma 13. *There are only finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system C and satisfy $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$.*

Proof. If a tuple $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system C and $(x_4 \in \{1,2\}) \vee (x_9 \in \{1,2\})$, then $x_1, \ldots, x_{16} \leqslant 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. □

Theorem 8. *The statement Ψ_{16} proves the following implication: if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.*

Proof. Suppose that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0,1,2\}$. By Lemma 12, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geqslant g(14)$. Therefore, $(x_9 - 1)! \geqslant g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement Ψ_{16} and the inequality $x_{16} > g(16)$ imply that the system C has infinitely many solutions in positive integers $x_1, \ldots, x_{16}$. According to Lemmas 12 and 13, there are infinitely many twin primes. □

Corollary 5. *(cf. [6]). Let $\mathcal{X}_{16}$ denote the set of twin primes. The statement Ψ_{16} implies that we know an algorithm such that it returns a threshold number of $\mathcal{X}_{16}$, and this number equals $\max(\mathcal{X}_{16})$, if $\mathcal{X}_{16}$ is finite. Assuming the statement Ψ_{16}, a single query to an oracle for the halting problem decides the infinity of $\mathcal{X}_{16}$. Assuming the statement Ψ_{16}, the infinity of $\mathcal{X}_{16}$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_{16} \cap [1, g(14)])$. □

9 Hypothetical statements $\Delta_5, \ldots, \Delta_{14}$ and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n+1) = \Gamma(\lambda(n))$ for every integer $n \geqslant 5$. For an integer $n \geqslant 5$, let $\mathcal{J}_n$ denote the following system of equations:

$$\left\{\begin{array}{rcl} \forall i \in \{1, \ldots, n-1\} \setminus \{3\}\ \Gamma(x_i) & = & x_{i+1} \\ x_1 \cdot x_1 & = & x_4 \\ x_2 \cdot x_3 & = & x_5 \end{array}\right.$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system $\mathcal{J}_n$.

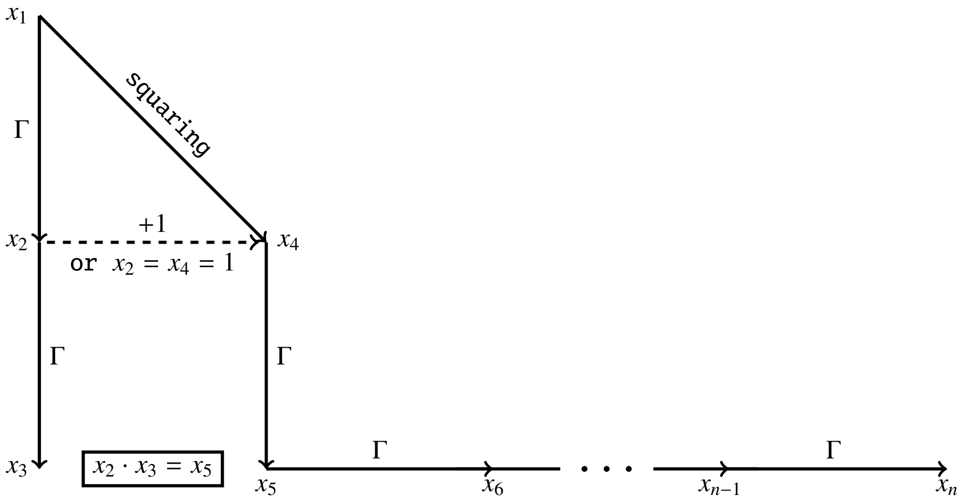

Fig. 5 Construction of the system $\mathcal{J}_n$

For every integer $n \geqslant 5$, the system $\mathcal{J}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$. For an integer $n \geqslant 5$, let Δ_n denote the following statement: *if a system of equations* $\mathcal{S} \subseteq \Big\{\Gamma(x_i) = x_k : i,k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i,j,k \in \{1, \ldots, n\}\Big\}$ *has only finitely many solutions in positive integers* $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant \lambda(n)$.

Hypothesis 2. *The statements* $\Delta_5, \ldots, \Delta_{14}$ *are true.*

Lemmas 3 and 5 imply that the statements Δ_n have similar consequences as the statements Ψ_n.

Theorem 9. *The statement* Δ_6 *implies that any prime number* $p \geqslant 25$ *proves the infinitude of primes.*

Proof. It follows from Lemmas 3 and 5. We leave the details to the reader. □

10 Hypothetical statements $\Sigma_3, \ldots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \ldots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : \; i,k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : \; i,j,k \in \{1, \ldots, n\}\}$$

For an integer $n \in \{3, \ldots, 16\}$, let P_n denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 &= x_1 \\ \Gamma_n(x_2) &= x_1 \\ \forall i \in \{2, \ldots, n-1\} \; x_i \cdot x_i &= x_{i+1} \end{cases}$$

Lemma 14. *For every integer* $n \in \{3, \ldots, 16\}$*,* $P_n \subseteq Q_n$ *and the system* P_n *with* Γ *instead of* Γ_n *has exactly one solution in positive integers* $x_1, \ldots, x_n$*, namely* $\Big(1, 2^{2^0}, \; 2^{2^1}, 2^{2^2}, \ldots, 2^{2^{n-2}}\Big)$.

For an integer $n \in \{3, \ldots, 16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq Q_n$ with Γ instead of Γ_n has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $\mathcal{S}$ satisfies $x_1, \ldots, x_n \leqslant 2^{2^{n-2}}$.

Hypothesis 3. *The statements* $\Sigma_3, \ldots, \Sigma_{16}$ *are true.*

Lemma 15. *(cf. Lemma 3). For every integer* $n \in \{4, \ldots, 16\}$ *and for every positive integers* x *and* y*,* $x \cdot \Gamma_n(x) = \Gamma_n(y)$ *if and only if* $(x + 1 = y) \wedge \Big(x \geqslant 2^{2^{n-3}} + 1\Big)$.

Let $\mathcal{Z}_9 \subseteq Q_9$ be the system of equations in Figure 6.

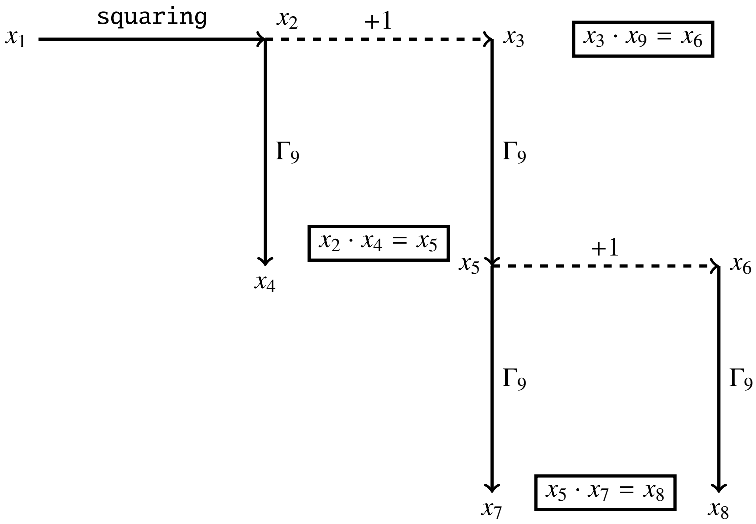

Fig. 6 Construction of the system $\mathcal{Z}_9$

Lemma 16. *For every positive integer x_1, the system $\mathcal{Z}_9$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers $x_2, \ldots, x_9$ are uniquely determined by x_1. For every positive integer n, at most finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with n and solve the system $\mathcal{Z}_9$ with Γ instead of Γ_9.*

Proof. It follows from Lemmas 3, 5, and 15. $\square$

Lemma 17. *([19]). The number $(13!)^2 + 1 = 38775788043632640001$ is prime.*

Lemma 18. $\left((13!)^2 \geqslant 2^{2^{9-3}} + 1 = 18446744073709551617\right) \wedge \left(\Gamma_9((13!)^2) > 2^{2^{9-2}}\right)$.

Theorem 10. *The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$.*

Proof. It follows from Lemmas 16–18. $\square$

Theorem 11. *(cf. Theorem 7). The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geqslant 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$.*

Proof. We leave the proof to the reader. $\square$

Corollary 6. *Let $\mathcal{Y}_9$ denote the set of primes of the form $n! + 1$. The statement Σ_9 implies that we know an algorithm such that it returns a threshold number of $\mathcal{Y}_9$, and this number equals $\max(\mathcal{Y}_9)$, if $\mathcal{Y}_9$ is finite. Assuming the statement Σ_9, a single query to an oracle for the halting problem decides the infinity of $\mathcal{Y}_9$. Assuming the statement Σ_9, the infinity of $\mathcal{Y}_9$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$. $\square$

Let $\mathcal{Z}_{14} \subseteq Q_{14}$ be the system of equations in Figure 7.

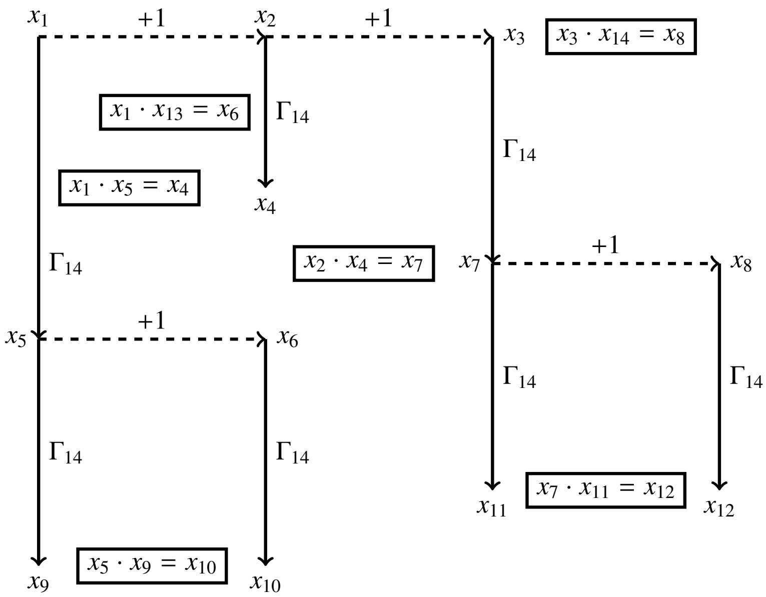

Fig. 7 Construction of the system $\mathcal{Z}_{14}$

Lemma 19. *For every positive integer x_1, the system $\mathcal{Z}_{14}$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if x_1 and $x_1 + 2$ are prime and $x_1 \geqslant 2^{2^{14-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{14}$ are uniquely determined by x_1. For every positive integer n, at most finitely many tuples $(x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with n and solve the system $\mathcal{Z}_{14}$ with Γ instead of Γ_{14}.*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 20. *([23, p. 87]). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).*

Lemma 21. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 12. *The statement Σ_{14} implies the infinitude of twin primes.*

Proof. It follows from Lemmas 19–21. □

A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [22]. It is conjectured that there are infinitely many Sophie Germain primes, see [17, p. 330]. Let $\mathcal{Z}_{16} \subseteq Q_{16}$ be the system of equations in Figure 8.

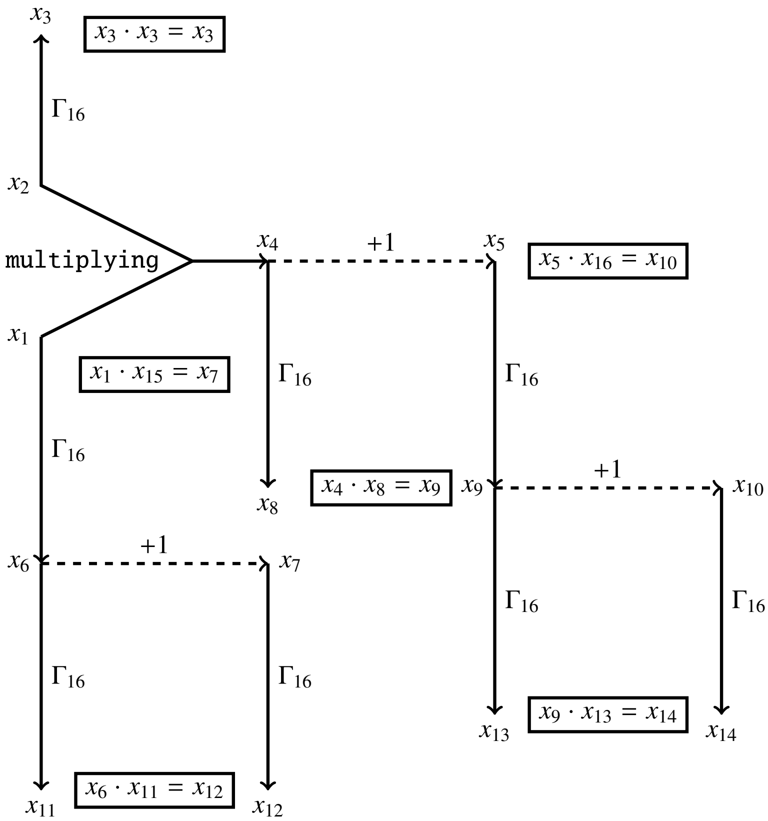

Fig. 8 Construction of the system $\mathcal{Z}_{16}$

Lemma 22. *For every positive integer x_1, the system $\mathcal{Z}_{16}$ is solvable in positive integers $x_2, \ldots, x_{16}$ if and only if x_1 is a Sophie Germain prime and $x_1 \geqslant 2^{2^{16-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{16}$ are uniquely determined by x_1. For every positive integer n, at most finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with n and solve the system $\mathcal{Z}_{16}$ with Γ instead of Γ_{16}.*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 23. *([17, p. 330]).* $8069496435 \cdot 10^{5072} - 1$ *is a Sophie Germain prime (Harvey Dubner).*

Lemma 24. $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$.

Theorem 13. *The statement Σ_{16} implies the infinitude of Sophie Germain primes.*

Proof. It follows from Lemmas 22–24. □

Theorem 14. *The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y, then each such solution (x, y) belongs to the set* $\{(1,2), (2,3)\}$.

Proof. We leave the proof to the reader. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdös, see [2]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [12].

Theorem 15. *The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y, then each such solution (x, y) belongs to the set* $\{(4,5), (5,11), (7,71)\}$.

Proof. We leave the proof to the reader. □

11 Hypothetical statements $\Omega_3, \ldots, \Omega_{16}$ and their consequences

For an integer $n \in \{3, \ldots, 16\}$, let Ω_n denote the following statement: *if a system of equations* $\mathcal{S} \subseteq \Big\{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\Big\}$ *has a solution in integers* $x_1, \ldots, x_n$ *greater than* $2^{2^{n-2}}$, *then* $\mathcal{S}$ *has infinitely many solutions in positive integers* $x_1, \ldots, x_n$. For every $n \in \{3, \ldots, 16\}$, the statement Σ_n implies the statement Ω_n.

Lemma 25. *The number* $(65!)^2 + 1$ *is prime and* $65! > 2^{2^{9-2}}$.

Proof. The following `PARI/GP` ([16]) command

```
(04:04) gp > isprime((65!)^2+1,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([23, p. 226]). It rigorously shows that the number $(65!)^2 + 1$ is prime. □

Lemma 26. *If positive integers* $x_1, \ldots, x_9$ *solve the system* $\mathcal{Z}_9$ *and* $x_1 > 2^{2^{9-2}}$, *then* $x_1 = \min(x_1, \ldots, x_9)$.

Theorem 16. *The statement* Ω_9 *implies the infinitude of primes of the form* $n^2 + 1$.

Proof. It follows from Lemmas 16 and 25–26. □

Lemma 27. *If positive integers* $x_1, \ldots, x_{14}$ *solve the system* $\mathcal{Z}_{14}$ *and* $x_1 > 2^{2^{14-2}}$, *then* $x_1 = \min(x_1, \ldots, x_{14})$.

Theorem 17. *The statement* Ω_{14} *implies the infinitude of twin primes.*

Proof. It follows from Lemmas 19–21 and 27. □

12 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [11, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [11, p. 1].

Open Problem. ([11, p. 159]). *Are there infinitely many composite numbers of the form* $2^{2^n} + 1$*?*
Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [10, p. 23]. Let

$$H_n = \Big\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\Big\} \cup \Big\{2^{2^{x_i}} = x_k : \ i, k \in \{1, \ldots, n\}\Big\}$$

Let $h(1) = 1$, and let $h(n + 1) = 2^{2^{h(n)}}$ for every positive integer n.

Lemma 28. *The following subsystem of* H_n

$$\begin{cases} x_1 \cdot x_1 & = & x_1 \\ \forall i \in \{1, \ldots, n-1\} \ 2^{2^{x_i}} & = & x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, *namely* $(h(1), \ldots, h(n))$.

For a positive integer n, let ξ_n denote the following statement: *if a system of equations $S \subseteq H_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant h(n)$.* The statement ξ_n says that for subsystems of H_n the largest known solution is indeed the largest possible.

Hypothesis 4. *The statements $\xi_1, \ldots, \xi_{13}$ are true.*

Proposition 5. *Every statement ξ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system H_n has a finite number of subsystems. □

Theorem 18. *The statement ξ_{13} proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers z.*

Proof. Let us consider the equation

$$(x+1)(y+1) = 2^{2^z} + 1 \tag{2}$$

in positive integers. By Lemma 4, we can transform equation (2) into an equivalent system of equations $\mathcal{G}$ which has 13 variables (x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 9.

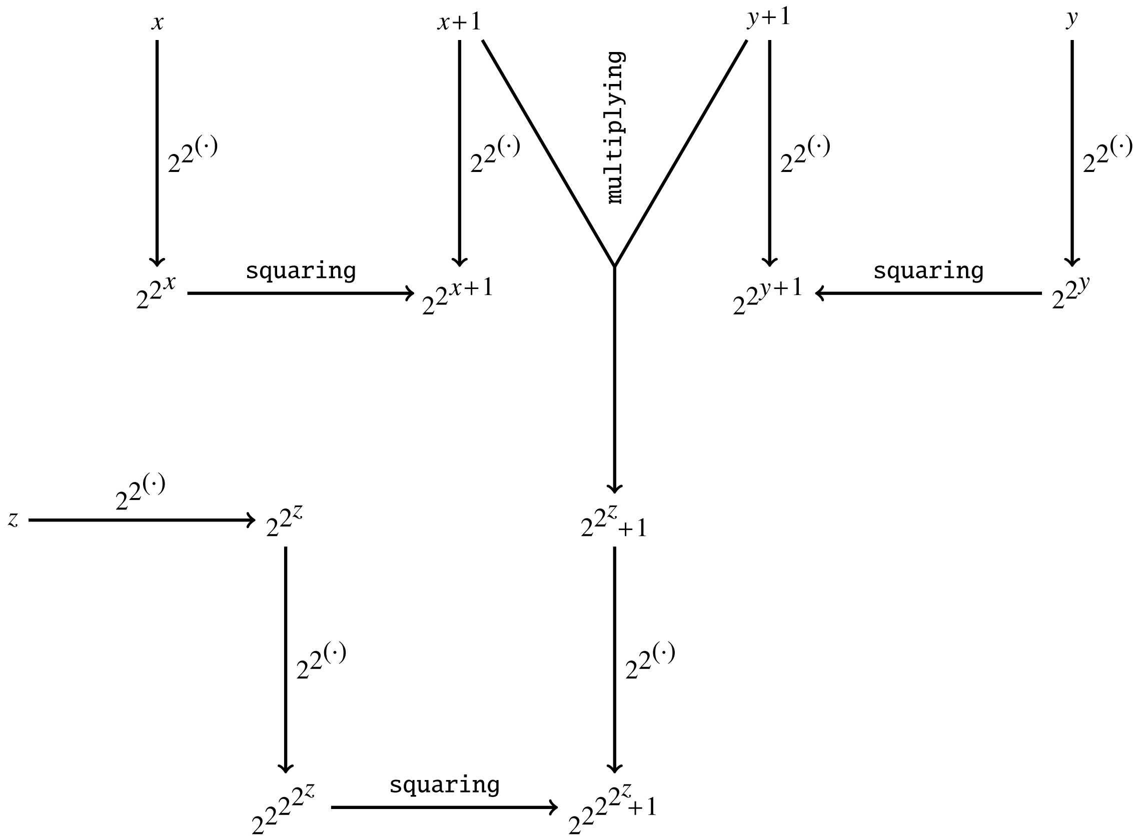

Fig. 9 Construction of the system $\mathcal{G}$

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z}+1}} > h(13)$. By this, the statement ξ_{13} implies that the system $\mathcal{G}$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

Corollary 7. *Let $\mathcal{W}_{13}$ denote the set of composite Fermat numbers. The statement ξ_{13} implies that we know an algorithm such that it returns a threshold number of $\mathcal{W}_{13}$, and this number equals* $\max(\mathcal{W}_{13})$*, if $\mathcal{W}_{13}$ is finite. Assuming the statement ξ_{13}, a single query to an oracle for the halting problem decides the infinity of $\mathcal{W}_{13}$. Assuming the statement ξ_{13}, the infinity of $\mathcal{W}_{13}$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{W}_{13} \cap [1, h(12)])$. □

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Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl