The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory ZFC as it refers to our current knowledge on $\mathcal{X}$

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Abstract. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite. Let $\beta = (((24!)!)!)!$, and let Φ denote the implication: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement Φ without invoking Landau's conjecture. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions `(1)`-`(4)`. `(1)` There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite. `(2)` No known algorithm decides the finiteness/infiniteness of $\mathcal{X}$. `(3)` There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. `(4)` There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(5)` There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ and some known definition of $\mathcal{X}$ is much simpler than every known definition of $\mathcal{X} \setminus (-\infty, n]$. The following problem is open: *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1)`-`(3)` *and* `(5)`*?* The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions `(1)`-`(3)`. The set $\mathcal{X} = \{k \in \mathbb{N}$: the number of digits of k belongs to $\mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions `(1)`-`(3)`. The statement Φ implies that both sets $\mathcal{X}$ satisfy condition `(5)`.

Key words and phrases: complexity of a mathematical definition, computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on $\mathcal{X}$, explicitly known integer n bounds $\mathcal{X}$ from above when $\mathcal{X}$ is finite, infiniteness of $\mathcal{X}$ remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, large number of elements of $\mathcal{X}$, mathematical statement that cannot be formalized in the set theory ZFC, no known algorithm decides the finiteness/infiniteness of $\mathcal{X}$, physical impossibility of machine computations on sufficiently large integers.

1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

Definition 1. *Conditions* (1)-(5) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1) *There are a large number of elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of* $\mathcal{X}$.

(3) *There is a known algorithm that for every* $n \in \mathbb{N}$ *decides whether or not* $n \in \mathcal{X}$.

(4) *There is an explicitly known integer* n *such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) *There is an explicitly known integer* n *such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ *and some known definition of* $\mathcal{X}$ *is much simpler than every known definition of* $\mathcal{X} \setminus (-\infty, n]$.

Definition 2. *We say that an integer* n *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$, *if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$, *cf.* [8] *and* [9].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [5] and [6].

Definition 3. *Let* Φ *denote the implication:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement Φ. In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *There is no explicitly known threshold number of* $\mathcal{P}_{n^2+1}$. *It means that there is no explicitly known integer* k *such that* $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.

Proving the statement Φ will falsify Statement 1. Statement 1 cannot be formalized in the set theory *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2 and 3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$.

2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

Definition 4. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Statement 2. *The set* $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ *satisfies conditions* `(1)-(4)`.

Proof. Condition `(1)` holds as $\mathcal{X} \supseteq \{0, \dots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β, see [2]. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

In Statement 2,

$$\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, \beta]$$

and the sets

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

and

$$\mathcal{X} \setminus (-\infty, \beta] = \{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

have definitions of similar complexity. The following problem arises:

Open Problem 1. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1)-(3)` *and* `(5)`*?*

3. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \dots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

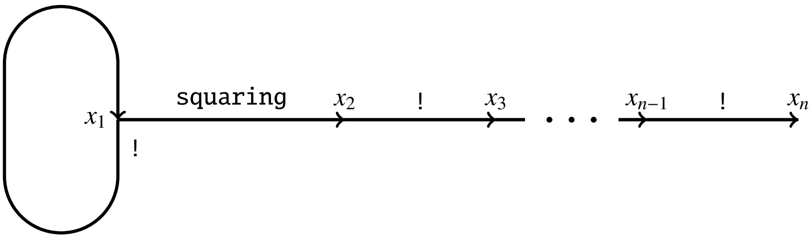

Fig. 1 Construction of the system $\mathcal{U}_n$

Lemma 2. *For every positive integer* n*, the system* $\mathcal{U}_n$ *has exactly two solutions in positive integers, namely* $(1, \dots, 1)$ *and* $\big(f(1), \dots, f(n)\big)$.

Let

$$B_n = \left\{x_i! = x_k : \ i,k \in \{1,\dots,n\}\right\} \cup \left\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\dots,n\}\right\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\dots,x_n$, then each such solution $(x_1,\dots,x_n)$ satisfies $x_1,\dots,x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement Ψ_9, cf. Conjecture 1 in Section 4.

Theorem 1. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 2. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\dots,x_n$, then for every integer $i \in \{1,\dots,n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1,\dots,x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 3. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 4. *(Wilson's theorem,* [1, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &=& x_3 \\ x_3! &=& x_4 \\ x_5! &=& x_6 \\ x_8! &=& x_9 \\ x_1 \cdot x_1 &=& x_2 \\ x_3 \cdot x_5 &=& x_6 \\ x_4 \cdot x_8 &=& x_9 \\ x_5 \cdot x_7 &=& x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

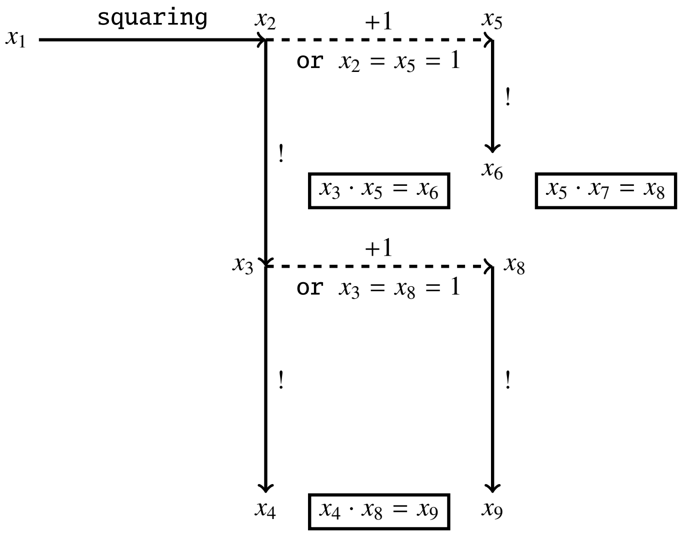

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 5. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{array}{rcl} x_2 & = & x_1^2 \\ x_3 & = & (x_1^2)! \\ x_4 & = & ((x_1^2)!)! \\ x_5 & = & x_1^2 + 1 \\ x_6 & = & (x_1^2 + 1)! \\ x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 & = & (x_1^2)! + 1 \\ x_9 & = & ((x_1^2)! + 1)! \end{array}$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Lemma 6. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system $\mathcal{A}$.*

Theorem 4. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N}\setminus\{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 5. *Conjecture 1 implies the statement* Φ.

Proof. It follows from Theorem 4 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 6. *The statement* Φ *implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Definition 5. *Let* $\mathcal{K} = \{k \in \mathbb{N} :$ *the number of digits of* k *belongs to* $\mathcal{P}_{n^2+1}\}$.

Lemma 7. $\operatorname{card}(\mathcal{K}) \geqslant 9 \cdot 10^{9 \cdot 4^{747}} \approx 10^{10^{450.6930560314272}}$.

Proof. The following PARI/GP ([4]) command

```
isprime(1+9*4^747,{flag=2})
```

returns $\%1 = 1$. This command performs the APRCL primality test, the best deterministic primality test algorithm ([7, p. 226]). It rigorously shows that the number $\left(3 \cdot 2^{747}\right)^2 + 1$ is prime. Since $9 \cdot 10^{9 \cdot 4^{747}}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10\hat{}(9 * 4\hat{}747))$. □

Statement 3. *The sets* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *and* $\mathcal{X} = \mathcal{K}$ *satisfy conditions* (1)-(3). *The statement* Φ *implies that both sets* $\mathcal{X}$ *satisfy condition* (5).

Proof. Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, Lemma 7 implies condition (1) for both sets $\mathcal{X}$. Condition (3) holds trivially for both sets $\mathcal{X}$. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds for both sets $\mathcal{X}$. Suppose that the statement Φ holds. This implies two facts:

$$\beta \text{ is a threshold number of } \mathcal{X} = \mathcal{P}_{n^2+1} \tag{6}$$

and

$$\underbrace{9 \ldots 9}_{\beta \text{ digits}} \text{ is a threshold number of } \mathcal{X} = \mathcal{K} \tag{7}$$

Thus condition (4) holds for both sets $\mathcal{X}$. The definition of $\mathcal{P}_{n^2+1}$ is much simpler than the definition of $\mathcal{P}_{n^2+1} \setminus (-\infty, \beta]$. The definition of $\mathcal{K}$ is much simpler than the definition of $\mathcal{K} \setminus (-\infty, \underbrace{9 \ldots 9}_{\beta \text{ digits}}]$. The last three sentences imply that condition (5) holds for both sets $\mathcal{X}$. □

Acknowledgment. Sławomir Kurpaska prepared two diagrams in *TikZ*. Apoloniusz Tyszka wrote the article.

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On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\{x \in \mathbb{N}\colon \varphi(x)\} \subseteq \{x \in \mathbb{N}\colon x \leqslant n-1\}$ if the set $\{x \in \mathbb{N}\colon \varphi(x)\}$ is finite

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Abstract

Let $\Gamma(k)$ denote $(k-1)!$, and let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3,\ldots,16\}$ and $k \in \{2\} \cup [2^{2^{n-3}}+1,\infty) \cap \mathbb{N}$. For an integer $n \in \{3,\ldots,16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq \{\Gamma_n(x_i)=x_k:\ i,k \in \{1,\ldots,n\}\} \cup \{x_i \cdot x_j = x_k:\ i,j,k \in \{1,\ldots,n\}\}$ with Γ instead of Γ_n has only finitely many solutions in positive integers $x_1,\ldots,x_n$, then every tuple $(x_1,\ldots,x_n) \in (\mathbb{N}\setminus\{0\})^n$ that solves the original system $\mathcal{S}$ satisfies $x_1,\ldots,x_n \leqslant 2^{2^{n-2}}$. Our hypothesis claims that the statements $\Sigma_3,\ldots,\Sigma_{16}$ are true. The statement Σ_6 proves the following implication: if the equation $x(x+1)=y!$ has only finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(1,2),(2,3)\}$. The statement Σ_6 proves the following implication: if the equation $x!+1=y^2$ has only finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(4,5),(5,11),(7,71)\}$. The statement Σ_9 implies the infinitude of primes of the form n^2+1. The statement Σ_9 implies that any prime of the form $n!+1$ with $n \geqslant 2^{2^{9-3}}$ proves the infinitude of primes of the form $n!+1$. The statement Σ_{14} implies the infinitude of twin primes. The statement Σ_{16} implies the infinitude of Sophie Germain primes.

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation $x!+1=y^2$, composite Fermat numbers, decidability in the limit, Erdös' equation $x(x+1)=y!$, finiteness of a set, infiniteness of a set, prime numbers of the form n^2+1, prime numbers of the form $n!+1$, single query to an oracle for the halting problem, Sophie Germain primes, twin primes.

2010 Mathematics Subject Classification: 03B30, 11A41.

1 Introduction and basic lemmas

The phrase "*we know a non-negative integer n*" in the title means that we know an algorithm which returns n. The title of the article cannot be formalised in ZFC because the phrase "*we know a non-negative integer n*" refers to currently known non-negative integers n with some property. A formally stated title may look like this: *On ZFC-formulae $\varphi(x)$ for which there exists a non-negative integer n such that ZFC proves that*

$$\mathrm{card}(\{x \in \mathbb{N}\colon \varphi(x)\}) < \infty \Longrightarrow \{x \in \mathbb{N}\colon \varphi(x)\} \subseteq \{x \in \mathbb{N}\colon x \leqslant n-1\}$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer n such that ZFC proves the above implication.

Lemma 1. *For every non-negative integer n,* $\mathrm{card}(\{x \in \mathbb{N}\colon x \leqslant n-1\}) = n$.

Corollary 1. *The title altered to* "On ZFC-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\mathrm{card}(\{x \in \mathbb{N}\colon \varphi(x)\}) \leqslant n$ if the set $\{x \in \mathbb{N}\colon \varphi(x)\}$ is finite" *involves a weaker assumption on $\varphi(x)$.*

Lemma 2. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Let $\Gamma(k)$ denote $(k-1)!$.

Lemma 3. *For every positive integers x and y, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 4. *For every non-negative integers b and c, $b+1=c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Lemma 5. *(Wilson's theorem, [8, p. 89]). For every positive integer x, x divides $(x-1)!+1$ if and only if $x=1$ or x is prime.*

2 Subsets of $\mathbb{N}$ and their threshold numbers

We say that a non-negative integer m is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than m, cf. [24] and [25]. If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer m is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $\{\max(\mathcal{X}), \max(\mathcal{X})+1, \max(\mathcal{X})+2, \ldots\}$.

It is conjectured that the set of prime numbers of the form n^2+1 is infinite, see [14, pp. 37–38]. It is conjectured that the set of prime numbers of the form $n!+1$ is infinite, see [3, p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [14, p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^n}+1$ is infinite, see [10, p. 23] and [11, pp. 158–159]. A prime p is said to be a Sophie Germain prime if both p and $2p+1$ are prime, see [22]. It is conjectured that the set of Sophie Germain primes is infinite, see [17, p. 330]. For each of these sets, we do not know any threshold number.

The following statement:

$$\text{for every non-negative integer } n \text{ there exist}$$
$$\text{prime numbers } p \text{ and } q \text{ such that } p+2=q \text{ and } p \in \left[10^n, 10^{n+1}\right] \tag{1}$$

is a Π_1 statement which strengthens the twin prime conjecture, see [4, p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger Π_1 statements, see [1]. Statement (1) is equivalent to the non-halting of a Turing machine. If a set $\mathcal{X} \subseteq \mathbb{N}$ is computable and we know a threshold number of $\mathcal{X}$, then the infinity of $\mathcal{X}$ is equivalent to the halting of a Turing machine.

The height of a rational number $\frac{p}{q}$ is denoted by $H\left(\frac{p}{q}\right)$ and equals $\max(|p|,|q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple $(x_1, \ldots, x_n)$ is denoted by $H(x_1, \ldots, x_n)$ and equals $\max(H(x_1), \ldots, H(x_n))$.

Lemma 6. *The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [13, p. 212]. The known rational solutions are $(x,y) = (-1,0)$, $(-1,1)$, $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(2,-5)$, $(2,6)$, $(3,-15)$, $(3,16)$, $(30,-4929)$, $(30,4930)$, $\left(\frac{1}{4}, \frac{15}{32}\right)$, $\left(\frac{1}{4}, \frac{17}{32}\right)$, $\left(-\frac{15}{16}, -\frac{185}{1024}\right)$, $\left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [18, pp. 223–224].*

Corollary 2. *The set $\mathcal{T} = \{n \in \mathbb{N} :$ the equation $x^5 - x = y^2 - y$ has a rational solution of height $n\}$ is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of $\mathcal{T}$.*

Let $\mathcal{L}$ denote the following system of equations:

$$\begin{cases} x^2 + y^2 &= s^2 \\ x^2 + z^2 &= t^2 \\ y^2 + z^2 &= u^2 \\ x^2 + y^2 + z^2 &= v^2 \end{cases}$$

Let

$$\mathcal{F} = \Big\{n \in \mathbb{N} \setminus \{0\} : \Big(\text{the system } \mathcal{L} \text{ has no solutions in } \{1, \ldots, n\}^7\Big) \wedge \Big(\text{the system } \mathcal{L} \text{ has a solution in } \{1, \ldots, n+1\}^7\Big)\Big\}$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

Lemma 7. *([21]). No perfect cuboids are known.*

Corollary 3. *We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{F}$. ZFC proves that* $\mathrm{card}(\mathcal{F}) \in \{0, 1\}$. *We do not know any algorithm which returns* $\mathrm{card}(\mathcal{F})$. *We do not know any algorithm which returns a threshold number of $\mathcal{F}$.*

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, \text{ if } \sin\left(9^{9^{9^{9^9}}}\right) < 0 \\ \mathbb{N} \cap \left[0, \ \sin\left(9^{9^{9^{9^9}}}\right) \cdot 9^{9^{9^{9^9}}}\right] \text{ otherwise} \end{cases}$$

We do not know whether or not the set $\mathcal{H}$ is finite.

Proposition 1. *The number $9^{9^{9^{9^9}}}$ is a threshold number of $\mathcal{H}$. We know an algorithm which decides the equality $\mathcal{H} = \mathbb{N}$. If $\mathcal{H} \neq \mathbb{N}$, then the set $\mathcal{H}$ consists of all integers from* 0 *to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{H}$.*

Let

$$\mathcal{K} = \begin{cases} \{n\}, \text{ if } (n \in \mathbb{N}) \wedge \left(2^{\aleph_0} = \aleph_{n+1}\right) \\ \{0\}, \text{ if } 2^{\aleph_0} \geqslant \aleph_\omega \end{cases}$$

Proposition 2. *ZFC proves that* $\mathrm{card}(\mathcal{K}) = 1$. *If ZFC is consistent, then for every $n \in \mathbb{N}$ the sentences* "n is a threshold number of $\mathcal{K}$" *and* "n is not a threshold number of $\mathcal{K}$" *are not provable in ZFC.*

Proof. It suffices to observe that $2^{\aleph_0}$ can attain every value from the set $\{\aleph_1, \aleph_2, \aleph_3, \ldots\}$, see [7] and [9, p. 232]. □

3 A Diophantine equation whose non-solvability expresses the consistency of *ZFC*

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

Theorem 1. *([5, p. 35]). There exists a polynomial $D(x_1, \ldots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences* "The equation $D(x_1, \ldots, x_m) = 0$ is solvable in non-negative integers" *and* "The equation $D(x_1, \ldots, x_m) = 0$ is not solvable in non-negative integers" *are not provable in ZFC.*

Let $\mathcal{Y}$ denote the set of all non-negative integers k such that the equation $D(x_1, \ldots, x_m) = 0$ has no solutions in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Theorem 1 implies the next theorem.

Theorem 2. *For every $n \in \mathbb{N}$, ZFC proves that $n \in \mathcal{Y}$. If ZFC is arithmetically consistent, then the sentences* "$\mathcal{Y}$ is finite" *and* "$\mathcal{Y}$ is infinite" *are not provable in ZFC. If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences* "n is a threshold number of $\mathcal{Y}$" *and* "n is not a threshold number of $\mathcal{Y}$" *are not provable in ZFC.*

Let $\mathcal{E}$ denote the set of all non-negative integers k such that the equation $D(x_1, \ldots, x_m) = 0$ has a solution in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{E}$. Theorem 1 implies the next theorem.

Theorem 3. *The set $\mathcal{E}$ is empty or infinite. In both cases, every non-negative integer n is a threshold number of $\mathcal{E}$. If ZFC is arithmetically consistent, then the sentences* "$\mathcal{E}$ is empty", "$\mathcal{E}$ is not empty", "$\mathcal{E}$ is finite", *and* "$\mathcal{E}$ is infinite" *are not provable in ZFC.*

Let

$$\mathcal{V} = \Big\{n \in \mathbb{N} : \Big(\text{the polynomial } D(x_1, \ldots, x_m) \text{ has no solutions in } \{0, \ldots, n\}^m\Big) \wedge$$
$$\Big(\text{the polynomial } D(x_1, \ldots, x_m) \text{ has a solution in } \{0, \ldots, n+1\}^m\Big)\Big\}$$

Since the sets $\{0, \ldots, n\}^m$ and $\{0, \ldots, n+1\}^m$ are finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{V}$. Theorem 1 implies the next theorem.

Theorem 4. *ZFC proves that* $\mathrm{card}(\mathcal{V}) \in \{0, 1\}$. *For every $n \in \mathbb{N}$, ZFC proves that $n \notin \mathcal{V}$. ZFC does not prove the emptiness of $\mathcal{V}$, if ZFC is arithmetically consistent. For every $n \in \mathbb{N}$, the sentence* "n is a threshold number of $\mathcal{V}$" *is not provable in ZFC, if ZFC is arithmetically consistent.*

4 Hypothetical statements $\Psi_3, \ldots, \Psi_{16}$

For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{2\}\ x_i! &= x_{i+1} \\ x_1 \cdot x_2 &= x_3 \\ x_2 \cdot x_2 &= x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

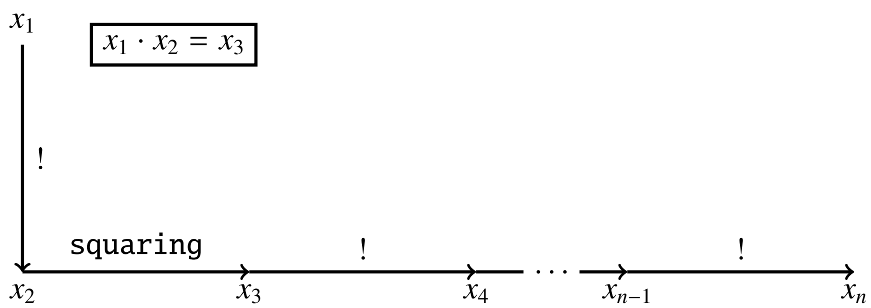

Fig. 1 Construction of the system $\mathcal{U}_n$

Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geqslant 3$.

Lemma 8. *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\big(2, 2, g(3), \ldots, g(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k : \ (i, k \in \{1, \ldots, n\}) \wedge (i \neq k)\Big\} \cup \Big\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\Big\}$$

For an integer $n \geqslant 3$, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant g(n)$.* The statement Ψ_n says that for subsystems of B_n the largest known solution is indeed the largest possible.

Hypothesis 1. *The statements* $\Psi_3, \ldots, \Psi_{16}$ *are true.*

Proposition 3. *Every statement* Ψ_n *is true with an unknown integer bound that depends on* n*.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

Proposition 4. *For every statement* Ψ_n*, the bound* $g(n)$ *cannot be decreased.*

Proof. It follows from Lemma 8 because $\mathcal{U}_n \subseteq B_n$. □

5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! & = & x_2 \\ x_2! & = & x_3 \\ x_5! & = & x_6 \\ x_4 \cdot x_4 & = & x_5 \\ x_3 \cdot x_5 & = & x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

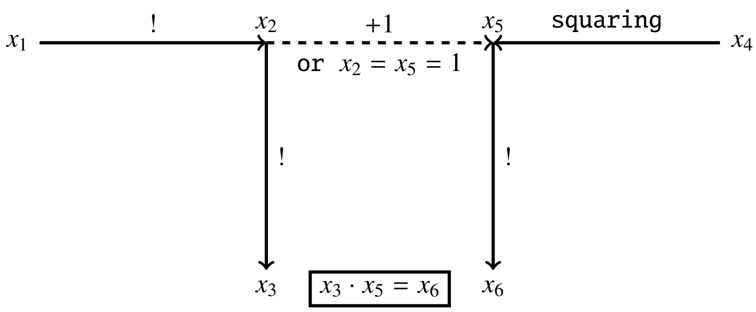

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 9. *For every* $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$*, the system* $\mathcal{A}$ *is solvable in positive integers* x_2*,* x_3*,* x_5*,* x_6 *if and only if* $x_1! + 1 = x_4^2$*. In this case, the integers* x_2, x_3, x_5, x_6 *are uniquely determined by the following equalities:*

$$\begin{array}{rcl} x_2 & = & x_1! \\ x_3 & = & (x_1!)! \\ x_5 & = & x_1! + 1 \\ x_6 & = & (x_1! + 1)! \end{array}$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [15].

Theorem 5. *If the equation* $x_1! + 1 = x_4^2$ *has only finitely many solutions in positive integers, then the statement* Ψ_6 *guarantees that each such solution* (x_1, x_4) *belongs to the set* $\{(4, 5), (5, 11), (7, 71)\}$*.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 9, the system $\mathcal{A}$ is solvable in positive integers x_2, x_3, x_5, x_6. Since $\mathcal{A} \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leqslant g(6) = g(5)!$. Hence, $x_1! + 1 \leqslant g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

6 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [14, pp. 37–38]. Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

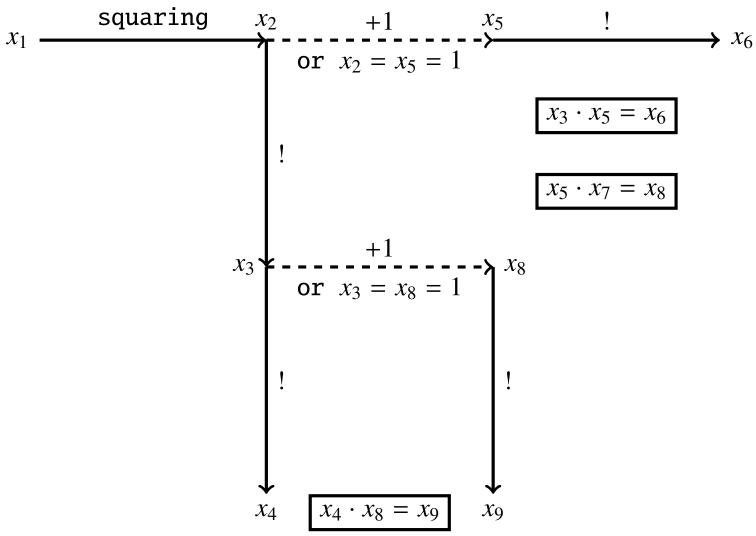

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 10. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 2, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 10 follows from Lemma 5. $\square$

Lemma 11. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. $\square$

Theorem 6. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Suppose that the antecedent holds. By Lemma 10, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geqslant g(7)$. Hence, $(x_1^2)! \geqslant g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 10 and 11, there are infinitely many primes of the form $n^2 + 1$. □

Corollary 4. *Let $\mathcal{X}_9$ denote the set of primes of the form $n^2 + 1$. The statement Ψ_9 implies that we know an algorithm such that it returns a threshold number of $\mathcal{X}_9$, and this number equals* $\max(\mathcal{X}_9)$, *if $\mathcal{X}_9$ is finite. Assuming the statement Ψ_9, a single query to an oracle for the halting problem decides the infinity of $\mathcal{X}_9$. Assuming the statement Ψ_9, the infinity of $\mathcal{X}_9$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_9 \cap [1, g(7)])$. □

7 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [3, p. 443].

Theorem 7. *(cf. Theorem 11). The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geqslant g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. We leave the analogous proof to the reader. □

8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [14, p. 39]. Let C denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{cases}$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system C.

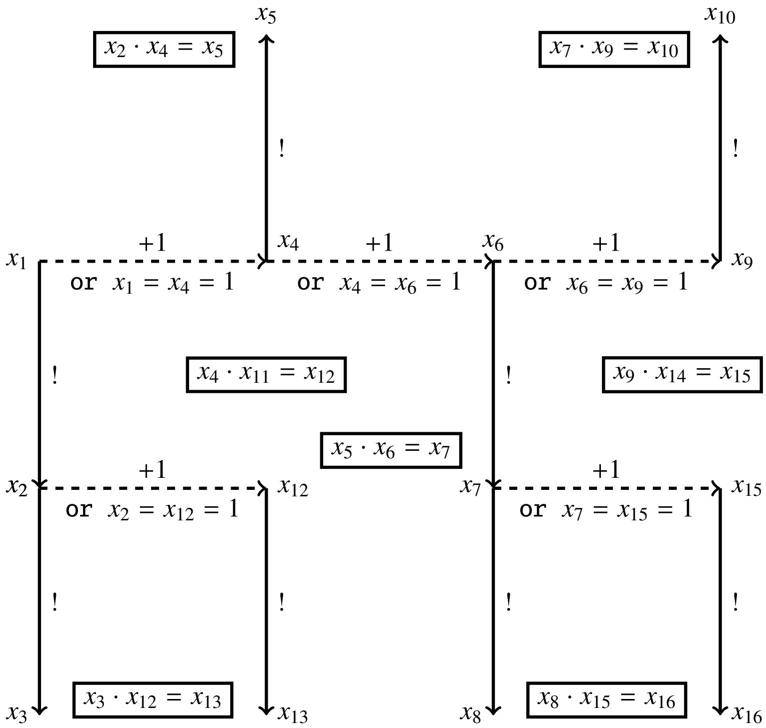

Fig. 4 Construction of the system C

Lemma 12. *For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$\begin{array}{rcl}
x_1 & = & x_4 - 1 \\
x_2 & = & (x_4 - 1)! \\
x_3 & = & ((x_4 - 1)!)! \\
x_5 & = & x_4! \\
x_6 & = & x_9 - 1 \\
x_7 & = & (x_9 - 1)! \\
x_8 & = & ((x_9 - 1)!)! \\
x_{10} & = & x_9! \\
x_{11} & = & \dfrac{(x_4 - 1)! + 1}{x_4} \\
x_{12} & = & (x_4 - 1)! + 1 \\
x_{13} & = & ((x_4 - 1)! + 1)! \\
x_{14} & = & \dfrac{(x_9 - 1)! + 1}{x_9} \\
x_{15} & = & (x_9 - 1)! + 1 \\
x_{16} & = & ((x_9 - 1)! + 1)!
\end{array}$$

Proof. By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 12 follows from Lemma 5. □

Lemma 13. *There are only finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system C and satisfy $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$.*

Proof. If a tuple $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system C and $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$, then $x_1, \dots, x_{16} \leqslant 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. □

Theorem 8. *The statement Ψ_{16} proves the following implication: if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.*

Proof. Suppose that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma 12, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$ such that the tuple $(x_1, \dots, x_{16})$ solves the system C. Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geqslant g(14)$. Therefore, $(x_9 - 1)! \geqslant g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement Ψ_{16} and the inequality $x_{16} > g(16)$ imply that the system C has infinitely many solutions in positive integers $x_1, \dots, x_{16}$. According to Lemmas 12 and 13, there are infinitely many twin primes. □

Corollary 5. *(cf. [6]). Let $\mathcal{X}_{16}$ denote the set of twin primes. The statement Ψ_{16} implies that we know an algorithm such that it returns a threshold number of $\mathcal{X}_{16}$, and this number equals $\max(\mathcal{X}_{16})$, if $\mathcal{X}_{16}$ is finite. Assuming the statement Ψ_{16}, a single query to an oracle for the halting problem decides the infinity of $\mathcal{X}_{16}$. Assuming the statement Ψ_{16}, the infinity of $\mathcal{X}_{16}$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_{16} \cap [1, g(14)])$. □

9 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n + 1) = \Gamma(\lambda(n))$ for every integer $n \geqslant 5$. For an integer $n \geqslant 5$, let $\mathcal{J}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{3\} \; \Gamma(x_i) & = & x_{i+1} \\ x_1 \cdot x_1 & = & x_4 \\ x_2 \cdot x_3 & = & x_5 \end{cases}$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system $\mathcal{J}_n$.

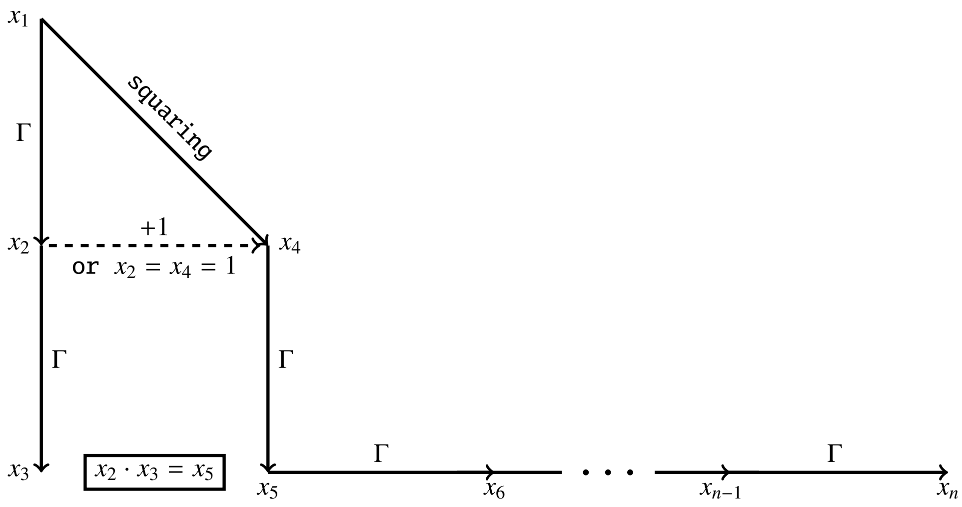

Fig. 5 Construction of the system $\mathcal{J}_n$

For every integer $n \geqslant 5$, the system $\mathcal{J}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$. For an integer $n \geqslant 5$, let Δ_n denote the following statement: *if a system of equations* $\mathcal{S} \subseteq \Big\{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\Big\}$ *has only finitely many solutions in positive integers* $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant \lambda(n)$.

Hypothesis 2. *The statements* $\Delta_5, \ldots, \Delta_{14}$ *are true.*

Lemmas 3 and 5 imply that the statements Δ_n have similar consequences as the statements Ψ_n.

Theorem 9. *The statement* Δ_6 *implies that any prime number* $p \geqslant 25$ *proves the infinitude of primes.*

Proof. It follows from Lemmas 3 and 5. We leave the details to the reader. □

10 Hypothetical statements $\Sigma_3, \ldots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \ldots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : \ i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\}$$

For an integer $n \in \{3, \ldots, 16\}$, let P_n denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 &= x_1 \\ \Gamma_n(x_2) &= x_1 \\ \forall i \in \{2, \ldots, n-1\} \ x_i \cdot x_i &= x_{i+1} \end{cases}$$

Lemma 14. *For every integer* $n \in \{3, \ldots, 16\}$*,* $P_n \subseteq Q_n$ *and the system* P_n *with* Γ *instead of* Γ_n *has exactly one solution in positive integers* $x_1, \ldots, x_n$*, namely* $\left(1, 2^{2^0}, \ 2^{2^1}, 2^{2^2}, \ldots, 2^{2^{n-2}}\right)$.

For an integer $n \in \{3, \ldots, 16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq Q_n$ with Γ instead of Γ_n has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $\mathcal{S}$ satisfies $x_1, \ldots, x_n \leqslant 2^{2^{n-2}}$.

Hypothesis 3. *The statements* $\Sigma_3, \ldots, \Sigma_{16}$ *are true.*

Lemma 15. *(cf. Lemma 3). For every integer* $n \in \{4, \ldots, 16\}$ *and for every positive integers* x *and* y*,* $x \cdot \Gamma_n(x) = \Gamma_n(y)$ *if and only if* $(x + 1 = y) \wedge \left(x \geqslant 2^{2^{n-3}} + 1\right)$.

Let $\mathcal{Z}_9 \subseteq Q_9$ be the system of equations in Figure 6.

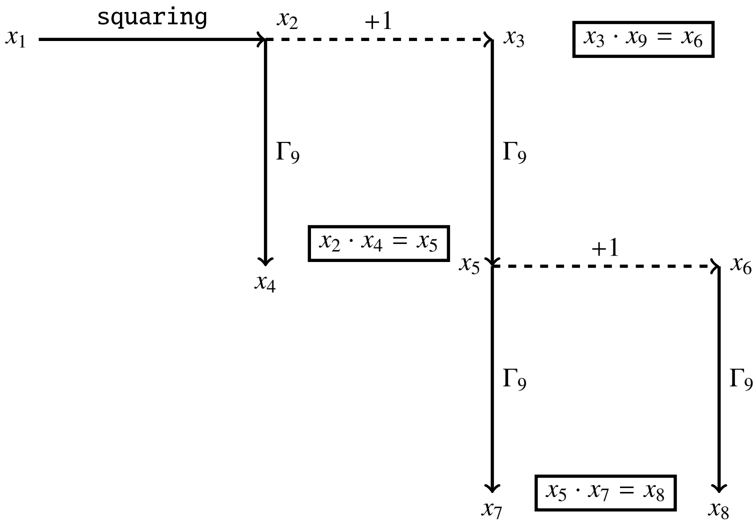

Fig. 6 Construction of the system $\mathcal{Z}_9$

Lemma 16. *For every positive integer x_1, the system $\mathcal{Z}_9$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers $x_2, \ldots, x_9$ are uniquely determined by x_1. For every positive integer n, at most finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with n and solve the system $\mathcal{Z}_9$ with Γ instead of Γ_9.*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 17. *([19]). The number $(13!)^2 + 1 = 38775788043632640001$ is prime.*

Lemma 18. $\left((13!)^2 \geqslant 2^{2^{9-3}} + 1 = 18446744073709551617\right) \wedge \left(\Gamma_9((13!)^2) > 2^{2^{9-2}}\right)$.

Theorem 10. *The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$.*

Proof. It follows from Lemmas 16–18. □

Theorem 11. *(cf. Theorem 7). The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geqslant 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$.*

Proof. We leave the proof to the reader. □

Corollary 6. *Let $\mathcal{Y}_9$ denote the set of primes of the form $n! + 1$. The statement Σ_9 implies that we know an algorithm such that it returns a threshold number of $\mathcal{Y}_9$, and this number equals $\max(\mathcal{Y}_9)$, if $\mathcal{Y}_9$ is finite. Assuming the statement Σ_9, a single query to an oracle for the halting problem decides the infinity of $\mathcal{Y}_9$. Assuming the statement Σ_9, the infinity of $\mathcal{Y}_9$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$. □

Let $\mathcal{Z}_{14} \subseteq Q_{14}$ be the system of equations in Figure 7.

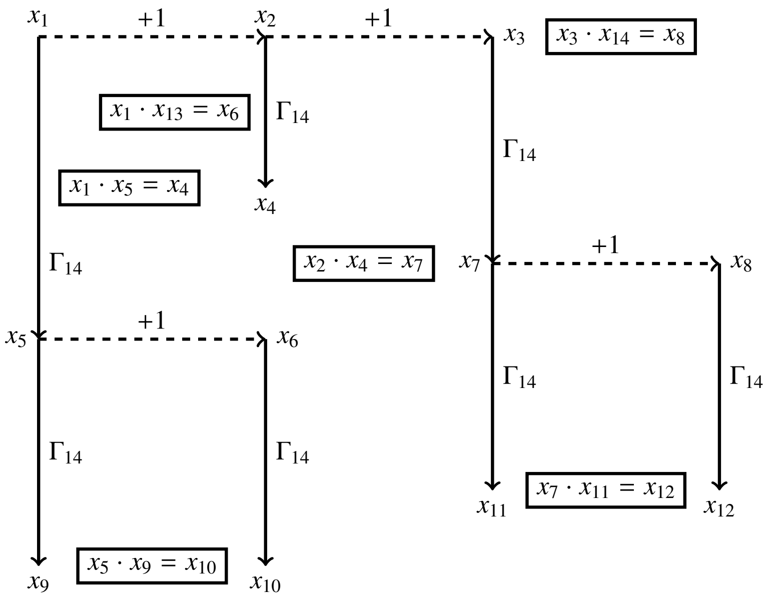

Fig. 7 Construction of the system $\mathcal{Z}_{14}$

Lemma 19. *For every positive integer x_1, the system $\mathcal{Z}_{14}$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if x_1 and $x_1 + 2$ are prime and $x_1 \geqslant 2^{2^{14-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{14}$ are uniquely determined by x_1. For every positive integer n, at most finitely many tuples $(x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with n and solve the system $\mathcal{Z}_{14}$ with Γ instead of Γ_{14}.*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 20. *([23, p. 87]). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).*

Lemma 21. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 12. *The statement Σ_{14} implies the infinitude of twin primes.*

Proof. It follows from Lemmas 19–21. □

A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [22]. It is conjectured that there are infinitely many Sophie Germain primes, see [17, p. 330]. Let $\mathcal{Z}_{16} \subseteq Q_{16}$ be the system of equations in Figure 8.

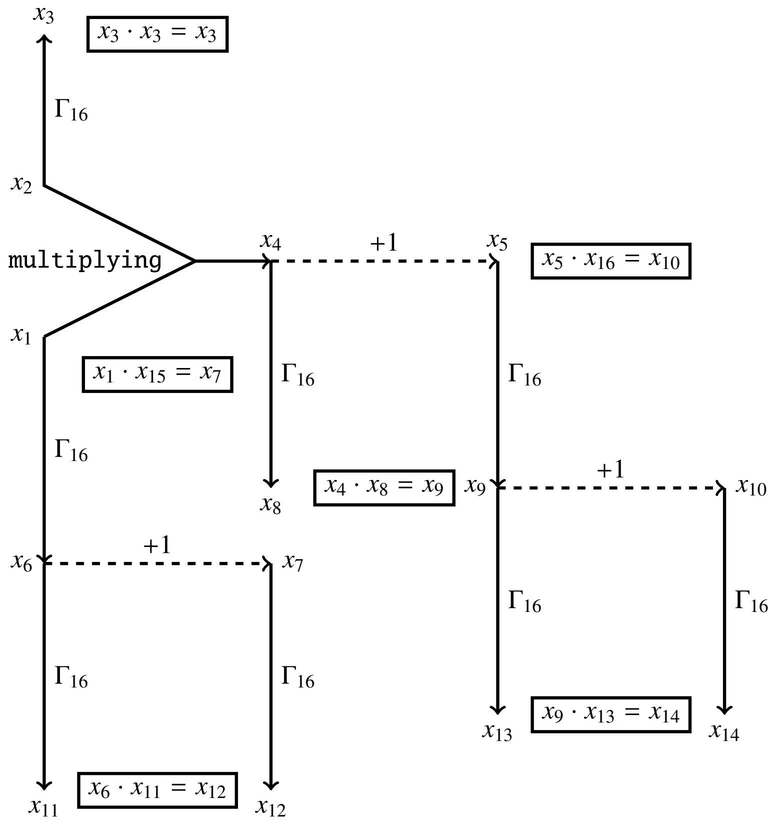

Fig. 8 Construction of the system $\mathcal{Z}_{16}$

Lemma 22. *For every positive integer x_1, the system $\mathcal{Z}_{16}$ is solvable in positive integers $x_2, \ldots, x_{16}$ if and only if x_1 is a Sophie Germain prime and $x_1 \geqslant 2^{2^{16-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{16}$ are uniquely determined by x_1. For every positive integer n, at most finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with n and solve the system $\mathcal{Z}_{16}$ with Γ instead of Γ_{16}.*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 23. *([17, p. 330]).* $8069496435 \cdot 10^{5072} - 1$ *is a Sophie Germain prime (Harvey Dubner).*

Lemma 24. $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$.

Theorem 13. *The statement Σ_{16} implies the infinitude of Sophie Germain primes.*

Proof. It follows from Lemmas 22–24. □

Theorem 14. *The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y, then each such solution (x, y) belongs to the set $\{(1, 2), (2, 3)\}$.*

Proof. We leave the proof to the reader. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdös, see [2]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [12].

Theorem 15. *The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y, then each such solution (x, y) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

Proof. We leave the proof to the reader. □

11 Hypothetical statements $\Omega_3, \ldots, \Omega_{16}$ and their consequences

For an integer $n \in \{3, \ldots, 16\}$, let Ω_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq \Big\{\Gamma(x_i) = x_k : i,k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i,j,k \in \{1, \ldots, n\}\Big\}$ has a solution in integers $x_1, \ldots, x_n$ greater than $2^{2^{n-2}}$, then $\mathcal{S}$ has infinitely many solutions in positive integers $x_1, \ldots, x_n$.* For every $n \in \{3, \ldots, 16\}$, the statement Σ_n implies the statement Ω_n.

Lemma 25. *The number* $(65!)^2 + 1$ *is prime and* $65! > 2^{2^{9-2}}$.

Proof. The following `PARI/GP` ([16]) command

```
(04:04) gp > isprime((65!)^2+1,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([23, p. 226]). It rigorously shows that the number $(65!)^2 + 1$ is prime. $\square$

Lemma 26. *If positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{Z}_9$ and $x_1 > 2^{2^{9-2}}$, then $x_1 = \min(x_1, \ldots, x_9)$.*

Theorem 16. *The statement Ω_9 implies the infinitude of primes of the form $n^2 + 1$.*

Proof. It follows from Lemmas 16 and 25–26. $\square$

Lemma 27. *If positive integers $x_1, \ldots, x_{14}$ solve the system $\mathcal{Z}_{14}$ and $x_1 > 2^{2^{14-2}}$, then $x_1 = \min(x_1, \ldots, x_{14})$.*

Theorem 17. *The statement Ω_{14} implies the infinitude of twin primes.*

Proof. It follows from Lemmas 19–21 and 27. $\square$

12 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [11, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [11, p. 1].

Open Problem. ([11, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*
Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [10, p. 23]. Let

$$H_n = \Big\{x_i \cdot x_j = x_k : \; i,j,k \in \{1, \ldots, n\}\Big\} \cup \Big\{2^{2^{x_i}} = x_k : \; i,k \in \{1, \ldots, n\}\Big\}$$

Let $h(1) = 1$, and let $h(n+1) = 2^{2^{h(n)}}$ for every positive integer n.

Lemma 28. *The following subsystem of H_n*

$$\begin{cases} \qquad\qquad\qquad\quad x_1 \cdot x_1 & = \quad x_1 \\ \forall i \in \{1, \ldots, n-1\} \; 2^{2^{x_i}} & = \quad x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \ldots, h(n))$.

For a positive integer n, let ξ_n denote the following statement: *if a system of equations $S \subseteq H_n$ has only finitely many solutions in positive integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant h(n)$.* The statement ξ_n says that for subsystems of H_n the largest known solution is indeed the largest possible.

Hypothesis 4. *The statements $\xi_1, \dots, \xi_{13}$ are true.*

Proposition 5. *Every statement ξ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system H_n has a finite number of subsystems. □

Theorem 18. *The statement ξ_{13} proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers z.*

Proof. Let us consider the equation

$$(x+1)(y+1) = 2^{2^z} + 1 \tag{2}$$

in positive integers. By Lemma 4, we can transform equation (2) into an equivalent system of equations $\mathcal{G}$ which has 13 variables (x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 9.

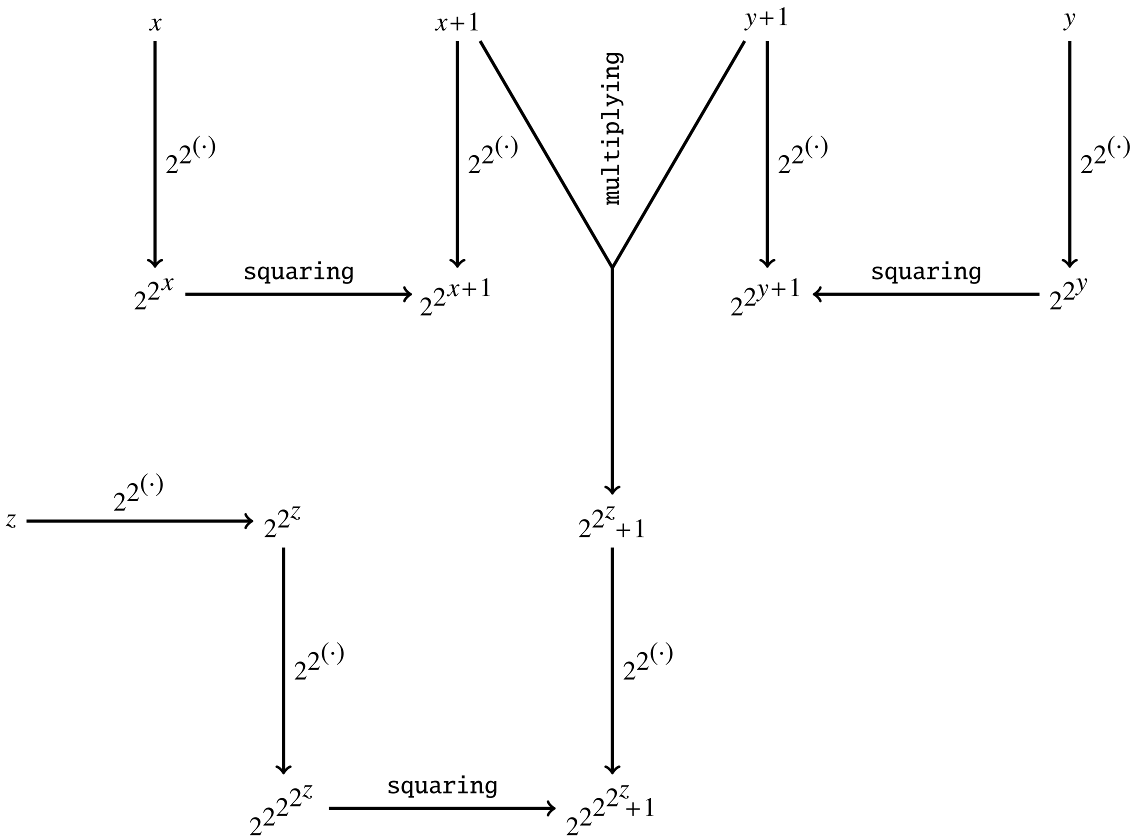

Fig. 9 Construction of the system $\mathcal{G}$

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z}+1}} > h(13)$. By this, the statement ξ_{13} implies that the system $\mathcal{G}$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

Corollary 7. *Let $\mathcal{W}_{13}$ denote the set of composite Fermat numbers. The statement ξ_{13} implies that we know an algorithm such that it returns a threshold number of $\mathcal{W}_{13}$, and this number equals* $\max(\mathcal{W}_{13})$*, if $\mathcal{W}_{13}$ is finite. Assuming the statement ξ_{13}, a single query to an oracle for the halting problem decides the infinity of $\mathcal{W}_{13}$. Assuming the statement ξ_{13}, the infinity of $\mathcal{W}_{13}$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{W}_{13} \cap [1, h(12)])$. □

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