

# The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in *ZFC* as it refers to our current knowledge on $\mathcal{X}$

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**Abstract.** Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite. Let  $\beta = (((24!)!)!)!$ , and let  $\Phi$  denote the implication:  $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$ . We heuristically justify the statement  $\Phi$  without invoking Landau's conjecture. Open problem: *Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies conditions (1)–(5)?* (1) There are a large number of elements of  $\mathcal{X}$  and it is conjectured that  $\mathcal{X}$  is infinite. (2) No known algorithm decides the finiteness/infiniteness of  $\mathcal{X}$ . (3) There is a known algorithm that for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ . (4) There is an explicitly known integer  $n$  such that  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ . (5) The set  $\mathcal{X}$  was studied in number theory. The set  $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$  satisfies conditions (1)–(4). The set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies conditions (1)–(3) and (5). The statement  $\Phi$  implies that the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies condition (4).

**Keywords:** computable set  $\mathcal{X} \subseteq \mathbb{N}$ , current knowledge on  $\mathcal{X}$ , explicitly known integer  $n$ , machine computations on large integers, mathematical statement about  $\mathcal{X}$  that refers to the current knowledge on  $\mathcal{X}$ , physical limits of computation.

## 1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3] p. 199].

**Definition 1.** *Conditions (1)–(5) concern sets  $\mathcal{X} \subseteq \mathbb{N}$ .*

- (1) *There are a large number of elements of  $\mathcal{X}$  and it is conjectured that  $\mathcal{X}$  is infinite.*
- (2) *No known algorithm decides the finiteness/infiniteness of  $\mathcal{X}$ .*
- (3) *There is a known algorithm that for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ .*

(4) *There is an explicitly known integer  $n$  such that  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .*

(5) *The set  $\mathcal{X}$  was studied in number theory.*

**Definition 2.** *We say that an integer  $n$  is a threshold number of a set  $\mathcal{X} \subseteq \mathbb{N}$ , if  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ , cf. [7] and [8].*

If a set  $\mathcal{X} \subseteq \mathbb{N}$  is empty or infinite, then any integer  $n$  is a threshold number of  $\mathcal{X}$ . If a set  $\mathcal{X} \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $\mathcal{X}$  form the set  $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$ .

Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite, see [4]–[6].

**Definition 3.** *Let  $\Phi$  denote the implication:*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)]$$

Landau's conjecture implies the statement  $\Phi$ . In Section 4, we heuristically justify the statement  $\Phi$  without invoking Landau's conjecture.

**Statement 1.** *There is no explicitly known threshold number of  $\mathcal{P}_{n^2+1}$ . It means that there is no explicitly known integer  $k$  such that  $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$ .*

Proving the statement  $\Phi$  will falsify Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Statements 2–3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets  $\mathcal{X} \subseteq \mathbb{N}$ .

## 2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

**Definition 4.** *Let  $\beta = (((24!)!)!)$ .*

**Lemma 1.**  $\beta \approx 10^{10^{10^{25.16114896940657}}}$ .

*Proof.* We ask Wolfram Alpha at <http://wolframalpha.com>. □

**Statement 2.** *The set  $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$  satisfies conditions (1)–(4).*

*Proof.* Condition (1) holds as  $\mathcal{X} \supseteq \{0, \dots, \beta\}$  and the set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $\beta$ , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

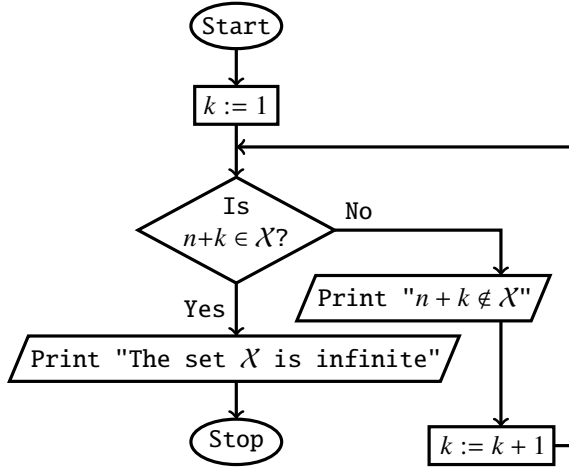
is empty or infinite, the integer  $\beta$  is a threshold number of  $\mathcal{X}$ . Thus condition (4) holds. □

**Open Problem 1.** *Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies conditions (1)–(5)?*

**Theorem 1.** Assume that for every positive integers  $b$  and  $s$ , at some future day, machine computations will be possible on every integers from the interval  $[-b, b]$  and this will be possible with the speed of  $s$  FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set  $X \subseteq \mathbb{N}$  will satisfy conditions (1)–(4) forever.

*Proof.* The proof goes by contradiction. Since conditons (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin X, n + 2 \notin X, n + 3 \notin X, \dots \quad (\text{T})$$



**Fig. 1** Algorithm whose execution never terminates if the set  $X$  is finite

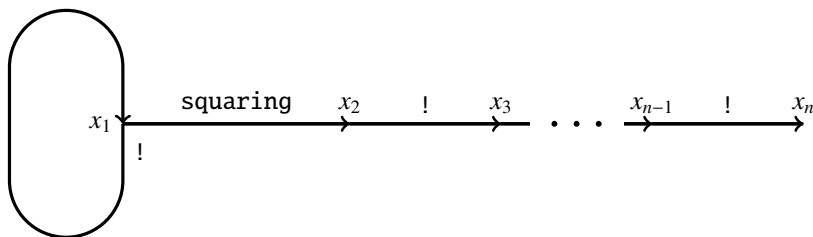
The sentences from the sequence (T) and our assumptions alone imply that for every explicitly given integer  $m > n$ , at some future day, a computer will be able to confirm in 1 second or less that  $(n, m] \cap X = \emptyset$ . Thus, at some future day, numerical evidence will support the conjecture that the set  $X$  is finite, contrary to the conjecture in condition (1).  $\square$

### 3. Number-theoretic statements $\Psi_n$

Let  $f(1) = 2$ ,  $f(2) = 4$ , and let  $f(n + 1) = f(n)!$  for every integer  $n \geq 2$ . Let  $\mathcal{U}_1$  denote the system of equations which consists of the equation  $x_1! = x_1$ . For an integer  $n \geq 2$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system  $\mathcal{U}_n$ .



**Fig. 2** Construction of the system  $\mathcal{U}_n$

**Lemma 2.** For every positive integer  $n$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(f(1), \dots, f(n))$ .

Let

$$B_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer  $n$ , let  $\Psi_n$  denote the following statement: if a system of equations  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq f(n)$ . The statement  $\Psi_n$  says that for subsystems of  $B_n$  with a finite number of solutions, the largest known solution is indeed the largest possible. The statements  $\Psi_1$  and  $\Psi_2$  hold trivially. There is no reason to assume the validity of the statement  $\Psi_9$ , cf. Conjecture [1](#) in Section [4](#).

**Theorem 2.** For every statement  $\Psi_n$ , the bound  $f(n)$  cannot be decreased.

*Proof.* It follows from Lemma [2](#) because  $\mathcal{U}_n \subseteq B_n$ . □

**Theorem 3.** For every integer  $n \geq 2$ , the statement  $\Psi_{n+1}$  implies the statement  $\Psi_n$ .

*Proof.* If a system  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then for every integer  $i \in \{1, \dots, n\}$  the system  $\mathcal{S} \cup \{x_i! = x_{n+1}\}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_{n+1}$ . The statement  $\Psi_{n+1}$  implies that  $x_i! = x_{n+1} \leq f(n+1) = f(n)!$ . Hence,  $x_i \leq f(n)$ . □

**Theorem 4.** Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

## 4. A conjectural solution to Open Problem [1](#)

**Lemma 3.** For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if

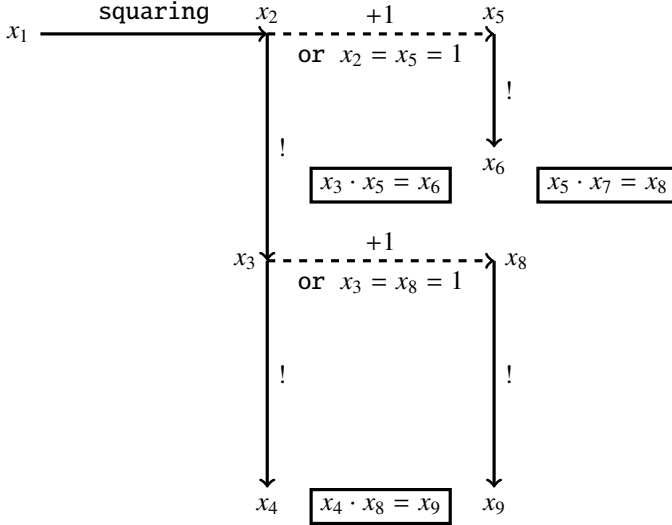
$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** (Wilson's theorem, [\[1\]](#) p. 89]). For every integer  $x \geq 2$ ,  $x$  is prime if and only if  $x$  divides  $(x - 1)! + 1$ .

Let  $\mathcal{A}$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system  $\mathcal{A}$ .



**Fig. 3** Construction of the system  $\mathcal{A}$

**Lemma 5.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 3, for every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 5 follows from Lemma 4.  $\square$

**Lemma 6.** *There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ , which solve the system  $\mathcal{A}$  and satisfy  $x_1 = 1$ . This is true as every such tuple  $(x_1, \dots, x_9)$  satisfies  $x_1, \dots, x_9 \in \{1, 2\}$ .*

*Proof.* The equality  $x_1 = 1$  implies that  $x_2 = x_1^2 = 1$ . Hence, for example,  $x_3 = x_2! = 1$ . Therefore,  $x_8 = x_3 + 1 = 2$  or  $x_8 = 1$ . Consequently,  $x_9 = x_8! \leq 2$ .  $\square$

**Conjecture 1.** *The statement  $\Psi_9$  is true when is restricted to the system  $\mathcal{A}$ .*

**Theorem 5.** *Conjecture 1 proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $f(7)$ , then the set  $\mathcal{P}_{n^2+1}$  is infinite.*

*Proof.* Suppose that the antecedent holds. By Lemma 5 there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{A}$ . Since  $x_1^2 + 1 > f(7)$ , we obtain that  $x_1^2 \geq f(7)$ . Hence,  $(x_1^2)! \geq f(7)! = f(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality  $x_9 > f(9)$  imply that the system  $\mathcal{A}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas 5 and 6, the set  $\mathcal{P}_{n^2+1}$  is infinite.  $\square$

**Theorem 6.** *Conjecture 1 implies the statement  $\Phi$ .*

*Proof.* It follows from Theorem 5 and the equality  $f(7) = (((24!)!)!)!$ .  $\square$

**Theorem 7.** *The statement  $\Phi$  implies Conjecture 1*

*Proof.* By Lemmas 5 and 6, if positive integers  $x_1, \dots, x_9$  solve the system  $\mathcal{A}$ , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or  $x_1, \dots, x_9 \in \{1, 2\}$ . In the first case, Lemma 5 and the statement  $\Phi$  imply that the inequality  $x_5 \leq (((24!)!)!)! = f(7)$  holds when the system  $\mathcal{A}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_9$ . Hence,  $x_2 = x_5 - 1 < f(7)$  and  $x_3 = x_2! < f(7)! = f(8)$ . Continuing this reasoning in the same manner, we can show that every  $x_i$  does not exceed  $f(9)$ .  $\square$

**Statement 3.** *The set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies conditions (1)–(3) and (5). The statement  $\Phi$  implies that the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies condition (4).*

*Proof.* The set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. There are 2199894223892 primes of the form  $n^2 + 1$  in the interval  $[2, 10^{28}]$ , see [5]. These two facts imply condition (1). Conditions (3) and (5) hold trivially. By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $f(7) = (((24!)!)!)! = \beta$ , see [2]. Thus condition (2) holds. Suppose the statement  $\Phi$ . It means that  $\beta$  is a threshold number of  $\mathcal{X} = \mathcal{P}_{n^2+1}$ . Thus condition (4) holds.  $\square$

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# On *ZFC*-formulae $\varphi(x)$ for which we know a non-negative integer $n$ such that $\{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite

Apoloniusz Tyszk

## Abstract

Let  $\Gamma(k)$  denote  $(k-1)!$ , and let  $\Gamma_n(k)$  denote  $(k-1)!$ , where  $n \in \{3, \dots, 16\}$  and  $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$ . For an integer  $n \in \{3, \dots, 16\}$ , let  $\Sigma_n$  denote the following statement: if a system of equations  $\mathcal{S} \subseteq \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  with  $\Gamma$  instead of  $\Gamma_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then every tuple  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$  that solves the original system  $\mathcal{S}$  satisfies  $x_1, \dots, x_n \leq 2^{2^{n-2}}$ . Our hypothesis claims that the statements  $\Sigma_3, \dots, \Sigma_{16}$  are true. The statement  $\Sigma_6$  proves the following implication: if the equation  $x(x+1) = y!$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(1, 2), (2, 3)\}$ . The statement  $\Sigma_6$  proves the following implication: if the equation  $x! + 1 = y^2$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ . The statement  $\Sigma_9$  implies the infinitude of primes of the form  $n^2 + 1$ . The statement  $\Sigma_9$  implies that any prime of the form  $n! + 1$  with  $n \geq 2^{2^{9-3}}$  proves the infinitude of primes of the form  $n! + 1$ . The statement  $\Sigma_{14}$  implies the infinitude of twin primes. The statement  $\Sigma_{16}$  implies the infinitude of Sophie Germain primes.

**Key words and phrases:** Brocard's problem, Brocard-Ramanujan equation  $x! + 1 = y^2$ , composite Fermat numbers, decidability in the limit, Erdős' equation  $x(x+1) = y!$ , finiteness of a set, infiniteness of a set, prime numbers of the form  $n^2 + 1$ , prime numbers of the form  $n! + 1$ , single query to an oracle for the halting problem, Sophie Germain primes, twin primes.

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## 1 Introduction and basic lemmas

The phrase “we know a non-negative integer  $n$ ” in the title means that we know an algorithm which returns  $n$ . The title of the article cannot be formalised in *ZFC* because the phrase “we know a non-negative integer  $n$ ” refers to currently known non-negative integers  $n$  with some property. A formally stated title may look like this: *On ZFC-formulae  $\varphi(x)$  for which there exists a non-negative integer  $n$  such that ZFC proves that*

$$\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) < \infty \implies \{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$$

Unfortunately, this formulation admits formulae  $\varphi(x)$  without any known non-negative integer  $n$  such that *ZFC* proves the above implication.

**Lemma 1.** *For every non-negative integer  $n$ ,  $\text{card}(\{x \in \mathbb{N} : x \leq n - 1\}) = n$ .*

**Corollary 1.** *The title altered to “On ZFC-formulae  $\varphi(x)$  for which we know a non-negative integer  $n$  such that  $\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) \leq n$  if the set  $\{x \in \mathbb{N} : \varphi(x)\}$  is finite” involves a weaker assumption on  $\varphi(x)$ .*

**Lemma 2.** *For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$



Let  $\Gamma(k)$  denote  $(k - 1)!$ .

**Lemma 3.** For every positive integers  $x$  and  $y$ ,  $x \cdot \Gamma(x) = \Gamma(y)$  if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** For every non-negative integers  $b$  and  $c$ ,  $b + 1 = c$  if and only if  $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$ .

**Lemma 5.** (Wilson's theorem, [8] p. 89]). For every positive integer  $x$ ,  $x$  divides  $(x - 1)! + 1$  if and only if  $x = 1$  or  $x$  is prime.

## 2 Subsets of $\mathbb{N}$ and their threshold numbers

We say that a non-negative integer  $m$  is a threshold number of a set  $X \subseteq \mathbb{N}$ , if  $X$  is infinite if and only if  $X$  contains an element greater than  $m$ , cf. [24] and [25]. If a set  $X \subseteq \mathbb{N}$  is empty or infinite, then any non-negative integer  $m$  is a threshold number of  $X$ . If a set  $X \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $X$  form the set  $\{\max(X), \max(X) + 1, \max(X) + 2, \dots\}$ .

It is conjectured that the set of prime numbers of the form  $n^2 + 1$  is infinite, see [14], pp. 37–38]. It is conjectured that the set of prime numbers of the form  $n! + 1$  is infinite, see [3], p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [14], p. 39]. It is conjectured that the set of composite numbers of the form  $2^{2^n} + 1$  is infinite, see [10], p. 23] and [11], pp. 158–159]. A prime  $p$  is said to be a Sophie Germain prime if both  $p$  and  $2p + 1$  are prime, see [22]. It is conjectured that the set of Sophie Germain primes is infinite, see [17], p. 330]. For each of these sets, we do not know any threshold number.

The following statement:

for every non-negative integer  $n$  there exist

$$\text{prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in [10^n, 10^n + 1] \quad (1)$$

is a  $\Pi_1$  statement which strengthens the twin prime conjecture, see [4], p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger  $\Pi_1$  statements, see [1]. Statement (1) is equivalent to the non-halting of a Turing machine. If a set  $X \subseteq \mathbb{N}$  is computable and we know a threshold number of  $X$ , then the infinity of  $X$  is equivalent to the halting of a Turing machine.

The height of a rational number  $\frac{p}{q}$  is denoted by  $H\left(\frac{p}{q}\right)$  and equals  $\max(|p|, |q|)$  provided  $\frac{p}{q}$  is written in lowest terms. The height of a rational tuple  $(x_1, \dots, x_n)$  is denoted by  $H(x_1, \dots, x_n)$  and equals  $\max(H(x_1), \dots, H(x_n))$ .

**Lemma 6.** The equation  $x^5 - x = y^2 - y$  has only finitely many rational solutions, see [13] p. 212]. The known rational solutions are  $(x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left(\frac{1}{4}, \frac{15}{32}\right), \left(\frac{1}{4}, \frac{17}{32}\right), \left(-\frac{15}{16}, -\frac{185}{1024}\right), \left(-\frac{15}{16}, \frac{1209}{1024}\right)$ , and the existence of other solutions is an open question, see [18] pp. 223–224].

**Corollary 2.** The set  $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$  is finite. We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{T}$ . We do not know any algorithm which returns a threshold number of  $\mathcal{T}$ .

Let  $\mathcal{L}$  denote the following system of equations:

$$\begin{cases} x^2 + y^2 = s^2 \\ x^2 + z^2 = t^2 \\ y^2 + z^2 = u^2 \\ x^2 + y^2 + z^2 = v^2 \end{cases}$$

Let

$$\mathcal{F} = \left\{ n \in \mathbb{N} \setminus \{0\} : \left( \text{the system } \mathcal{L} \text{ has no solutions in } \{1, \dots, n\}^7 \right) \wedge \right. \\ \left. \left( \text{the system } \mathcal{L} \text{ has a solution in } \{1, \dots, n+1\}^7 \right) \right\}$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

**Lemma 7.** ([21]). *No perfect cuboids are known.*

**Corollary 3.** *We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{F}$ . ZFC proves that  $\text{card}(\mathcal{F}) \in \{0, 1\}$ . We do not know any algorithm which returns  $\text{card}(\mathcal{F})$ . We do not know any algorithm which returns a threshold number of  $\mathcal{F}$ .*

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, & \text{if } \sin\left(999999\right) < 0 \\ \mathbb{N} \cap \left[ 0, \sin\left(999999\right) \cdot 999999 \right) & \text{otherwise} \end{cases}$$

We do not know whether or not the set  $\mathcal{H}$  is finite.

**Proposition 1.** *The number  $999999$  is a threshold number of  $\mathcal{H}$ . We know an algorithm which decides the equality  $\mathcal{H} = \mathbb{N}$ . If  $\mathcal{H} \neq \mathbb{N}$ , then the set  $\mathcal{H}$  consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{H}$ .*

Let

$$\mathcal{K} = \begin{cases} \{n\}, & \text{if } (n \in \mathbb{N}) \wedge (2^{\aleph_0} = \aleph_{n+1}) \\ \{0\}, & \text{if } 2^{\aleph_0} \geq \aleph_{\omega} \end{cases}$$

**Proposition 2.** *ZFC proves that  $\text{card}(\mathcal{K}) = 1$ . If ZFC is consistent, then for every  $n \in \mathbb{N}$  the sentences "n is a threshold number of  $\mathcal{K}$ " and "n is not a threshold number of  $\mathcal{K}$ " are not provable in ZFC.*

*Proof.* It suffices to observe that  $2^{\aleph_0}$  can attain every value from the set  $\{\aleph_1, \aleph_2, \aleph_3, \dots\}$ , see [7] and [9] p. 232]. □

### 3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matijasevich theorem imply the following theorem.

**Theorem 1.** ([5] p. 35]. *There exists a polynomial  $D(x_1, \dots, x_m)$  with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation  $D(x_1, \dots, x_m) = 0$  is solvable in non-negative integers" and "The equation  $D(x_1, \dots, x_m) = 0$  is not solvable in non-negative integers" are not provable in ZFC.*

Let  $\mathcal{Y}$  denote the set of all non-negative integers  $k$  such that the equation  $D(x_1, \dots, x_m) = 0$  has no solutions in  $\{0, \dots, k\}^m$ . Since the set  $\{0, \dots, k\}^m$  is finite, we know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{Y}$ . Theorem 1 implies the next theorem.

**Theorem 2.** For every  $n \in \mathbb{N}$ , ZFC proves that  $n \in \mathcal{Y}$ . If ZFC is arithmetically consistent, then the sentences “ $\mathcal{Y}$  is finite” and “ $\mathcal{Y}$  is infinite” are not provable in ZFC. If ZFC is arithmetically consistent, then for every  $n \in \mathbb{N}$  the sentences “ $n$  is a threshold number of  $\mathcal{Y}$ ” and “ $n$  is not a threshold number of  $\mathcal{Y}$ ” are not provable in ZFC.

Let  $\mathcal{E}$  denote the set of all non-negative integers  $k$  such that the equation  $D(x_1, \dots, x_m) = 0$  has a solution in  $\{0, \dots, k\}^m$ . Since the set  $\{0, \dots, k\}^m$  is finite, we know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{E}$ . Theorem 1 implies the next theorem.

**Theorem 3.** The set  $\mathcal{E}$  is empty or infinite. In both cases, every non-negative integer  $n$  is a threshold number of  $\mathcal{E}$ . If ZFC is arithmetically consistent, then the sentences “ $\mathcal{E}$  is empty”, “ $\mathcal{E}$  is not empty”, “ $\mathcal{E}$  is finite”, and “ $\mathcal{E}$  is infinite” are not provable in ZFC.

Let

$$\mathcal{V} = \left\{ n \in \mathbb{N} : \left( \text{the polynomial } D(x_1, \dots, x_m) \text{ has no solutions in } \{0, \dots, n\}^m \right) \wedge \right. \\ \left. \left( \text{the polynomial } D(x_1, \dots, x_m) \text{ has a solution in } \{0, \dots, n+1\}^m \right) \right\}$$

Since the sets  $\{0, \dots, n\}^m$  and  $\{0, \dots, n+1\}^m$  are finite, we know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{V}$ . Theorem 1 implies the next theorem.

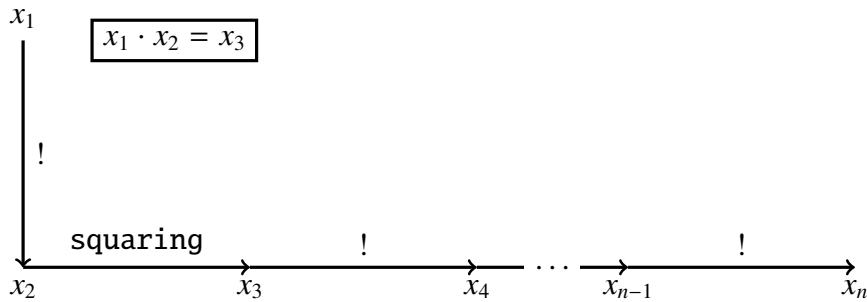
**Theorem 4.** ZFC proves that  $\text{card}(\mathcal{V}) \in \{0, 1\}$ . For every  $n \in \mathbb{N}$ , ZFC proves that  $n \notin \mathcal{V}$ . ZFC does not prove the emptiness of  $\mathcal{V}$ , if ZFC is arithmetically consistent. For every  $n \in \mathbb{N}$ , the sentence “ $n$  is a threshold number of  $\mathcal{V}$ ” is not provable in ZFC, if ZFC is arithmetically consistent.

## 4 Hypothetical statements $\Psi_3, \dots, \Psi_{16}$

For an integer  $n \geq 3$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{2\} \quad x_i! = x_{i+1} \\ x_1 \cdot x_2 = x_3 \\ x_2 \cdot x_2 = x_3 \end{array} \right\}$$

The diagram in Figure 1 illustrates the construction of the system  $\mathcal{U}_n$ .



**Fig. 1** Construction of the system  $\mathcal{U}_n$

Let  $g(3) = 4$ , and let  $g(n+1) = g(n)!$  for every integer  $n \geq 3$ .

**Lemma 8.** For every integer  $n \geq 3$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(2, 2, g(3), \dots, g(n))$ .

Let

$$B_n = \{x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer  $n \geq 3$ , let  $\Psi_n$  denote the following statement: if a system of equations  $\mathcal{S} \subseteq B_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq g(n)$ . The statement  $\Psi_n$  says that for subsystems of  $B_n$  the largest known solution is indeed the largest possible.

**Hypothesis 1.** The statements  $\Psi_3, \dots, \Psi_{16}$  are true.

**Proposition 3.** Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

**Proposition 4.** For every statement  $\Psi_n$ , the bound  $g(n)$  cannot be decreased.

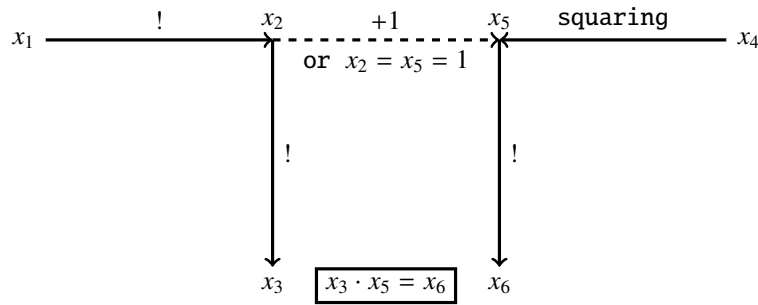
*Proof.* It follows from Lemma 8 because  $\mathcal{U}_n \subseteq B_n$ . □

## 5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let  $\mathcal{A}$  denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system  $\mathcal{A}$ .



**Fig. 2** Construction of the system  $\mathcal{A}$

**Lemma 9.** For every  $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, x_3, x_5, x_6$  if and only if  $x_1! + 1 = x_4^2$ . In this case, the integers  $x_2, x_3, x_5, x_6$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

*Proof.* It follows from Lemma 2. □

It is conjectured that  $x! + 1$  is a perfect square only for  $x \in \{4, 5, 7\}$ , see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation  $x! + 1 = y^2$ , see [15].

**Theorem 5.** If the equation  $x_1! + 1 = x_4^2$  has only finitely many solutions in positive integers, then the statement  $\Psi_6$  guarantees that each such solution  $(x_1, x_4)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ .

*Proof.* Suppose that the antecedent holds. Let positive integers  $x_1$  and  $x_4$  satisfy  $x_1! + 1 = x_4^2$ . Then,  $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$ . By Lemma 9, the system  $\mathcal{A}$  is solvable in positive integers  $x_2, x_3, x_5, x_6$ . Since  $\mathcal{A} \subseteq B_6$ , the statement  $\Psi_6$  implies that  $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$ . Hence,  $x_1! + 1 \leq g(5) = g(4)!$ . Consequently,  $x_1 < g(4) = 24$ . If  $x_1 \in \{1, \dots, 23\}$ , then  $x_1! + 1$  is a perfect square only for  $x_1 \in \{4, 5, 7\}$ . □

## 6 Are there infinitely many prime numbers of the form $n^2 + 1$ ?

Edmund Landau's conjecture states that there are infinitely many primes of the form  $n^2 + 1$ , see [14, pp. 37–38]. Let  $\mathcal{B}$  denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system  $\mathcal{B}$ .

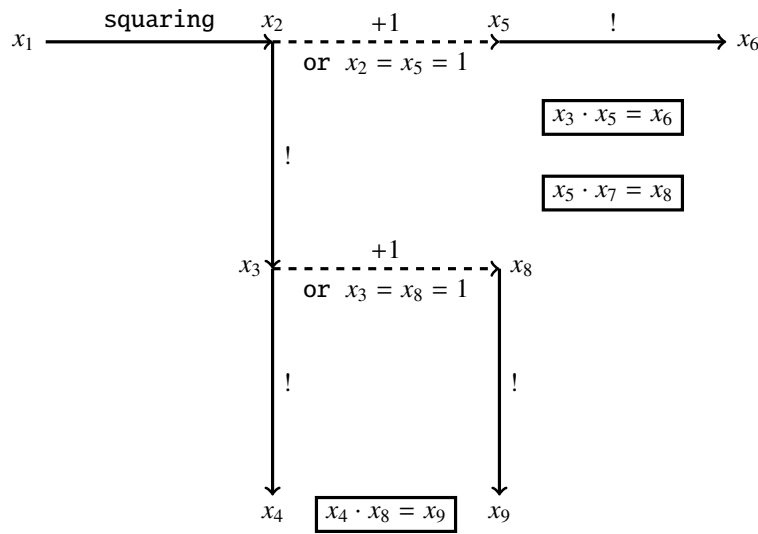


Fig. 3 Construction of the system  $\mathcal{B}$

**Lemma 10.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{B}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 2, for every integer  $x_1 \geq 2$ , the system  $\mathcal{B}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 10 follows from Lemma 5.  $\square$

**Lemma 11.** There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  which solve the system  $\mathcal{B}$  and satisfy  $x_1 = 1$ .

*Proof.* If a tuple  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  solves the system  $\mathcal{B}$  and  $x_1 = 1$ , then  $x_1, \dots, x_9 \leq 2$ . Indeed,  $x_1 = 1$  implies that  $x_2 = x_1^2 = 1$ . Hence, for example,  $x_3 = x_2! = 1$ . Therefore,  $x_8 = x_3 + 1 = 2$  or  $x_8 = 1$ . Consequently,  $x_9 = x_8! \leq 2$ .  $\square$

**Theorem 6.** *The statement  $\Psi_9$  proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $g(7)$ , then there are infinitely many primes of the form  $n^2 + 1$ .*

*Proof.* Suppose that the antecedent holds. By Lemma [10](#), there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{B}$ . Since  $x_1^2 + 1 > g(7)$ , we obtain that  $x_1^2 \geq g(7)$ . Hence,  $(x_1^2)! \geq g(7)! = g(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)$$

Since  $\mathcal{B} \subseteq B_9$ , the statement  $\Psi_9$  and the inequality  $x_9 > g(9)$  imply that the system  $\mathcal{B}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas [10](#) and [11](#), there are infinitely many primes of the form  $n^2 + 1$ .  $\square$

**Corollary 4.** *Let  $X_9$  denote the set of primes of the form  $n^2 + 1$ . The statement  $\Psi_9$  implies that we know an algorithm such that it returns a threshold number of  $X_9$ , and this number equals  $\max(X_9)$ , if  $X_9$  is finite. Assuming the statement  $\Psi_9$ , a single query to an oracle for the halting problem decides the infinity of  $X_9$ . Assuming the statement  $\Psi_9$ , the infinity of  $X_9$  is decidable in the limit.*

*Proof.* We consider an algorithm which computes  $\max(X_9 \cap [1, g(7)])$ .  $\square$

## 7 Are there infinitely many prime numbers of the form $n! + 1$ ?

It is conjectured that there are infinitely many primes of the form  $n! + 1$ , see [\[3\]](#) p. 443].

**Theorem 7.** *(cf. Theorem [11](#)). The statement  $\Psi_9$  proves the following implication: if there exists an integer  $x_1 \geq g(6)$  such that  $x_1! + 1$  is prime, then there are infinitely many primes of the form  $n! + 1$ .*

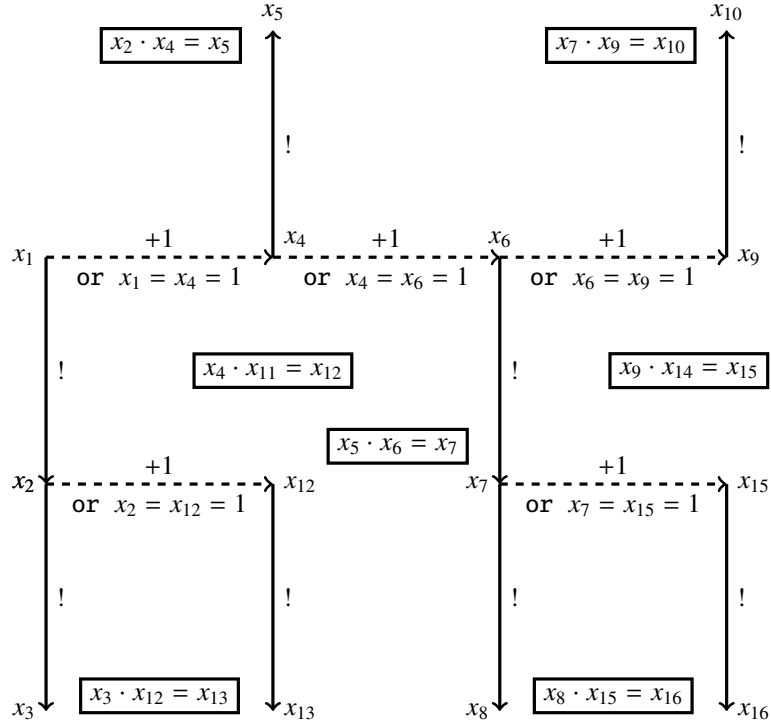
*Proof.* We leave the analogous proof to the reader.  $\square$

## 8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [\[14\]](#) p. 39]. Let  $C$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{array} \right.$$

Lemma [2](#) and the diagram in Figure 4 explain the construction of the system  $C$ .



**Fig. 4** Construction of the system  $C$

**Lemma 12.** For every  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ , the system  $C$  is solvable in positive integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  if and only if  $x_4$  and  $x_9$  are prime and  $x_4 + 2 = x_9$ . In this case, the integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  are uniquely determined by the following equalities:

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

*Proof.* By Lemma 2, for every  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ , the system  $C$  is solvable in positive integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | ((x_4 - 1)! + 1)) \wedge (x_9 | ((x_9 - 1)! + 1))$$

Hence, the claim of Lemma 12 follows from Lemma 5.  $\square$

**Lemma 13.** There are only finitely many tuples  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  which solve the system  $C$  and satisfy  $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$ .

*Proof.* If a tuple  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  solves the system  $C$  and  $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$ , then  $x_1, \dots, x_{16} \leq 7!$ . Indeed, for example, if  $x_4 = 2$  then  $x_6 = x_4 + 1 = 3$ . Hence,  $x_7 = x_6! = 6$ . Therefore,  $x_{15} = x_7 + 1 = 7$ . Consequently,  $x_{16} = x_{15}! = 7!$ .  $\square$

**Theorem 8.** *The statement  $\Psi_{16}$  proves the following implication: if there exists a twin prime greater than  $g(14)$ , then there are infinitely many twin primes.*

*Proof.* Suppose that the antecedent holds. Then, there exist prime numbers  $x_4$  and  $x_9$  such that  $x_9 = x_4 + 2 > g(14)$ . Hence,  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ . By Lemma [12](#), there exists a unique tuple  $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$  such that the tuple  $(x_1, \dots, x_{16})$  solves the system  $C$ . Since  $x_9 > g(14)$ , we obtain that  $x_9 - 1 \geq g(14)$ . Therefore,  $(x_9 - 1)! \geq g(14)! = g(15)$ . Hence,  $(x_9 - 1)! + 1 > g(15)$ . Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since  $C \subseteq B_{16}$ , the statement  $\Psi_{16}$  and the inequality  $x_{16} > g(16)$  imply that the system  $C$  has infinitely many solutions in positive integers  $x_1, \dots, x_{16}$ . According to Lemmas [12](#) and [13](#), there are infinitely many twin primes.  $\square$

**Corollary 5.** (cf. [\[6\]](#)). *Let  $\mathcal{X}_{16}$  denote the set of twin primes. The statement  $\Psi_{16}$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{X}_{16}$ , and this number equals  $\max(\mathcal{X}_{16})$ , if  $\mathcal{X}_{16}$  is finite. Assuming the statement  $\Psi_{16}$ , a single query to an oracle for the halting problem decides the infinity of  $\mathcal{X}_{16}$ . Assuming the statement  $\Psi_{16}$ , the infinity of  $\mathcal{X}_{16}$  is decidable in the limit.*

*Proof.* We consider an algorithm which computes  $\max(\mathcal{X}_{16} \cap [1, g(14)])$ .  $\square$

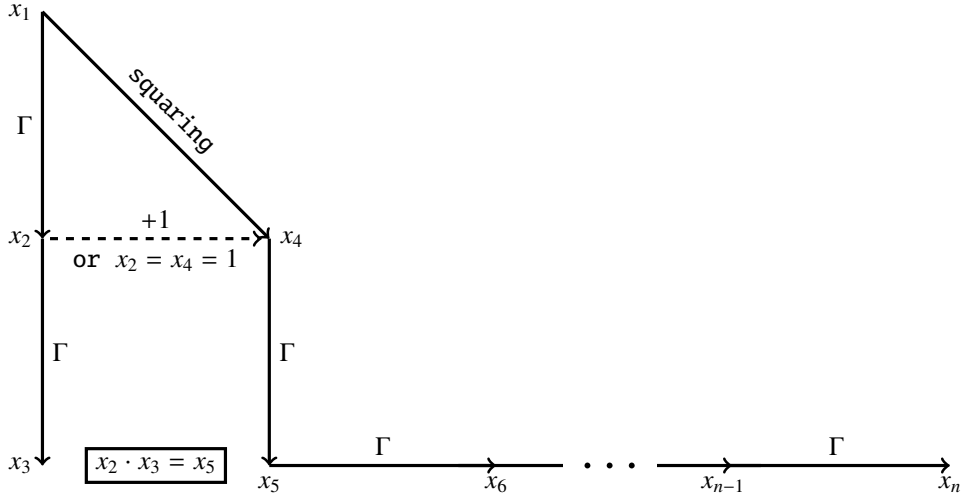
## 9 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ and their consequences

Let  $\lambda(5) = \Gamma(25)$ , and let  $\lambda(n + 1) = \Gamma(\lambda(n))$  for every integer  $n \geq 5$ . For an integer  $n \geq 5$ , let  $\mathcal{J}_n$  denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{3\} \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{array} \right.$$

Lemma [3](#) and the diagram in Figure 5 explain the construction of the system  $\mathcal{J}_n$ .





**Fig. 5** Construction of the system  $\mathcal{J}_n$

For every integer  $n \geq 5$ , the system  $\mathcal{J}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(5, 24, 23!, 25, \lambda(5), \dots, \lambda(n))$ . For an integer  $n \geq 5$ , let  $\Delta_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq \lambda(n)$ .*

**Hypothesis 2.** *The statements  $\Delta_5, \dots, \Delta_{14}$  are true.*

Lemmas [3](#) and [5](#) imply that the statements  $\Delta_n$  have similar consequences as the statements  $\Psi_n$ .

**Theorem 9.** *The statement  $\Delta_6$  implies that any prime number  $p \geq 25$  proves the infinitude of primes.*

*Proof.* It follows from Lemmas [3](#) and [5](#). We leave the details to the reader.  $\square$

## 10 Hypothetical statements $\Sigma_3, \dots, \Sigma_{16}$ and their consequences

Let  $\Gamma_n(k)$  denote  $(k-1)!$ , where  $n \in \{3, \dots, 16\}$  and  $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$ . For an integer  $n \in \{3, \dots, 16\}$ , let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer  $n \in \{3, \dots, 16\}$ , let  $P_n$  denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \Gamma_n(x_2) = x_1 \\ \forall i \in \{2, \dots, n-1\} x_i \cdot x_i = x_{i+1} \end{cases}$$

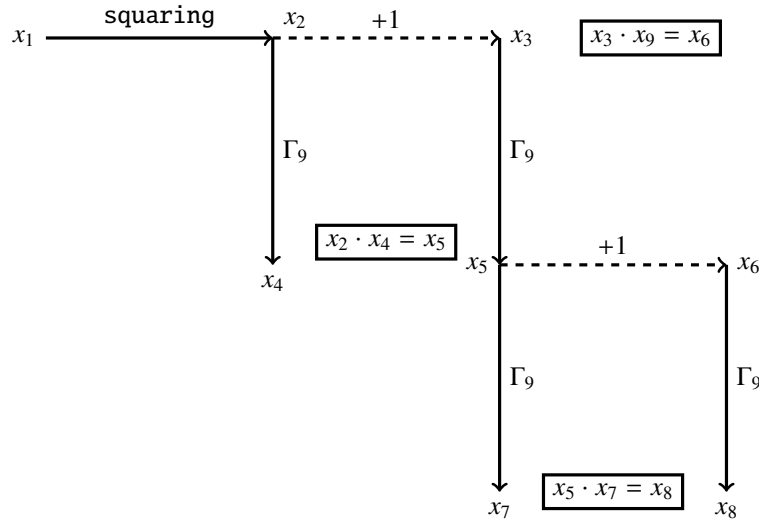
**Lemma 14.** *For every integer  $n \in \{3, \dots, 16\}$ ,  $P_n \subseteq Q_n$  and the system  $P_n$  with  $\Gamma$  instead of  $\Gamma_n$  has exactly one solution in positive integers  $x_1, \dots, x_n$ , namely  $(1, 2^{2^0}, 2^{2^1}, 2^{2^2}, \dots, 2^{2^{n-2}})$ .*

For an integer  $n \in \{3, \dots, 16\}$ , let  $\Sigma_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq Q_n$  with  $\Gamma$  instead of  $\Gamma_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then every tuple  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$  that solves the original system  $\mathcal{S}$  satisfies  $x_1, \dots, x_n \leq 2^{2^{n-2}}$ .*

**Hypothesis 3.** *The statements  $\Sigma_3, \dots, \Sigma_{16}$  are true.*

**Lemma 15.** (cf. Lemma [3](#)). *For every integer  $n \in \{4, \dots, 16\}$  and for every positive integers  $x$  and  $y$ ,  $x \cdot \Gamma_n(x) = \Gamma_n(y)$  if and only if  $(x+1 = y) \wedge (x \geq 2^{2^{n-3}} + 1)$ .*

Let  $\mathcal{Z}_9 \subseteq \mathcal{Q}_9$  be the system of equations in Figure 6.



**Fig. 6** Construction of the system  $\mathcal{Z}_9$

**Lemma 16.** For every positive integer  $x_1$ , the system  $\mathcal{Z}_9$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1 > 2^{2^{9-4}}$  and  $x_1^2 + 1$  is prime. In this case, positive integers  $x_2, \dots, x_9$  are uniquely determined by  $x_1$ . For every positive integer  $n$ , at most finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  begin with  $n$  and solve the system  $\mathcal{Z}_9$  with  $\Gamma$  instead of  $\Gamma_9$ .

*Proof.* It follows from Lemmas [3](#), [5](#), and [15](#). □

**Lemma 17.** ([\[19\]](#)). The number  $(13!)^2 + 1 = 38775788043632640001$  is prime.

**Lemma 18.**  $\left( (13!)^2 \geq 2^{2^{9-3}} + 1 = 18446744073709551617 \right) \wedge \left( \Gamma_9((13!)^2) > 2^{2^{9-2}} \right)$ .

**Theorem 10.** The statement  $\Sigma_9$  implies the infinitude of primes of the form  $n^2 + 1$ .

*Proof.* It follows from Lemmas [16](#)-[18](#). □

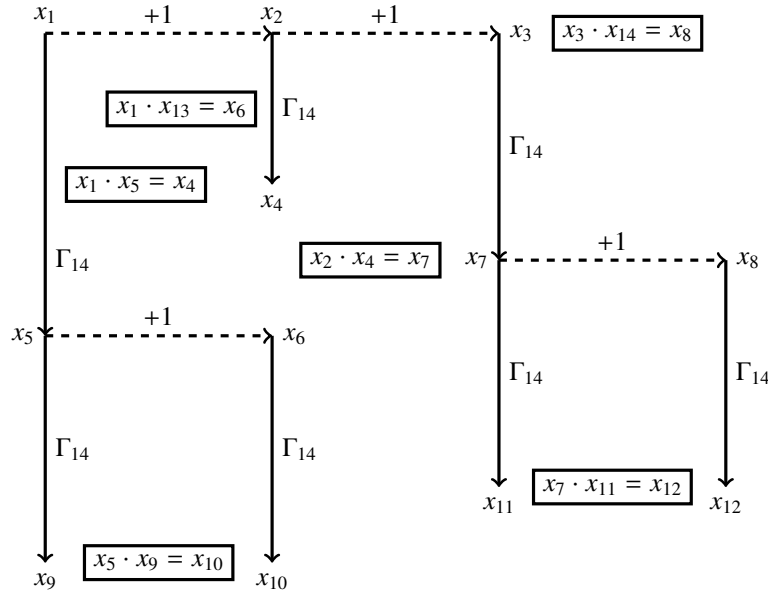
**Theorem 11.** (cf. Theorem [7](#)). The statement  $\Sigma_9$  implies that any prime of the form  $n! + 1$  with  $n \geq 2^{2^{9-3}}$  proves the infinitude of primes of the form  $n! + 1$ .

*Proof.* We leave the proof to the reader. □

**Corollary 6.** Let  $\mathcal{Y}_9$  denote the set of primes of the form  $n! + 1$ . The statement  $\Sigma_9$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{Y}_9$ , and this number equals  $\max(\mathcal{Y}_9)$ , if  $\mathcal{Y}_9$  is finite. Assuming the statement  $\Sigma_9$ , a single query to an oracle for the halting problem decides the infinity of  $\mathcal{Y}_9$ . Assuming the statement  $\Sigma_9$ , the infinity of  $\mathcal{Y}_9$  is decidable in the limit.

*Proof.* We consider an algorithm which computes  $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$ . □

Let  $\mathcal{Z}_{14} \subseteq \mathcal{Q}_{14}$  be the system of equations in Figure 7.



**Fig. 7** Construction of the system  $\mathcal{Z}_{14}$

**Lemma 19.** For every positive integer  $x_1$ , the system  $\mathcal{Z}_{14}$  is solvable in positive integers  $x_2, \dots, x_{14}$  if and only if  $x_1$  and  $x_1 + 2$  are prime and  $x_1 \geq 2^{2^{14-3}} + 1$ . In this case, positive integers  $x_2, \dots, x_{14}$  are uniquely determined by  $x_1$ . For every positive integer  $n$ , at most finitely many tuples  $(x_1, \dots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$  begin with  $n$  and solve the system  $\mathcal{Z}_{14}$  with  $\Gamma$  instead of  $\Gamma_{14}$ .

*Proof.* It follows from Lemmas [3](#), [5](#), and [15](#). □

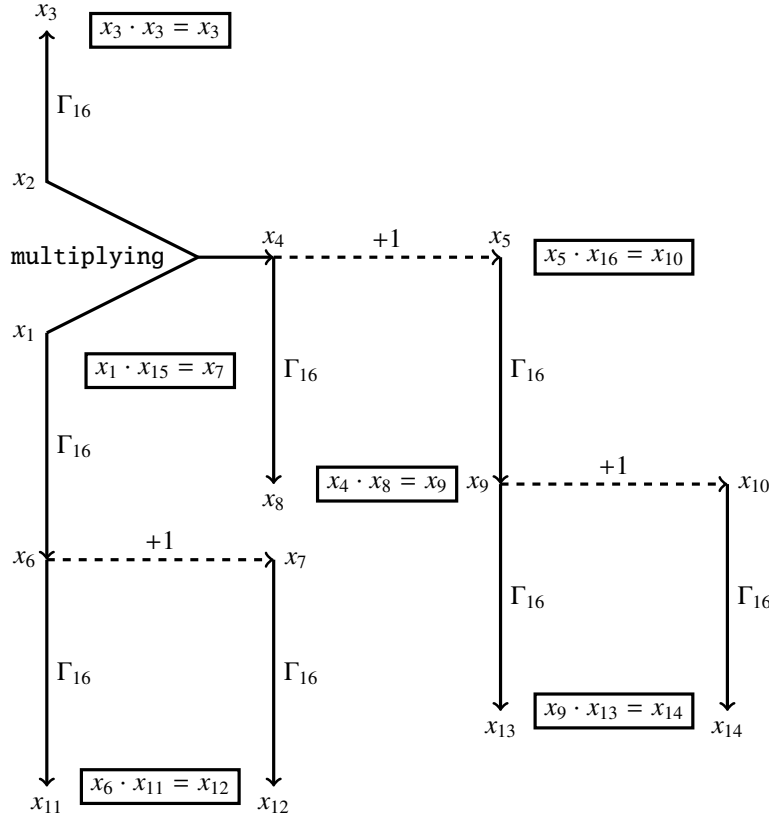
**Lemma 20.** ([\[23\]](#) p. 87]). The numbers  $459 \cdot 2^{8529} - 1$  and  $459 \cdot 2^{8529} + 1$  are prime (Harvey Dubner).

**Lemma 21.**  $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$ .

**Theorem 12.** The statement  $\Sigma_{14}$  implies the infinitude of twin primes.

*Proof.* It follows from Lemmas [19](#)-[21](#). □

A prime  $p$  is said to be a Sophie Germain prime if both  $p$  and  $2p + 1$  are prime, see [\[22\]](#). It is conjectured that there are infinitely many Sophie Germain primes, see [\[17\]](#) p. 330]. Let  $\mathcal{Z}_{16} \subseteq \mathcal{Q}_{16}$  be the system of equations in Figure 8.



**Fig. 8** Construction of the system  $\mathcal{Z}_{16}$

**Lemma 22.** For every positive integer  $x_1$ , the system  $\mathcal{Z}_{16}$  is solvable in positive integers  $x_2, \dots, x_{16}$  if and only if  $x_1$  is a Sophie Germain prime and  $x_1 \geq 2^{2^{16-3}} + 1$ . In this case, positive integers  $x_2, \dots, x_{16}$  are uniquely determined by  $x_1$ . For every positive integer  $n$ , at most finitely many tuples  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  begin with  $n$  and solve the system  $\mathcal{Z}_{16}$  with  $\Gamma$  instead of  $\Gamma_{16}$ .

*Proof.* It follows from Lemmas [3](#), [5](#), and [15](#). □

**Lemma 23.** ([\[17\]](#) p. 330).  $8069496435 \cdot 10^{5072} - 1$  is a Sophie Germain prime (Harvey Dubner).

**Lemma 24.**  $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$ .

**Theorem 13.** The statement  $\Sigma_{16}$  implies the infinitude of Sophie Germain primes.

*Proof.* It follows from Lemmas [22](#)-[24](#). □

**Theorem 14.** The statement  $\Sigma_6$  proves the following implication: if the equation  $x(x+1) = y!$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(1, 2), (2, 3)\}$ .

*Proof.* We leave the proof to the reader. □

The question of solving the equation  $x(x+1) = y!$  was posed by P. Erdős, see [\[2\]](#). F. Luca proved that the *abc* conjecture implies that the equation  $x(x+1) = y!$  has only finitely many solutions in positive integers, see [\[12\]](#).

**Theorem 15.** The statement  $\Sigma_6$  proves the following implication: if the equation  $x! + 1 = y^2$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ .

*Proof.* We leave the proof to the reader. □

## 11 Hypothetical statements $\Omega_3, \dots, \Omega_{16}$ and their consequences

For an integer  $n \in \{3, \dots, 16\}$ , let  $\Omega_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  has a solution in integers  $x_1, \dots, x_n$  greater than  $2^{2^{n-2}}$ , then  $\mathcal{S}$  has infinitely many solutions in positive integers  $x_1, \dots, x_n$ .* For every  $n \in \{3, \dots, 16\}$ , the statement  $\Sigma_n$  implies the statement  $\Omega_n$ .

**Lemma 25.** *The number  $(65!)^2 + 1$  is prime and  $65! > 2^{2^{9-2}}$ .*

*Proof.* The following PARI/GP ([16]) command

```
(04:04) gp > isprime((65!)^2+1,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([23] p. 226). It rigorously shows that the number  $(65!)^2 + 1$  is prime.  $\square$

**Lemma 26.** *If positive integers  $x_1, \dots, x_9$  solve the system  $\mathcal{Z}_9$  and  $x_1 > 2^{2^{9-2}}$ , then  $x_1 = \min(x_1, \dots, x_9)$ .*

**Theorem 16.** *The statement  $\Omega_9$  implies the infinitude of primes of the form  $n^2 + 1$ .*

*Proof.* It follows from Lemmas [16] and [25-26].  $\square$

**Lemma 27.** *If positive integers  $x_1, \dots, x_{14}$  solve the system  $\mathcal{Z}_{14}$  and  $x_1 > 2^{2^{14-2}}$ , then  $x_1 = \min(x_1, \dots, x_{14})$ .*

**Theorem 17.** *The statement  $\Omega_{14}$  implies the infinitude of twin primes.*

*Proof.* It follows from Lemmas [19-21] and [27].  $\square$

## 12 Are there infinitely many composite Fermat numbers?

Integers of the form  $2^{2^n} + 1$  are called Fermat numbers. Primes of the form  $2^{2^n} + 1$  are called Fermat primes, as Fermat conjectured that every integer of the form  $2^{2^n} + 1$  is prime, see [11] p. 1]. Fermat correctly remarked that  $2^{2^0} + 1 = 3$ ,  $2^{2^1} + 1 = 5$ ,  $2^{2^2} + 1 = 17$ ,  $2^{2^3} + 1 = 257$ , and  $2^{2^4} + 1 = 65537$  are all prime, see [11] p. 1].

**Open Problem.** ([11] p. 159]. *Are there infinitely many composite numbers of the form  $2^{2^n} + 1$ ? Most mathematicians believe that  $2^{2^n} + 1$  is composite for every integer  $n \geq 5$ , see [10] p. 23]. Let*

$$H_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Let  $h(1) = 1$ , and let  $h(n+1) = 2^{2^{h(n)}}$  for every positive integer  $n$ .

**Lemma 28.** *The following subsystem of  $H_n$*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

*has exactly one solution  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ , namely  $(h(1), \dots, h(n))$ .*

For a positive integer  $n$ , let  $\xi_n$  denote the following statement: *if a system of equations  $S \subseteq H_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq h(n)$* . The statement  $\xi_n$  says that for subsystems of  $H_n$  the largest known solution is indeed the largest possible.

**Hypothesis 4.** *The statements  $\xi_1, \dots, \xi_{13}$  are true.*

**Proposition 5.** *Every statement  $\xi_n$  is true with an unknown integer bound that depends on  $n$ .*

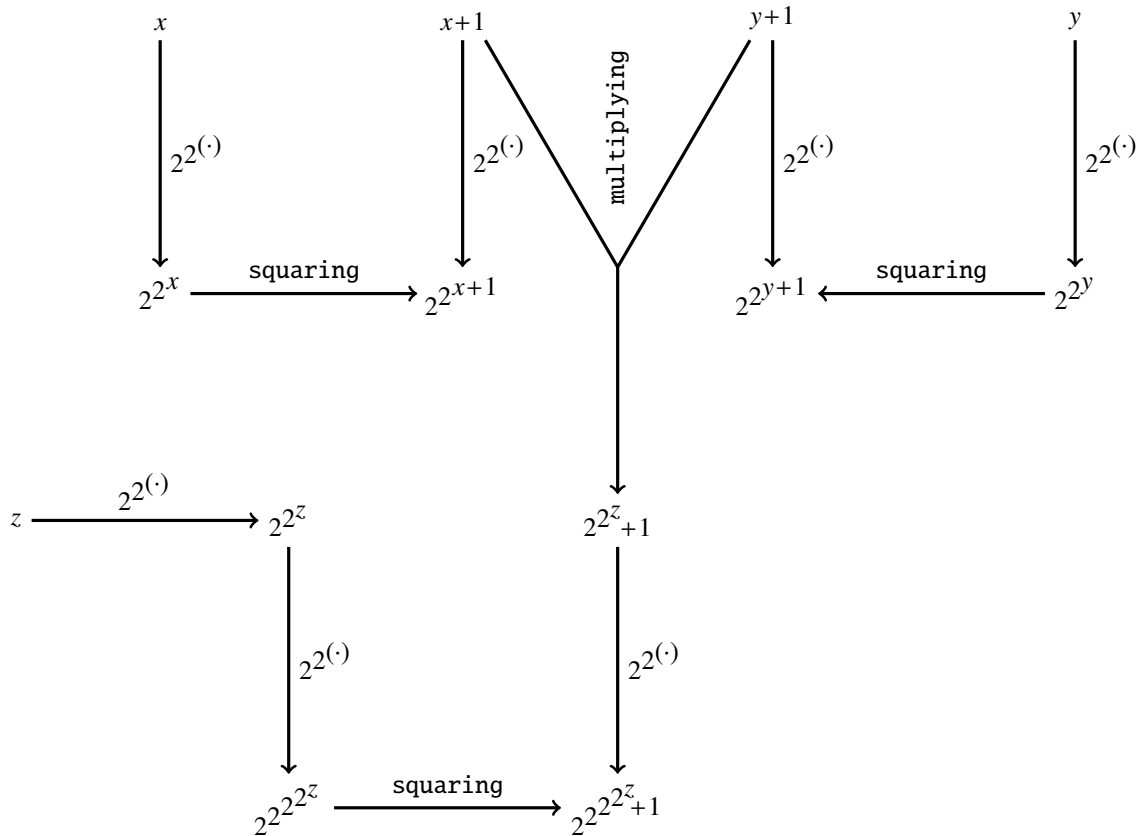
*Proof.* For every positive integer  $n$ , the system  $H_n$  has a finite number of subsystems. □

**Theorem 18.** *The statement  $\xi_{13}$  proves the following implication: if  $z \in \mathbb{N} \setminus \{0\}$  and  $2^{2^z} + 1$  is composite and greater than  $h(12)$ , then  $2^{2^z} + 1$  is composite for infinitely many positive integers  $z$ .*

*Proof.* Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{2}$$

in positive integers. By Lemma 4, we can transform equation (2) into an equivalent system of equations  $\mathcal{G}$  which has 13 variables ( $x, y, z$ , and 10 other variables) and which consists of equations of the forms  $\alpha \cdot \beta = \gamma$  and  $2^{2^\alpha} = \gamma$ , see the diagram in Figure 9.



**Fig. 9** Construction of the system  $\mathcal{G}$

Since  $2^{2^z} + 1 > h(12)$ , we obtain that  $2^{2^{2^z} + 1} > h(13)$ . By this, the statement  $\xi_{13}$  implies that the system  $\mathcal{G}$  has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

**Corollary 7.** Let  $\mathcal{W}_{13}$  denote the set of composite Fermat numbers. The statement  $\xi_{13}$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{W}_{13}$ , and this number equals  $\max(\mathcal{W}_{13})$ , if  $\mathcal{W}_{13}$  is finite. Assuming the statement  $\xi_{13}$ , a single query to an oracle for the halting problem decides the infinity of  $\mathcal{W}_{13}$ . Assuming the statement  $\xi_{13}$ , the infinity of  $\mathcal{W}_{13}$  is decidable in the limit.

*Proof.* We consider an algorithm which computes  $\max(\mathcal{W}_{13} \cap [1, h(12)])$ . □

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