The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in ZFC as it refers to our current knowledge on $\mathcal{X}$

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Abstract. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite. Let $\beta = (((24!)!)!)!$, and let Φ denote the implication: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement Φ without invoking Landau's conjecture. Open problem: Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions `(1)-(5)`? `(1)` *There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.* `(2)` *No known algorithm decides the finiteness/infiniteness of $\mathcal{X}$.* `(3)` *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* `(4)` *There is an explicitly known integer n such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(5)` *We know an algorithm such that for every input $k \in \mathbb{N}$ it returns the sentence "$k \in \mathcal{X}$" or the sentence "$k \notin \mathcal{X}$" and every returned sentence is true when k is sufficiently large. The simplest known such algorithm may return a false sentence only if k is small.* We prove: `(i)` the set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions `(1)-(4)`, `(ii)` the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions `(1)-(3)` and `(5)`, `(iii)` the statement Φ implies that the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies condition `(4)`.

Keywords: computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on $\mathcal{X}$, explicitly known integer n, machine computations on large integers, mathematical statement about $\mathcal{X}$ that refers to the current knowledge on $\mathcal{X}$, physical limits of computation.

1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

Definition 1. *Conditions* `(1)-(5)` *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

`(1)` *There are a large number of elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is infinite.*

`(2)` *No known algorithm decides the finiteness/infiniteness of* $\mathcal{X}$.

`(3)` *There is a known algorithm that for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.

`(4)` *There is an explicitly known integer n such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

`(5)` *We know an algorithm such that for every input* $k \in \mathbb{N}$ *it returns the sentence "*$k \in \mathcal{X}$*" or the sentence "*$k \notin \mathcal{X}$*" and every returned sentence is true when k is sufficiently large. The simplest known such algorithm may return a false sentence only if k is small.*

Definition 2. *We say that an integer n is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$, *if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$, *cf.* [7] *and* [8].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [4]–[6].

Definition 3. *Let* Φ *denote the implication:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement Φ. In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *There is no explicitly known threshold number of* $\mathcal{P}_{n^2+1}$. *It means that there is no explicitly known integer k such that* $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.

Proving the statement Φ will falsify Statement 1. Statement 1 cannot be formalized in *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2–3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$.

2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

Definition 4. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Statement 2. *The set* $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ *satisfies conditions* `(1)-(4)`.

Proof. Condition `(1)` holds as $\mathcal{X} \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β, see [2]. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

Open Problem 1. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1)-(5)`*?*

Theorem 1. *Assume that for every positive integers b and s, at some future day, machine computations will be possible on every integers from the interval* $[-b, b]$ *and this will be possible with the speed of s FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy conditions* `(1)-(4)` *forever.*

Proof. The proof goes by contradiction. Since conditons `(2)-(4)` will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots \qquad \texttt{(T)}$$

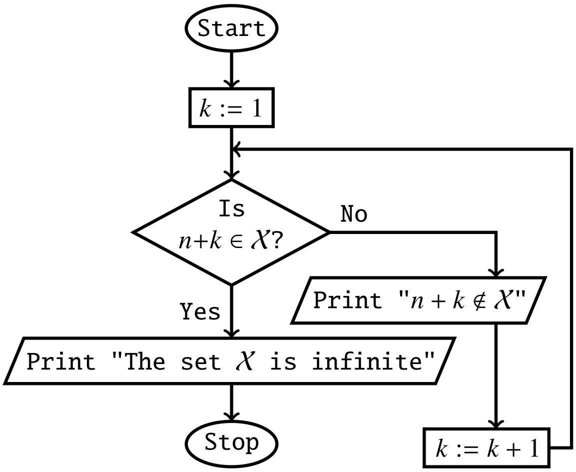

Fig. 1 An algorithm whose execution never terminates if the set $\mathcal{X}$ is finite

The sentences from the sequence `(T)` and our assumptions alone imply that for every explicitly given integer $m > n$, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in condition `(1)`. □

3. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

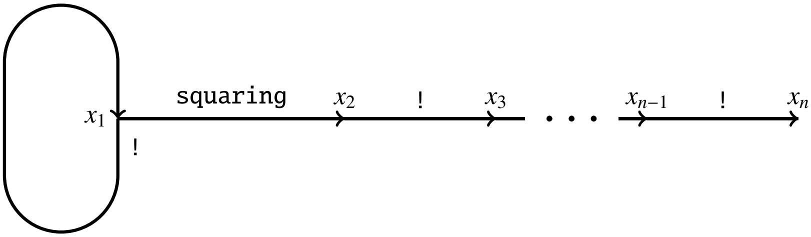

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 2. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\big(f(1), \ldots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k :\ i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \ldots, n\}\Big\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement Ψ_9, cf. Conjecture 1 in Section 4.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. *(Wilson's theorem,* [1, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)! + 1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

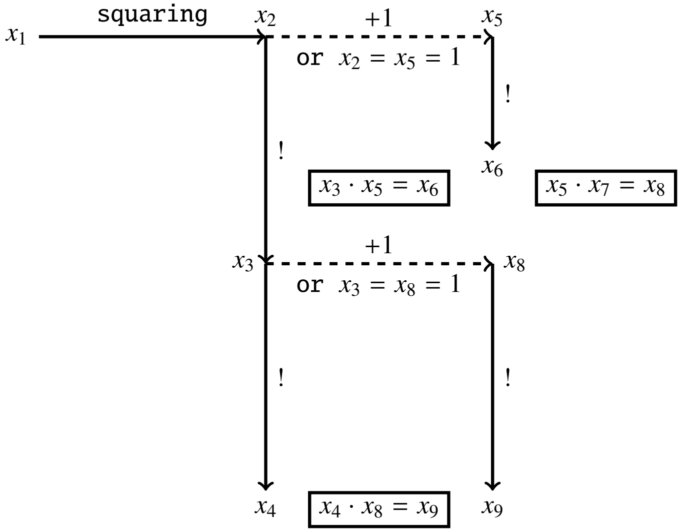

Fig. 3 Construction of the system $\mathcal{A}$

Lemma 5. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{array}{lcl} x_2 & = & x_1^2 \\ x_3 & = & (x_1^2)! \\ x_4 & = & ((x_1^2)!)! \\ x_5 & = & x_1^2 + 1 \\ x_6 & = & (x_1^2 + 1)! \\ x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 & = & (x_1^2)! + 1 \\ x_9 & = & ((x_1^2)! + 1)! \end{array}$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Lemma 6. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system $\mathcal{A}$.*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement Φ.*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 3. *The set* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *satisfies conditions* (1)-(3) *and* (5). *The statement* Φ *implies that the set* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *satisfies condition* (4).

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [5]. These two facts imply condition (1). Condition (3) holds trivially. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds. Condition (5) holds as the algorithm in Figure 4 returns a true sentence for every input $k \in \mathbb{N} \setminus \{1, 2\}$.

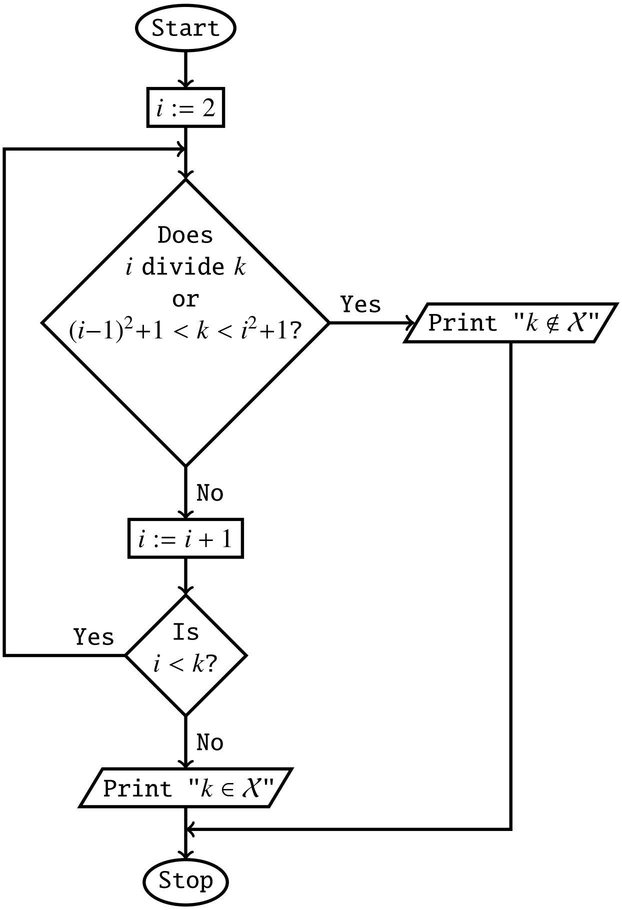

Fig. 4 An algorithm that satisfies condition (5) for $\mathcal{X} = \mathcal{P}_{n^2+1}$

Suppose that the statement Φ is true. This means that β is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. Thus condition (4) holds. □

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References

[1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory,* 2nd ed., CRC Press, Boca Raton, FL, 2016.

[2] S. Lloyd, *Ultimate physical limits to computation,* Nature 406 (2000), 1047–1054, `http://doi.org/10.1038/35023282`.

[3] W. Marciszewski, *Logic, modern, history of,* in: *Dictionary of logic as applied in the study of language* (ed. W. Marciszewski), pp. 183–200, Springer, Dordrecht, 1981.

[4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002496, Primes of the form* $n^2 + 1$, `http://oeis.org/A002496`.

[5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A083844, Number of primes of the form* $x^2 + 1 < 10^n$, `http://oeis.org/A083844`.

[6] Wolfram MathWorld, *Landau's Problems,* `http://mathworld.wolfram.com/LandausProblems.html`.

[7] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity,* Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, `http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm`.

[8] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer,* in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

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