

# The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in *ZFC* as it refers to our current knowledge on $\mathcal{X}$

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**Abstract.** Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite. Let  $\beta = (((24!)!)!)!$ , and let  $\Phi$  denote the implication:  $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$ . We heuristically justify the statement  $\Phi$  without invoking Landau's conjecture. Open problem: Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies conditions (1)-(5)? (1) *There are a large number of elements of  $\mathcal{X}$  and it is conjectured that  $\mathcal{X}$  is infinite.* (2) *No known algorithm decides the finiteness/infiniteness of  $\mathcal{X}$ .* (3) *There is a known algorithm that for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ .* (4) *There is an explicitly known integer  $n$  such that  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .* (5)  *$\mathcal{X}$  is simply defined and we know an algorithm such that for every input  $k \in \mathbb{N}$  it returns the sentence " $k \in \mathcal{X}$ " or the sentence " $k \notin \mathcal{X}$ " and every returned sentence is true when  $k$  is sufficiently large. The simplest known such algorithm may return a false sentence only if  $k$  is small.* We prove: (i) the set  $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$  satisfies conditions (1)-(4), (ii) the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies conditions (1)-(3) and (5), (iii) the statement  $\Phi$  implies that the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies condition (4).

**Keywords:** computable set  $\mathcal{X} \subseteq \mathbb{N}$ , current knowledge on  $\mathcal{X}$ , explicitly known integer  $n$ , machine computations on large integers, mathematical statement about  $\mathcal{X}$  that refers to the current knowledge on  $\mathcal{X}$ , physical limits of computation.

## 1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

**Definition 1.** Conditions (1)–(5) concern sets  $\mathcal{X} \subseteq \mathbb{N}$ .

(1) There are a large number of elements of  $\mathcal{X}$  and it is conjectured that  $\mathcal{X}$  is infinite.

(2) No known algorithm decides the finiteness/infiniteness of  $\mathcal{X}$ .

(3) There is a known algorithm that for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ .

(4) There is an explicitly known integer  $n$  such that  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .

(5)  $\mathcal{X}$  is simply defined and we know an algorithm such that for every input  $k \in \mathbb{N}$  it returns the sentence " $k \in \mathcal{X}$ " or the sentence " $k \notin \mathcal{X}$ " and every returned sentence is true when  $k$  is sufficiently large. The simplest known such algorithm may return a false sentence only if  $k$  is small.

**Definition 2.** We say that an integer  $n$  is a threshold number of a set  $\mathcal{X} \subseteq \mathbb{N}$ , if  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ , cf. [7] and [8].

If a set  $\mathcal{X} \subseteq \mathbb{N}$  is empty or infinite, then any integer  $n$  is a threshold number of  $\mathcal{X}$ . If a set  $\mathcal{X} \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $\mathcal{X}$  form the set  $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$ .

Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite, see [4]–[6].

**Definition 3.** Let  $\Phi$  denote the implication:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement  $\Phi$ . In Section 4, we heuristically justify the statement  $\Phi$  without invoking Landau's conjecture.

**Statement 1.** There is no explicitly known threshold number of  $\mathcal{P}_{n^2+1}$ . It means that there is no explicitly known integer  $k$  such that  $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$ .

Proving the statement  $\Phi$  will falsify Statement 1. Statement 1 cannot be formalized in *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2–3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets  $\mathcal{X} \subseteq \mathbb{N}$ .

## 2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

**Definition 4.** Let  $\beta = (((24!)!)!)!$ .

**Lemma 1.**  $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta)))))) \approx 1.42298$ .

*Proof.* We ask Wolfram Alpha at <http://wolframalpha.com>. □

**Statement 2.** The set  $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$  satisfies conditions (1)–(4).

*Proof.* Condition (1) holds as  $\mathcal{X} \supseteq \{0, \dots, \beta\}$  and the set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $\beta$ , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

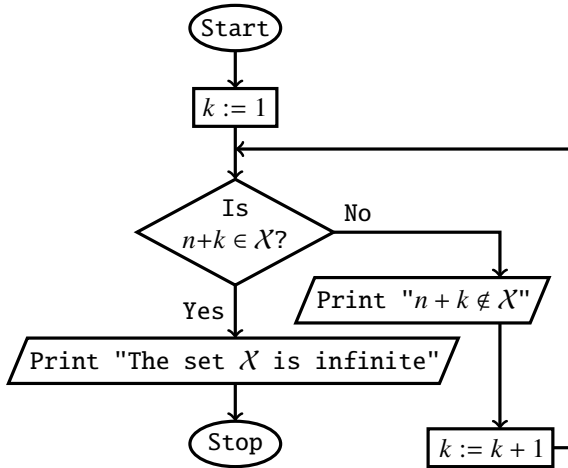
is empty or infinite, the integer  $\beta$  is a threshold number of  $\mathcal{X}$ . Thus condition (4) holds.  $\square$

**Open Problem 1.** Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies conditions (1)–(5)?

**Theorem 1.** Assume that for every positive integers  $b$  and  $s$ , at some future day, machine computations will be possible on every integers from the interval  $[-b, b]$  and this will be possible with the speed of  $s$  FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set  $\mathcal{X} \subseteq \mathbb{N}$  will satisfy conditions (1)–(4) forever.

*Proof.* The proof goes by contradiction. Since conditions (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots \quad (\text{T})$$



**Fig. 1** An algorithm whose execution never terminates if the set  $\mathcal{X}$  is finite

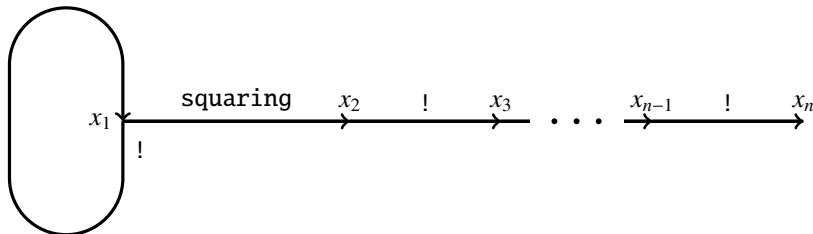
The sentences from the sequence (T) and our assumptions alone imply that for every explicitly given integer  $m > n$ , at some future day, a computer will be able to confirm in 1 second or less that  $(n, m] \cap \mathcal{X} = \emptyset$ . Thus, at some future day, numerical evidence will support the conjecture that the set  $\mathcal{X}$  is finite, contrary to the conjecture in condition (1).  $\square$

### 3. Number-theoretic statements $\Psi_n$

Let  $f(1) = 2$ ,  $f(2) = 4$ , and let  $f(n+1) = f(n)!$  for every integer  $n \geq 2$ . Let  $\mathcal{U}_1$  denote the system of equations which consists of the equation  $x_1! = x_1$ . For an integer  $n \geq 2$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system  $\mathcal{U}_n$ .



**Fig. 2** Construction of the system  $\mathcal{U}_n$

**Lemma 2.** For every positive integer  $n$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(f(1), \dots, f(n))$ .

Let

$$B_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer  $n$ , let  $\Psi_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq f(n)$ .* The statement  $\Psi_n$  says that for subsystems of  $B_n$  with a finite number of solutions, the largest known solution is indeed the largest possible. The statements  $\Psi_1$  and  $\Psi_2$  hold trivially. There is no reason to assume the validity of the statement  $\Psi_9$ , cf. Conjecture 1 in Section 4.

**Theorem 2.** For every statement  $\Psi_n$ , the bound  $f(n)$  cannot be decreased.

*Proof.* It follows from Lemma 2 because  $\mathcal{U}_n \subseteq B_n$ . □

**Theorem 3.** For every integer  $n \geq 2$ , the statement  $\Psi_{n+1}$  implies the statement  $\Psi_n$ .

*Proof.* If a system  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then for every integer  $i \in \{1, \dots, n\}$  the system  $\mathcal{S} \cup \{x_i! = x_{n+1}\}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_{n+1}$ . The statement  $\Psi_{n+1}$  implies that  $x_i! = x_{n+1} \leq f(n+1) = f(n)!$ . Hence,  $x_i \leq f(n)$ . □

**Theorem 4.** Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

#### 4. A conjectural solution to Open Problem 1

**Lemma 3.** For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if

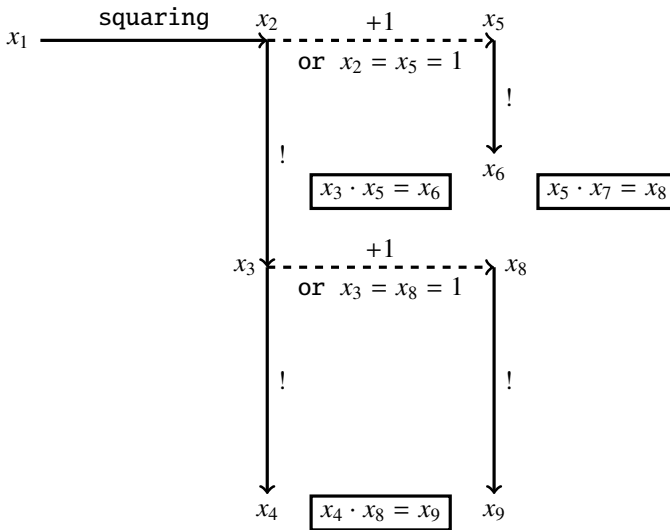
$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** (Wilson's theorem, [1, p. 89]). For every integer  $x \geq 2$ ,  $x$  is prime if and only if  $x$  divides  $(x - 1)! + 1$ .

Let  $\mathcal{A}$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system  $\mathcal{A}$ .



**Fig. 3** Construction of the system  $\mathcal{A}$

**Lemma 5.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 3, for every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 5 follows from Lemma 4.  $\square$

**Lemma 6.** There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ , which solve the system  $\mathcal{A}$  and satisfy  $x_1 = 1$ . This is true as every such tuple  $(x_1, \dots, x_9)$  satisfies  $x_1, \dots, x_9 \in \{1, 2\}$ .

*Proof.* The equality  $x_1 = 1$  implies that  $x_2 = x_1^2 = 1$ . Hence, for example,  $x_3 = x_2! = 1$ . Therefore,  $x_8 = x_3 + 1 = 2$  or  $x_8 = 1$ . Consequently,  $x_9 = x_8! \leq 2$ .  $\square$

**Conjecture 1.** The statement  $\Psi_9$  is true when is restricted to the system  $\mathcal{A}$ .

**Theorem 5.** Conjecture 1 proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $f(7)$ , then the set  $\mathcal{P}_{n^2+1}$  is infinite.

*Proof.* Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{A}$ . Since  $x_1^2 + 1 > f(7)$ , we obtain that  $x_1^2 \geq f(7)$ . Hence,  $(x_1^2)! \geq f(7)! = f(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality  $x_9 > f(9)$  imply that the system  $\mathcal{A}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas 5 and 6, the set  $\mathcal{P}_{n^2+1}$  is infinite.  $\square$

**Theorem 6.** Conjecture 1 implies the statement  $\Phi$ .

*Proof.* It follows from Theorem 5 and the equality  $f(7) = (((24!)!)!)!$ .  $\square$

**Theorem 7.** The statement  $\Phi$  implies Conjecture 1.

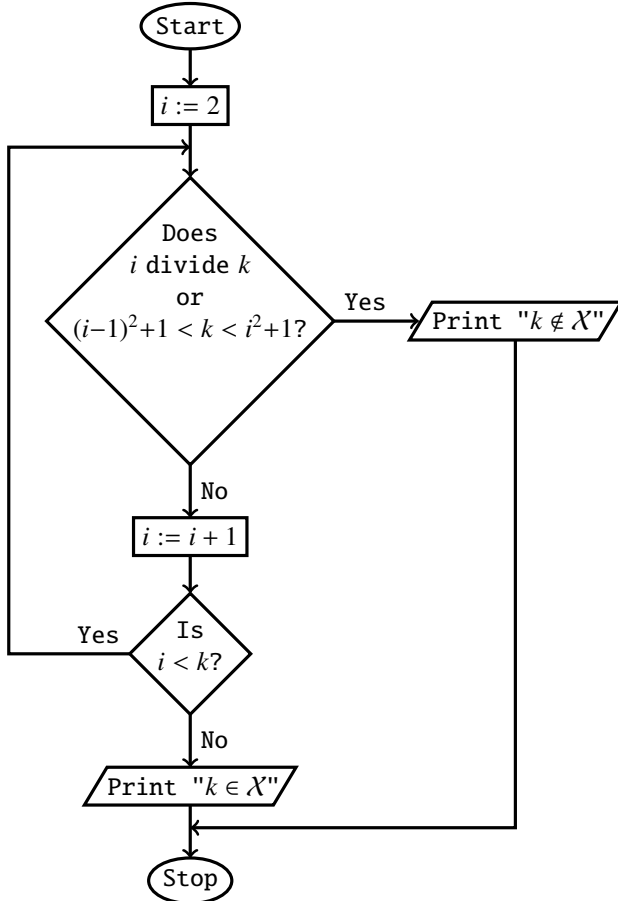
*Proof.* By Lemmas 5 and 6, if positive integers  $x_1, \dots, x_9$  solve the system  $\mathcal{A}$ , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or  $x_1, \dots, x_9 \in \{1, 2\}$ . In the first case, Lemma 5 and the statement  $\Phi$  imply that the inequality  $x_5 \leq (((24!)!)!)! = f(7)$  holds when the system  $\mathcal{A}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_9$ . Hence,  $x_2 = x_5 - 1 < f(7)$  and  $x_3 = x_2! < f(7)! = f(8)$ . Continuing this reasoning in the same manner, we can show that every  $x_i$  does not exceed  $f(9)$ .  $\square$

**Statement 3.** The set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies conditions (1)–(3) and (5). The statement  $\Phi$  implies that the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies condition (4).

*Proof.* The set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. There are 2199894223892 primes of the form  $n^2 + 1$  in the interval  $[2, 10^{28})$ , see [5]. These two facts imply condition (1). Condition (3) holds trivially. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $f(7) = (((24!)!)!) = \beta$ , see [2]. Thus condition (2) holds. Condition (5) holds as the algorithm in Figure 4 returns a true sentence for every input  $k \in \mathbb{N} \setminus \{1, 2\}$ .



**Fig. 4** An algorithm that satisfies condition (5) for  $\mathcal{X} = \mathcal{P}_{n^2+1}$

Suppose that the statement  $\Phi$  is true. This means that  $\beta$  is a threshold number of  $\mathcal{X} = \mathcal{P}_{n^2+1}$ . Thus condition (4) holds.  $\square$

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