

# The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in *ZFC* as it refers to our current knowledge on $\mathcal{X}$

Sławomir Kurpaska, Apoloniusz Tyszka

**Abstract.** Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite. Let  $\beta = (((24!)!)!)!$ , and let  $\Phi$  denote the implication:  $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$ . We heuristically justify the statement  $\Phi$  without invoking Landau's conjecture. Open problem: Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies conditions (1)-(5)? (1) *There are a large number of elements of  $\mathcal{X}$  and it is conjectured that  $\mathcal{X}$  is infinite.* (2) *No known algorithm decides the finiteness/infiniteness of  $\mathcal{X}$ .* (3) *There is a known algorithm that for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ .* (4) *There is an explicitly known integer  $n$  such that  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .* (5)  *$\mathcal{X}$  is simply defined and we know an algorithm such that for every input  $k \in \mathbb{N}$  it returns the sentence " $k \in \mathcal{X}$ " or the sentence " $k \notin \mathcal{X}$ " and every returned sentence is true when  $k$  is sufficiently large. The simplest known such algorithm may return a false sentence only if  $k$  is small.* We prove: (i) the set  $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$  satisfies conditions (1)-(4), (ii) the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies conditions (1)-(3) and (5), (iii) the statement  $\Phi$  implies that the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies condition (4).

**Keywords:** computable set  $\mathcal{X} \subseteq \mathbb{N}$ , current knowledge on  $\mathcal{X}$ , explicitly known integer  $n$ , machine computations on large integers, mathematical statement about  $\mathcal{X}$  that refers to the current knowledge on  $\mathcal{X}$ , physical limits of computation.

## 1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

**Definition 1.** Conditions (1)–(5) concern sets  $\mathcal{X} \subseteq \mathbb{N}$ .

(1) There are a large number of elements of  $\mathcal{X}$  and it is conjectured that  $\mathcal{X}$  is infinite.

(2) No known algorithm decides the finiteness/infiniteness of  $\mathcal{X}$ .

(3) There is a known algorithm that for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ .

(4) There is an explicitly known integer  $n$  such that  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .

(5)  $\mathcal{X}$  is simply defined and we know an algorithm such that for every input  $k \in \mathbb{N}$  it returns the sentence " $k \in \mathcal{X}$ " or the sentence " $k \notin \mathcal{X}$ " and every returned sentence is true when  $k$  is sufficiently large. The simplest known such algorithm may return a false sentence only if  $k$  is small.

**Definition 2.** We say that an integer  $n$  is a threshold number of a set  $\mathcal{X} \subseteq \mathbb{N}$ , if  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ , cf. [7] and [8].

If a set  $\mathcal{X} \subseteq \mathbb{N}$  is empty or infinite, then any integer  $n$  is a threshold number of  $\mathcal{X}$ . If a set  $\mathcal{X} \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $\mathcal{X}$  form the set  $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$ .

Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite, see [4]–[6].

**Definition 3.** Let  $\Phi$  denote the implication:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement  $\Phi$ . In Section 4, we heuristically justify the statement  $\Phi$  without invoking Landau's conjecture.

**Statement 1.** There is no explicitly known threshold number of  $\mathcal{P}_{n^2+1}$ . It means that there is no explicitly known integer  $k$  such that  $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$ .

Proving the statement  $\Phi$  will falsify Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Statements 2–3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets  $\mathcal{X} \subseteq \mathbb{N}$ .

## 2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

**Definition 4.** Let  $\beta = (((24!)!)!)!$ .

**Lemma 1.**  $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$ .

*Proof.* We ask Wolfram Alpha at <http://wolframalpha.com>. □

**Statement 2.** The set  $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$  satisfies conditions (1)–(4).

*Proof.* Condition (1) holds as  $\mathcal{X} \supseteq \{0, \dots, \beta\}$  and the set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $\beta$ , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

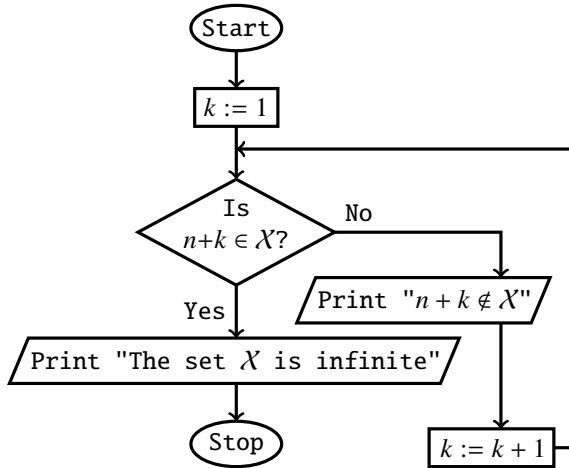
is empty or infinite, the integer  $\beta$  is a threshold number of  $\mathcal{X}$ . Thus condition (4) holds.  $\square$

**Open Problem 1.** Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies conditions (1)–(5)?

**Theorem 1.** Assume that for every positive integers  $b$  and  $s$ , at some future day, machine computations will be possible on every integers from the interval  $[-b, b]$  and this will be possible with the speed of  $s$  FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set  $\mathcal{X} \subseteq \mathbb{N}$  will satisfy conditions (1)–(4) forever.

*Proof.* The proof goes by contradiction. Since conditions (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots \quad (\text{T})$$



**Fig. 1** An algorithm whose execution never terminates if the set  $\mathcal{X}$  is finite

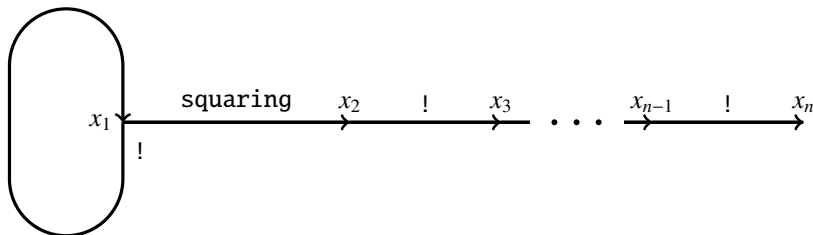
The sentences from the sequence (T) and our assumptions alone imply that for every explicitly given integer  $m > n$ , at some future day, a computer will be able to confirm in 1 second or less that  $(n, m] \cap \mathcal{X} = \emptyset$ . Thus, at some future day, numerical evidence will support the conjecture that the set  $\mathcal{X}$  is finite, contrary to the conjecture in condition (1).  $\square$

### 3. Number-theoretic statements $\Psi_n$

Let  $f(1) = 2$ ,  $f(2) = 4$ , and let  $f(n+1) = f(n)!$  for every integer  $n \geq 2$ . Let  $\mathcal{U}_1$  denote the system of equations which consists of the equation  $x_1! = x_1$ . For an integer  $n \geq 2$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system  $\mathcal{U}_n$ .



**Fig. 2** Construction of the system  $\mathcal{U}_n$

**Lemma 2.** For every positive integer  $n$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(f(1), \dots, f(n))$ .

Let

$$B_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer  $n$ , let  $\Psi_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq f(n)$ .* The statement  $\Psi_n$  says that for subsystems of  $B_n$  with a finite number of solutions, the largest known solution is indeed the largest possible. The statements  $\Psi_1$  and  $\Psi_2$  hold trivially. There is no reason to assume the validity of the statement  $\Psi_9$ , cf. Conjecture [1](#) in Section [4](#).

**Theorem 2.** For every statement  $\Psi_n$ , the bound  $f(n)$  cannot be decreased.

*Proof.* It follows from Lemma [2](#) because  $\mathcal{U}_n \subseteq B_n$ . □

**Theorem 3.** For every integer  $n \geq 2$ , the statement  $\Psi_{n+1}$  implies the statement  $\Psi_n$ .

*Proof.* If a system  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then for every integer  $i \in \{1, \dots, n\}$  the system  $\mathcal{S} \cup \{x_i! = x_{n+1}\}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_{n+1}$ . The statement  $\Psi_{n+1}$  implies that  $x_i! = x_{n+1} \leq f(n+1) = f(n)!$ . Hence,  $x_i \leq f(n)$ . □

**Theorem 4.** Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

#### 4. A conjectural solution to Open Problem 1

**Lemma 3.** For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if

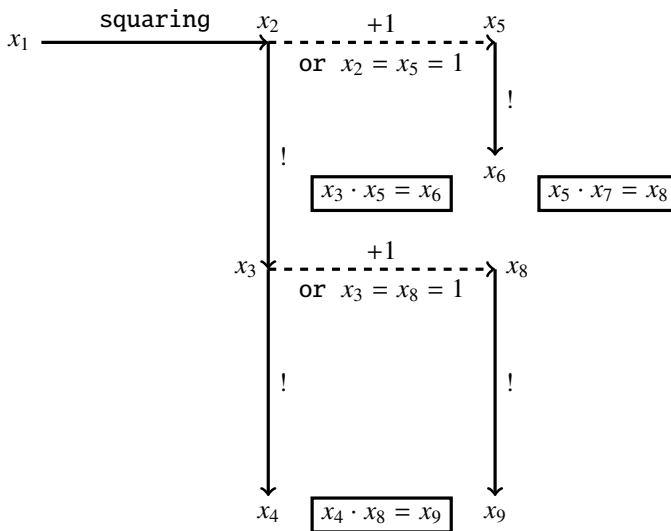
$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** (Wilson's theorem, [1] p. 89). For every integer  $x \geq 2$ ,  $x$  is prime if and only if  $x$  divides  $(x - 1)! + 1$ .

Let  $\mathcal{A}$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system  $\mathcal{A}$ .



**Fig. 3** Construction of the system  $\mathcal{A}$

**Lemma 5.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 3, for every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 5 follows from Lemma 4.  $\square$

**Lemma 6.** There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ , which solve the system  $\mathcal{A}$  and satisfy  $x_1 = 1$ . This is true as every such tuple  $(x_1, \dots, x_9)$  satisfies  $x_1, \dots, x_9 \in \{1, 2\}$ .

*Proof.* The equality  $x_1 = 1$  implies that  $x_2 = x_1^2 = 1$ . Hence, for example,  $x_3 = x_2! = 1$ . Therefore,  $x_8 = x_3 + 1 = 2$  or  $x_8 = 1$ . Consequently,  $x_9 = x_8! \leq 2$ .  $\square$

**Conjecture 1.** The statement  $\Psi_9$  is true when is restricted to the system  $\mathcal{A}$ .

**Theorem 5.** Conjecture 1 proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $f(7)$ , then the set  $\mathcal{P}_{n^2+1}$  is infinite.

*Proof.* Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{A}$ . Since  $x_1^2 + 1 > f(7)$ , we obtain that  $x_1^2 \geq f(7)$ . Hence,  $(x_1^2)! \geq f(7)! = f(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality  $x_9 > f(9)$  imply that the system  $\mathcal{A}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas 5 and 6, the set  $\mathcal{P}_{n^2+1}$  is infinite.  $\square$

**Theorem 6.** Conjecture 1 implies the statement  $\Phi$ .

*Proof.* It follows from Theorem 5 and the equality  $f(7) = (((24)!)!)!$ .  $\square$

**Theorem 7.** The statement  $\Phi$  implies Conjecture 1

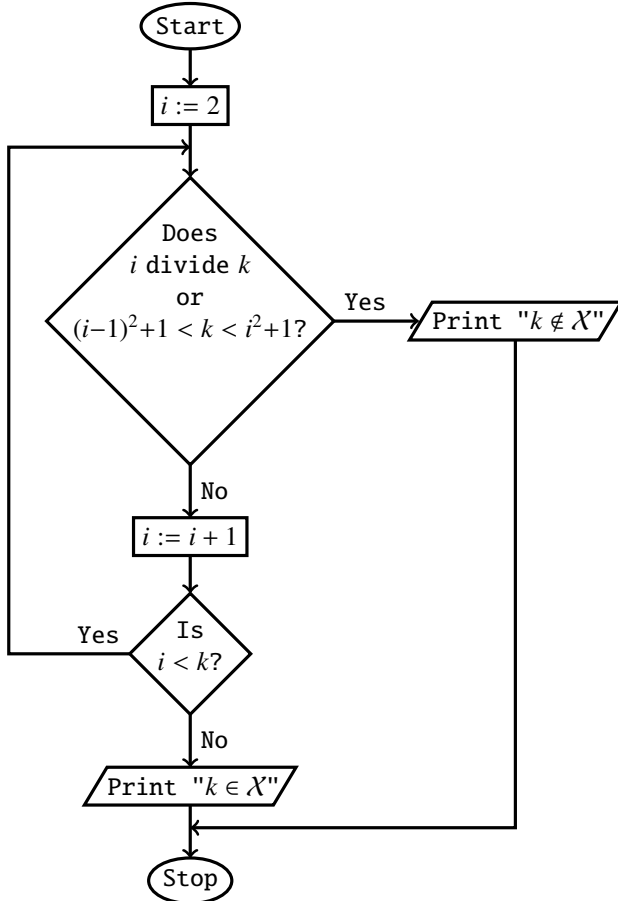
*Proof.* By Lemmas 5 and 6, if positive integers  $x_1, \dots, x_9$  solve the system  $\mathcal{A}$ , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or  $x_1, \dots, x_9 \in \{1, 2\}$ . In the first case, Lemma 5 and the statement  $\Phi$  imply that the inequality  $x_5 \leq (((24)!)!)! = f(7)$  holds when the system  $\mathcal{A}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_9$ . Hence,  $x_2 = x_5 - 1 < f(7)$  and  $x_3 = x_2! < f(7)! = f(8)$ . Continuing this reasoning in the same manner, we can show that every  $x_i$  does not exceed  $f(9)$ .  $\square$

**Statement 3.** The set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies conditions (1)–(3) and (5). The statement  $\Phi$  implies that the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies condition (4).

*Proof.* The set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. There are 2199894223892 primes of the form  $n^2 + 1$  in the interval  $[2, 10^{28})$ , see [5]. These two facts imply condition (1). Condition (3) holds trivially. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $f(7) = (((24!)!)!) = \beta$ , see [2]. Thus condition (2) holds. Condition (5) holds as the algorithm in Figure 4 returns a true sentence for every input  $k \in \mathbb{N} \setminus \{1, 2\}$ .



**Fig. 4** An algorithm that satisfies condition (5) for  $\mathcal{X} = \mathcal{P}_{n^2+1}$

Suppose that the statement  $\Phi$  is true. This means that  $\beta$  is a threshold number of  $\mathcal{X} = \mathcal{P}_{n^2+1}$ . Thus condition (4) holds.  $\square$

**Acknowledgment.** Sławomir Kurpaska prepared four diagrams in *TikZ*. Apoloniusz Tyszka wrote the article.

## References

- [1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [2] S. Lloyd, *Ultimate physical limits to computation*, *Nature* 406 (2000), 1047–1054, <http://doi.org/10.1038/35023282>.
- [3] W. Marciszewski, *Logic, modern, history of*, in: *Dictionary of logic as applied in the study of language* (ed. W. Marciszewski), pp. 183–200, Springer, Dordrecht, 1981.
- [4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A002496, *Primes of the form  $n^2 + 1$* , <http://oeis.org/A002496>.
- [5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A083844, *Number of primes of the form  $x^2 + 1 < 10^n$* , <http://oeis.org/A083844>.
- [6] Wolfram MathWorld, *Landau's Problems*, <http://mathworld.wolfram.com/LandausProblems.html>.
- [7] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity*, Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, <http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm>.
- [8] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer*, in: J. Cachro and K. Kijania-Placek (eds.), *Volume of Abstracts*, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

Sławomir Kurpaska  
Technical Faculty  
Hugo Kołłątaj University  
Balicka 116B, 30-149 Kraków, Poland  
E-mail: [rtkurpas@cyf-kr.edu.pl](mailto:rtkurpas@cyf-kr.edu.pl)

Apoloniusz Tyszka  
Technical Faculty  
Hugo Kołłątaj University  
Balicka 116B, 30-149 Kraków, Poland  
E-mail: [rttyszka@cyf-kr.edu.pl](mailto:rttyszka@cyf-kr.edu.pl)



# On *ZFC*-formulae $\varphi(x)$ for which we know a non-negative integer $n$ such that $\{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite

Apoloniusz Tyszk

## Abstract

Let  $\Gamma(k)$  denote  $(k-1)!$ , and let  $\Gamma_n(k)$  denote  $(k-1)!$ , where  $n \in \{3, \dots, 16\}$  and  $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$ . For an integer  $n \in \{3, \dots, 16\}$ , let  $\Sigma_n$  denote the following statement: if a system of equations  $\mathcal{S} \subseteq \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  with  $\Gamma$  instead of  $\Gamma_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then every tuple  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$  that solves the original system  $\mathcal{S}$  satisfies  $x_1, \dots, x_n \leq 2^{2^{n-2}}$ . Our hypothesis claims that the statements  $\Sigma_3, \dots, \Sigma_{16}$  are true. The statement  $\Sigma_6$  proves the following implication: if the equation  $x(x+1) = y!$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(1, 2), (2, 3)\}$ . The statement  $\Sigma_6$  proves the following implication: if the equation  $x! + 1 = y^2$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ . The statement  $\Sigma_9$  implies the infinitude of primes of the form  $n^2 + 1$ . The statement  $\Sigma_9$  implies that any prime of the form  $n! + 1$  with  $n \geq 2^{2^{9-3}}$  proves the infinitude of primes of the form  $n! + 1$ . The statement  $\Sigma_{14}$  implies the infinitude of twin primes. The statement  $\Sigma_{16}$  implies the infinitude of Sophie Germain primes.

**Key words and phrases:** Brocard's problem, Brocard-Ramanujan equation  $x! + 1 = y^2$ , composite Fermat numbers, decidability in the limit, Erdős' equation  $x(x+1) = y!$ , finiteness of a set, infiniteness of a set, prime numbers of the form  $n^2 + 1$ , prime numbers of the form  $n! + 1$ , single query to an oracle for the halting problem, Sophie Germain primes, twin primes.

**2010 Mathematics Subject Classification:** 03B30, 11A41.

## 1 Introduction and basic lemmas

The phrase “we know a non-negative integer  $n$ ” in the title means that we know an algorithm which returns  $n$ . The title of the article cannot be formalised in *ZFC* because the phrase “we know a non-negative integer  $n$ ” refers to currently known non-negative integers  $n$  with some property. A formally stated title may look like this: *On ZFC-formulae  $\varphi(x)$  for which there exists a non-negative integer  $n$  such that ZFC proves that*

$$\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) < \infty \implies \{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$$

Unfortunately, this formulation admits formulae  $\varphi(x)$  without any known non-negative integer  $n$  such that *ZFC* proves the above implication.

**Lemma 1.** *For every non-negative integer  $n$ ,  $\text{card}(\{x \in \mathbb{N} : x \leq n - 1\}) = n$ .*

**Corollary 1.** *The title altered to “On ZFC-formulae  $\varphi(x)$  for which we know a non-negative integer  $n$  such that  $\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) \leq n$  if the set  $\{x \in \mathbb{N} : \varphi(x)\}$  is finite” involves a weaker assumption on  $\varphi(x)$ .*

**Lemma 2.** *For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Let  $\Gamma(k)$  denote  $(k - 1)!$ .

**Lemma 3.** For every positive integers  $x$  and  $y$ ,  $x \cdot \Gamma(x) = \Gamma(y)$  if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** For every non-negative integers  $b$  and  $c$ ,  $b + 1 = c$  if and only if  $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$ .

**Lemma 5.** (Wilson's theorem, [8] p. 89]). For every positive integer  $x$ ,  $x$  divides  $(x - 1)! + 1$  if and only if  $x = 1$  or  $x$  is prime.

## 2 Subsets of $\mathbb{N}$ and their threshold numbers

We say that a non-negative integer  $m$  is a threshold number of a set  $X \subseteq \mathbb{N}$ , if  $X$  is infinite if and only if  $X$  contains an element greater than  $m$ , cf. [24] and [25]. If a set  $X \subseteq \mathbb{N}$  is empty or infinite, then any non-negative integer  $m$  is a threshold number of  $X$ . If a set  $X \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $X$  form the set  $\{\max(X), \max(X) + 1, \max(X) + 2, \dots\}$ .

It is conjectured that the set of prime numbers of the form  $n^2 + 1$  is infinite, see [14], pp. 37–38]. It is conjectured that the set of prime numbers of the form  $n! + 1$  is infinite, see [3], p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [14], p. 39]. It is conjectured that the set of composite numbers of the form  $2^{2^n} + 1$  is infinite, see [10], p. 23] and [11], pp. 158–159]. A prime  $p$  is said to be a Sophie Germain prime if both  $p$  and  $2p + 1$  are prime, see [22]. It is conjectured that the set of Sophie Germain primes is infinite, see [17], p. 330]. For each of these sets, we do not know any threshold number.

The following statement:

for every non-negative integer  $n$  there exist

$$\text{prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in [10^n, 10^n + 1] \quad (1)$$

is a  $\Pi_1$  statement which strengthens the twin prime conjecture, see [4], p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger  $\Pi_1$  statements, see [1]. Statement (1) is equivalent to the non-halting of a Turing machine. If a set  $X \subseteq \mathbb{N}$  is computable and we know a threshold number of  $X$ , then the infinity of  $X$  is equivalent to the halting of a Turing machine.

The height of a rational number  $\frac{p}{q}$  is denoted by  $H\left(\frac{p}{q}\right)$  and equals  $\max(|p|, |q|)$  provided  $\frac{p}{q}$  is written in lowest terms. The height of a rational tuple  $(x_1, \dots, x_n)$  is denoted by  $H(x_1, \dots, x_n)$  and equals  $\max(H(x_1), \dots, H(x_n))$ .

**Lemma 6.** The equation  $x^5 - x = y^2 - y$  has only finitely many rational solutions, see [13] p. 212]. The known rational solutions are  $(x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left(\frac{1}{4}, \frac{15}{32}\right), \left(\frac{1}{4}, \frac{17}{32}\right), \left(-\frac{15}{16}, -\frac{185}{1024}\right), \left(-\frac{15}{16}, \frac{1209}{1024}\right)$ , and the existence of other solutions is an open question, see [18] pp. 223–224].

**Corollary 2.** The set  $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$  is finite. We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{T}$ . We do not know any algorithm which returns a threshold number of  $\mathcal{T}$ .

Let  $\mathcal{L}$  denote the following system of equations:

$$\begin{cases} x^2 + y^2 = s^2 \\ x^2 + z^2 = t^2 \\ y^2 + z^2 = u^2 \\ x^2 + y^2 + z^2 = v^2 \end{cases}$$

Let

$$\mathcal{F} = \left\{ n \in \mathbb{N} \setminus \{0\} : \left( \text{the system } \mathcal{L} \text{ has no solutions in } \{1, \dots, n\}^7 \right) \wedge \right. \\ \left. \left( \text{the system } \mathcal{L} \text{ has a solution in } \{1, \dots, n+1\}^7 \right) \right\}$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

**Lemma 7.** ([21]). *No perfect cuboids are known.*

**Corollary 3.** *We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{F}$ . ZFC proves that  $\text{card}(\mathcal{F}) \in \{0, 1\}$ . We do not know any algorithm which returns  $\text{card}(\mathcal{F})$ . We do not know any algorithm which returns a threshold number of  $\mathcal{F}$ .*

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, & \text{if } \sin\left(999999\right) < 0 \\ \mathbb{N} \cap \left[ 0, \sin\left(999999\right) \cdot 999999 \right) & \text{otherwise} \end{cases}$$

We do not know whether or not the set  $\mathcal{H}$  is finite.

**Proposition 1.** *The number  $999999$  is a threshold number of  $\mathcal{H}$ . We know an algorithm which decides the equality  $\mathcal{H} = \mathbb{N}$ . If  $\mathcal{H} \neq \mathbb{N}$ , then the set  $\mathcal{H}$  consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{H}$ .*

Let

$$\mathcal{K} = \begin{cases} \{n\}, & \text{if } (n \in \mathbb{N}) \wedge (2^{\aleph_0} = \aleph_{n+1}) \\ \{0\}, & \text{if } 2^{\aleph_0} \geq \aleph_{\omega} \end{cases}$$

**Proposition 2.** *ZFC proves that  $\text{card}(\mathcal{K}) = 1$ . If ZFC is consistent, then for every  $n \in \mathbb{N}$  the sentences "n is a threshold number of  $\mathcal{K}$ " and "n is not a threshold number of  $\mathcal{K}$ " are not provable in ZFC.*

*Proof.* It suffices to observe that  $2^{\aleph_0}$  can attain every value from the set  $\{\aleph_1, \aleph_2, \aleph_3, \dots\}$ , see [7] and [9] p. 232]. □

### 3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

**Theorem 1.** ([5] p. 35]. *There exists a polynomial  $D(x_1, \dots, x_m)$  with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation  $D(x_1, \dots, x_m) = 0$  is solvable in non-negative integers" and "The equation  $D(x_1, \dots, x_m) = 0$  is not solvable in non-negative integers" are not provable in ZFC.*

Let  $\mathcal{Y}$  denote the set of all non-negative integers  $k$  such that the equation  $D(x_1, \dots, x_m) = 0$  has no solutions in  $\{0, \dots, k\}^m$ . Since the set  $\{0, \dots, k\}^m$  is finite, we know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{Y}$ . Theorem 1 implies the next theorem.

**Theorem 2.** For every  $n \in \mathbb{N}$ , ZFC proves that  $n \in \mathcal{Y}$ . If ZFC is arithmetically consistent, then the sentences “ $\mathcal{Y}$  is finite” and “ $\mathcal{Y}$  is infinite” are not provable in ZFC. If ZFC is arithmetically consistent, then for every  $n \in \mathbb{N}$  the sentences “ $n$  is a threshold number of  $\mathcal{Y}$ ” and “ $n$  is not a threshold number of  $\mathcal{Y}$ ” are not provable in ZFC.

Let  $\mathcal{E}$  denote the set of all non-negative integers  $k$  such that the equation  $D(x_1, \dots, x_m) = 0$  has a solution in  $\{0, \dots, k\}^m$ . Since the set  $\{0, \dots, k\}^m$  is finite, we know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{E}$ . Theorem 1 implies the next theorem.

**Theorem 3.** The set  $\mathcal{E}$  is empty or infinite. In both cases, every non-negative integer  $n$  is a threshold number of  $\mathcal{E}$ . If ZFC is arithmetically consistent, then the sentences “ $\mathcal{E}$  is empty”, “ $\mathcal{E}$  is not empty”, “ $\mathcal{E}$  is finite”, and “ $\mathcal{E}$  is infinite” are not provable in ZFC.

Let

$$\mathcal{V} = \left\{ n \in \mathbb{N} : \left( \text{the polynomial } D(x_1, \dots, x_m) \text{ has no solutions in } \{0, \dots, n\}^m \right) \wedge \right. \\ \left. \left( \text{the polynomial } D(x_1, \dots, x_m) \text{ has a solution in } \{0, \dots, n+1\}^m \right) \right\}$$

Since the sets  $\{0, \dots, n\}^m$  and  $\{0, \dots, n+1\}^m$  are finite, we know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{V}$ . Theorem 1 implies the next theorem.

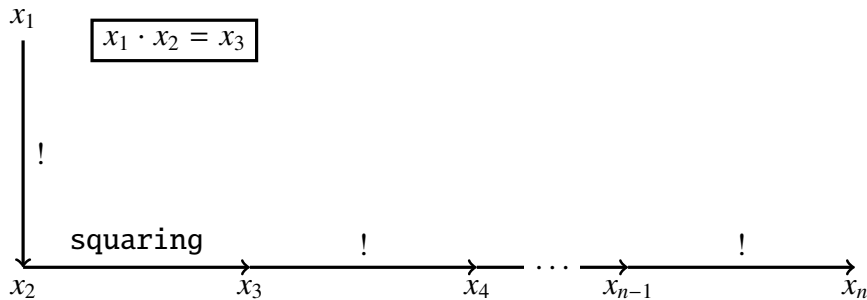
**Theorem 4.** ZFC proves that  $\text{card}(\mathcal{V}) \in \{0, 1\}$ . For every  $n \in \mathbb{N}$ , ZFC proves that  $n \notin \mathcal{V}$ . ZFC does not prove the emptiness of  $\mathcal{V}$ , if ZFC is arithmetically consistent. For every  $n \in \mathbb{N}$ , the sentence “ $n$  is a threshold number of  $\mathcal{V}$ ” is not provable in ZFC, if ZFC is arithmetically consistent.

## 4 Hypothetical statements $\Psi_3, \dots, \Psi_{16}$

For an integer  $n \geq 3$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{2\} \quad x_i! = x_{i+1} \\ x_1 \cdot x_2 = x_3 \\ x_2 \cdot x_2 = x_3 \end{array} \right\}$$

The diagram in Figure 1 illustrates the construction of the system  $\mathcal{U}_n$ .



**Fig. 1** Construction of the system  $\mathcal{U}_n$

Let  $g(3) = 4$ , and let  $g(n+1) = g(n)!$  for every integer  $n \geq 3$ .

**Lemma 8.** For every integer  $n \geq 3$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(2, 2, g(3), \dots, g(n))$ .

Let

$$B_n = \left\{ x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k) \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\} \right\}$$

For an integer  $n \geq 3$ , let  $\Psi_n$  denote the following statement: if a system of equations  $\mathcal{S} \subseteq B_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq g(n)$ . The statement  $\Psi_n$  says that for subsystems of  $B_n$  the largest known solution is indeed the largest possible.

**Hypothesis 1.** The statements  $\Psi_3, \dots, \Psi_{16}$  are true.

**Proposition 3.** Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

**Proposition 4.** For every statement  $\Psi_n$ , the bound  $g(n)$  cannot be decreased.

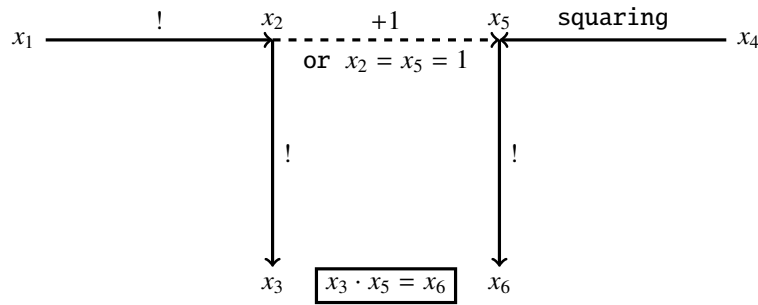
*Proof.* It follows from Lemma 8 because  $\mathcal{U}_n \subseteq B_n$ . □

## 5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let  $\mathcal{A}$  denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system  $\mathcal{A}$ .



**Fig. 2** Construction of the system  $\mathcal{A}$

**Lemma 9.** For every  $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, x_3, x_5, x_6$  if and only if  $x_1! + 1 = x_4^2$ . In this case, the integers  $x_2, x_3, x_5, x_6$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

*Proof.* It follows from Lemma 2. □

It is conjectured that  $x! + 1$  is a perfect square only for  $x \in \{4, 5, 7\}$ , see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation  $x! + 1 = y^2$ , see [15].

**Theorem 5.** If the equation  $x_1! + 1 = x_4^2$  has only finitely many solutions in positive integers, then the statement  $\Psi_6$  guarantees that each such solution  $(x_1, x_4)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ .

*Proof.* Suppose that the antecedent holds. Let positive integers  $x_1$  and  $x_4$  satisfy  $x_1! + 1 = x_4^2$ . Then,  $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$ . By Lemma 9, the system  $\mathcal{A}$  is solvable in positive integers  $x_2, x_3, x_5, x_6$ . Since  $\mathcal{A} \subseteq B_6$ , the statement  $\Psi_6$  implies that  $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$ . Hence,  $x_1! + 1 \leq g(5) = g(4)!$ . Consequently,  $x_1 < g(4) = 24$ . If  $x_1 \in \{1, \dots, 23\}$ , then  $x_1! + 1$  is a perfect square only for  $x_1 \in \{4, 5, 7\}$ . □

## 6 Are there infinitely many prime numbers of the form $n^2 + 1$ ?

Edmund Landau's conjecture states that there are infinitely many primes of the form  $n^2 + 1$ , see [14, pp. 37–38]. Let  $\mathcal{B}$  denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system  $\mathcal{B}$ .

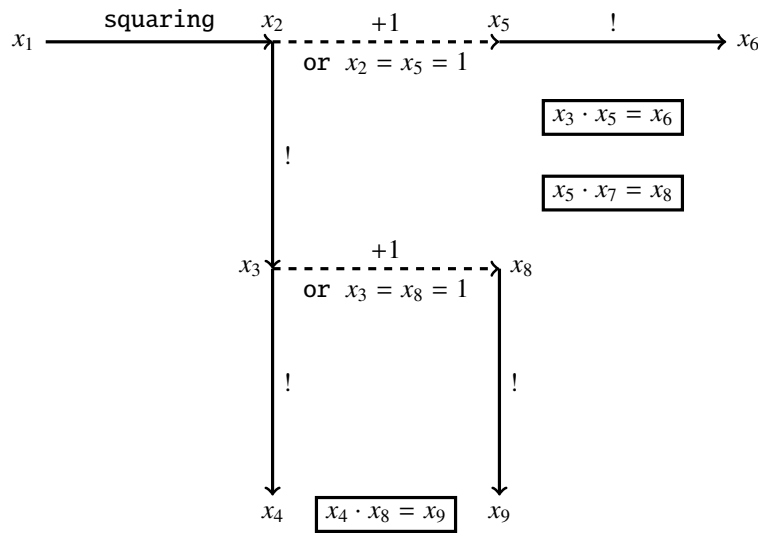


Fig. 3 Construction of the system  $\mathcal{B}$

**Lemma 10.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{B}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 2, for every integer  $x_1 \geq 2$ , the system  $\mathcal{B}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 10 follows from Lemma 5.  $\square$

**Lemma 11.** There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  which solve the system  $\mathcal{B}$  and satisfy  $x_1 = 1$ .

*Proof.* If a tuple  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  solves the system  $\mathcal{B}$  and  $x_1 = 1$ , then  $x_1, \dots, x_9 \leq 2$ . Indeed,  $x_1 = 1$  implies that  $x_2 = x_1^2 = 1$ . Hence, for example,  $x_3 = x_2! = 1$ . Therefore,  $x_8 = x_3 + 1 = 2$  or  $x_8 = 1$ . Consequently,  $x_9 = x_8! \leq 2$ .  $\square$

**Theorem 6.** *The statement  $\Psi_9$  proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $g(7)$ , then there are infinitely many primes of the form  $n^2 + 1$ .*

*Proof.* Suppose that the antecedent holds. By Lemma [10](#), there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{B}$ . Since  $x_1^2 + 1 > g(7)$ , we obtain that  $x_1^2 \geq g(7)$ . Hence,  $(x_1^2)! \geq g(7)! = g(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)$$

Since  $\mathcal{B} \subseteq B_9$ , the statement  $\Psi_9$  and the inequality  $x_9 > g(9)$  imply that the system  $\mathcal{B}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas [10](#) and [11](#), there are infinitely many primes of the form  $n^2 + 1$ .  $\square$

**Corollary 4.** *Let  $X_9$  denote the set of primes of the form  $n^2 + 1$ . The statement  $\Psi_9$  implies that we know an algorithm such that it returns a threshold number of  $X_9$ , and this number equals  $\max(X_9)$ , if  $X_9$  is finite. Assuming the statement  $\Psi_9$ , a single query to an oracle for the halting problem decides the infinity of  $X_9$ . Assuming the statement  $\Psi_9$ , the infinity of  $X_9$  is decidable in the limit.*

*Proof.* We consider an algorithm which computes  $\max(X_9 \cap [1, g(7)])$ .  $\square$

## 7 Are there infinitely many prime numbers of the form $n! + 1$ ?

It is conjectured that there are infinitely many primes of the form  $n! + 1$ , see [\[3\]](#) p. 443].

**Theorem 7.** *(cf. Theorem [11](#)). The statement  $\Psi_9$  proves the following implication: if there exists an integer  $x_1 \geq g(6)$  such that  $x_1! + 1$  is prime, then there are infinitely many primes of the form  $n! + 1$ .*

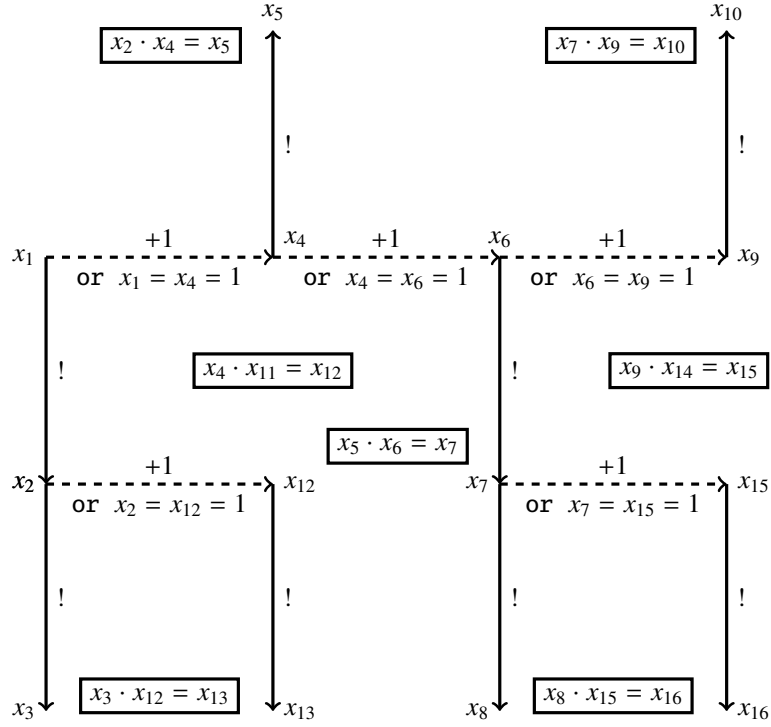
*Proof.* We leave the analogous proof to the reader.  $\square$

## 8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [\[14\]](#) p. 39]. Let  $C$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{array} \right.$$

Lemma [2](#) and the diagram in Figure 4 explain the construction of the system  $C$ .



**Fig. 4** Construction of the system  $C$

**Lemma 12.** For every  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ , the system  $C$  is solvable in positive integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  if and only if  $x_4$  and  $x_9$  are prime and  $x_4 + 2 = x_9$ . In this case, the integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  are uniquely determined by the following equalities:

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

*Proof.* By Lemma 2, for every  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ , the system  $C$  is solvable in positive integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | ((x_4 - 1)! + 1)) \wedge (x_9 | ((x_9 - 1)! + 1))$$

Hence, the claim of Lemma 12 follows from Lemma 5.  $\square$

**Lemma 13.** There are only finitely many tuples  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  which solve the system  $C$  and satisfy  $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$ .



*Proof.* If a tuple  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  solves the system  $C$  and  $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$ , then  $x_1, \dots, x_{16} \leq 7!$ . Indeed, for example, if  $x_4 = 2$  then  $x_6 = x_4 + 1 = 3$ . Hence,  $x_7 = x_6! = 6$ . Therefore,  $x_{15} = x_7 + 1 = 7$ . Consequently,  $x_{16} = x_{15}! = 7!$ .  $\square$

**Theorem 8.** *The statement  $\Psi_{16}$  proves the following implication: if there exists a twin prime greater than  $g(14)$ , then there are infinitely many twin primes.*

*Proof.* Suppose that the antecedent holds. Then, there exist prime numbers  $x_4$  and  $x_9$  such that  $x_9 = x_4 + 2 > g(14)$ . Hence,  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ . By Lemma [12](#), there exists a unique tuple  $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$  such that the tuple  $(x_1, \dots, x_{16})$  solves the system  $C$ . Since  $x_9 > g(14)$ , we obtain that  $x_9 - 1 \geq g(14)$ . Therefore,  $(x_9 - 1)! \geq g(14)! = g(15)$ . Hence,  $(x_9 - 1)! + 1 > g(15)$ . Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since  $C \subseteq B_{16}$ , the statement  $\Psi_{16}$  and the inequality  $x_{16} > g(16)$  imply that the system  $C$  has infinitely many solutions in positive integers  $x_1, \dots, x_{16}$ . According to Lemmas [12](#) and [13](#), there are infinitely many twin primes.  $\square$

**Corollary 5.** (cf. [\[6\]](#)). *Let  $\mathcal{X}_{16}$  denote the set of twin primes. The statement  $\Psi_{16}$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{X}_{16}$ , and this number equals  $\max(\mathcal{X}_{16})$ , if  $\mathcal{X}_{16}$  is finite. Assuming the statement  $\Psi_{16}$ , a single query to an oracle for the halting problem decides the infinity of  $\mathcal{X}_{16}$ . Assuming the statement  $\Psi_{16}$ , the infinity of  $\mathcal{X}_{16}$  is decidable in the limit.*

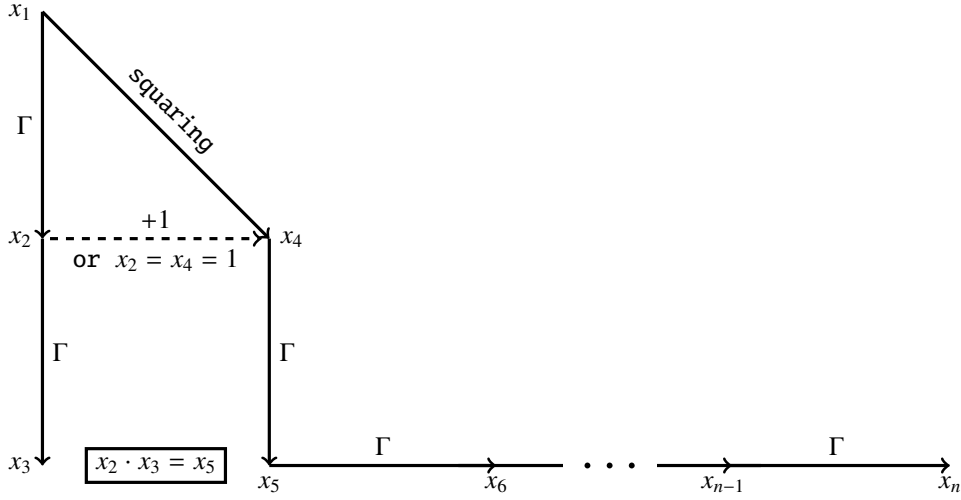
*Proof.* We consider an algorithm which computes  $\max(\mathcal{X}_{16} \cap [1, g(14)])$ .  $\square$

## 9 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ and their consequences

Let  $\lambda(5) = \Gamma(25)$ , and let  $\lambda(n + 1) = \Gamma(\lambda(n))$  for every integer  $n \geq 5$ . For an integer  $n \geq 5$ , let  $\mathcal{J}_n$  denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{3\} \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{array} \right.$$

Lemma [3](#) and the diagram in Figure 5 explain the construction of the system  $\mathcal{J}_n$ .



**Fig. 5** Construction of the system  $\mathcal{J}_n$

For every integer  $n \geq 5$ , the system  $\mathcal{J}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(5, 24, 23!, 25, \lambda(5), \dots, \lambda(n))$ . For an integer  $n \geq 5$ , let  $\Delta_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq \lambda(n)$ .*

**Hypothesis 2.** *The statements  $\Delta_5, \dots, \Delta_{14}$  are true.*

Lemmas [3](#) and [5](#) imply that the statements  $\Delta_n$  have similar consequences as the statements  $\Psi_n$ .

**Theorem 9.** *The statement  $\Delta_6$  implies that any prime number  $p \geq 25$  proves the infinitude of primes.*

*Proof.* It follows from Lemmas [3](#) and [5](#). We leave the details to the reader.  $\square$

## 10 Hypothetical statements $\Sigma_3, \dots, \Sigma_{16}$ and their consequences

Let  $\Gamma_n(k)$  denote  $(k-1)!$ , where  $n \in \{3, \dots, 16\}$  and  $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$ . For an integer  $n \in \{3, \dots, 16\}$ , let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer  $n \in \{3, \dots, 16\}$ , let  $P_n$  denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \Gamma_n(x_2) = x_1 \\ \forall i \in \{2, \dots, n-1\} x_i \cdot x_i = x_{i+1} \end{cases}$$

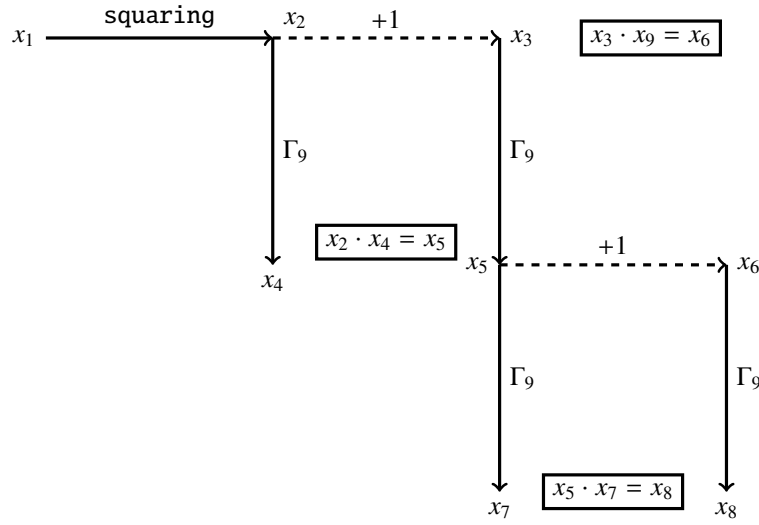
**Lemma 14.** *For every integer  $n \in \{3, \dots, 16\}$ ,  $P_n \subseteq Q_n$  and the system  $P_n$  with  $\Gamma$  instead of  $\Gamma_n$  has exactly one solution in positive integers  $x_1, \dots, x_n$ , namely  $(1, 2^{2^0}, 2^{2^1}, 2^{2^2}, \dots, 2^{2^{n-2}})$ .*

For an integer  $n \in \{3, \dots, 16\}$ , let  $\Sigma_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq Q_n$  with  $\Gamma$  instead of  $\Gamma_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then every tuple  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$  that solves the original system  $\mathcal{S}$  satisfies  $x_1, \dots, x_n \leq 2^{2^{n-2}}$ .*

**Hypothesis 3.** *The statements  $\Sigma_3, \dots, \Sigma_{16}$  are true.*

**Lemma 15.** (cf. Lemma [3](#)). *For every integer  $n \in \{4, \dots, 16\}$  and for every positive integers  $x$  and  $y$ ,  $x \cdot \Gamma_n(x) = \Gamma_n(y)$  if and only if  $(x+1 = y) \wedge (x \geq 2^{2^{n-3}} + 1)$ .*

Let  $\mathcal{Z}_9 \subseteq \mathcal{Q}_9$  be the system of equations in Figure 6.



**Fig. 6** Construction of the system  $\mathcal{Z}_9$

**Lemma 16.** For every positive integer  $x_1$ , the system  $\mathcal{Z}_9$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1 > 2^{2^{9-4}}$  and  $x_1^2 + 1$  is prime. In this case, positive integers  $x_2, \dots, x_9$  are uniquely determined by  $x_1$ . For every positive integer  $n$ , at most finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  begin with  $n$  and solve the system  $\mathcal{Z}_9$  with  $\Gamma$  instead of  $\Gamma_9$ .

*Proof.* It follows from Lemmas [3](#), [5](#), and [15](#). □

**Lemma 17.** ([\[19\]](#)). The number  $(13!)^2 + 1 = 38775788043632640001$  is prime.

**Lemma 18.**  $\left( (13!)^2 \geq 2^{2^{9-3}} + 1 = 18446744073709551617 \right) \wedge \left( \Gamma_9((13!)^2) > 2^{2^{9-2}} \right)$ .

**Theorem 10.** The statement  $\Sigma_9$  implies the infinitude of primes of the form  $n^2 + 1$ .

*Proof.* It follows from Lemmas [16](#)-[18](#). □

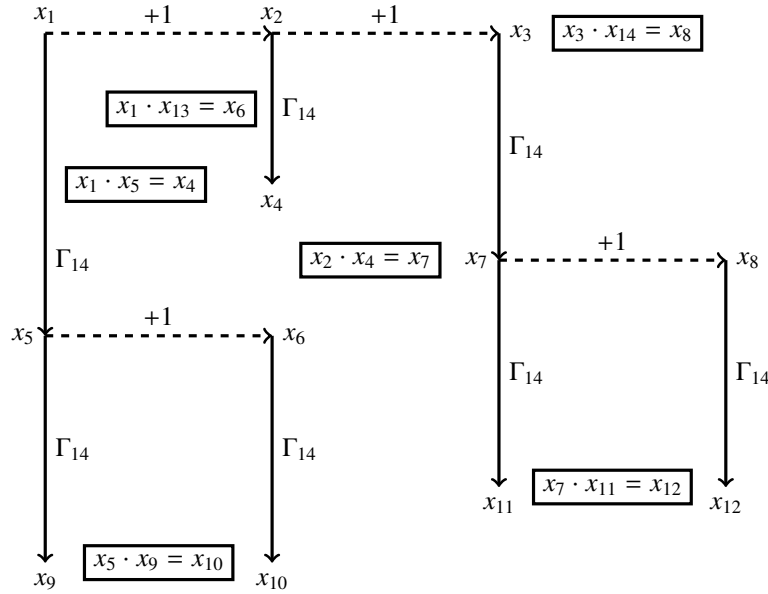
**Theorem 11.** (cf. Theorem [7](#)). The statement  $\Sigma_9$  implies that any prime of the form  $n! + 1$  with  $n \geq 2^{2^{9-3}}$  proves the infinitude of primes of the form  $n! + 1$ .

*Proof.* We leave the proof to the reader. □

**Corollary 6.** Let  $\mathcal{Y}_9$  denote the set of primes of the form  $n! + 1$ . The statement  $\Sigma_9$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{Y}_9$ , and this number equals  $\max(\mathcal{Y}_9)$ , if  $\mathcal{Y}_9$  is finite. Assuming the statement  $\Sigma_9$ , a single query to an oracle for the halting problem decides the infinity of  $\mathcal{Y}_9$ . Assuming the statement  $\Sigma_9$ , the infinity of  $\mathcal{Y}_9$  is decidable in the limit.

*Proof.* We consider an algorithm which computes  $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$ . □

Let  $\mathcal{Z}_{14} \subseteq \mathcal{Q}_{14}$  be the system of equations in Figure 7.



**Fig. 7** Construction of the system  $\mathcal{Z}_{14}$

**Lemma 19.** For every positive integer  $x_1$ , the system  $\mathcal{Z}_{14}$  is solvable in positive integers  $x_2, \dots, x_{14}$  if and only if  $x_1$  and  $x_1 + 2$  are prime and  $x_1 \geq 2^{2^{14-3}} + 1$ . In this case, positive integers  $x_2, \dots, x_{14}$  are uniquely determined by  $x_1$ . For every positive integer  $n$ , at most finitely many tuples  $(x_1, \dots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$  begin with  $n$  and solve the system  $\mathcal{Z}_{14}$  with  $\Gamma$  instead of  $\Gamma_{14}$ .

*Proof.* It follows from Lemmas [3](#), [5](#), and [15](#). □

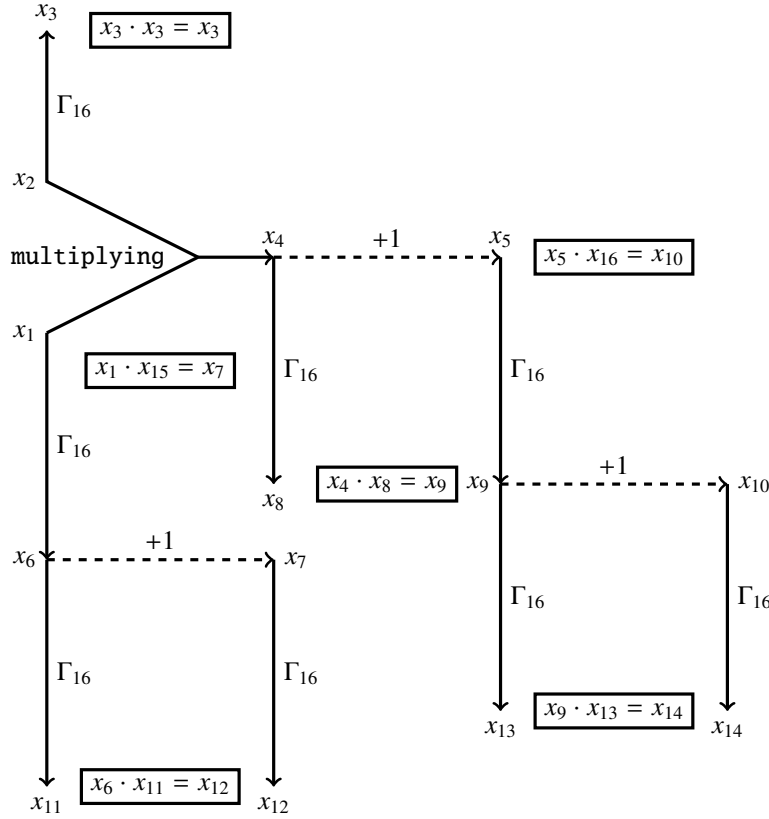
**Lemma 20.** ([\[23\]](#) p. 87]). The numbers  $459 \cdot 2^{8529} - 1$  and  $459 \cdot 2^{8529} + 1$  are prime (Harvey Dubner).

**Lemma 21.**  $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$ .

**Theorem 12.** The statement  $\Sigma_{14}$  implies the infinitude of twin primes.

*Proof.* It follows from Lemmas [19](#)-[21](#). □

A prime  $p$  is said to be a Sophie Germain prime if both  $p$  and  $2p + 1$  are prime, see [\[22\]](#). It is conjectured that there are infinitely many Sophie Germain primes, see [\[17\]](#) p. 330]. Let  $\mathcal{Z}_{16} \subseteq \mathcal{Q}_{16}$  be the system of equations in Figure 8.



**Fig. 8** Construction of the system  $\mathcal{Z}_{16}$

**Lemma 22.** For every positive integer  $x_1$ , the system  $\mathcal{Z}_{16}$  is solvable in positive integers  $x_2, \dots, x_{16}$  if and only if  $x_1$  is a Sophie Germain prime and  $x_1 \geq 2^{2^{16-3}} + 1$ . In this case, positive integers  $x_2, \dots, x_{16}$  are uniquely determined by  $x_1$ . For every positive integer  $n$ , at most finitely many tuples  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  begin with  $n$  and solve the system  $\mathcal{Z}_{16}$  with  $\Gamma$  instead of  $\Gamma_{16}$ .

*Proof.* It follows from Lemmas [3](#), [5](#), and [15](#). □

**Lemma 23.** ([\[17\]](#) p. 330).  $8069496435 \cdot 10^{5072} - 1$  is a Sophie Germain prime (Harvey Dubner).

**Lemma 24.**  $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$ .

**Theorem 13.** The statement  $\Sigma_{16}$  implies the infinitude of Sophie Germain primes.

*Proof.* It follows from Lemmas [22](#)-[24](#). □

**Theorem 14.** The statement  $\Sigma_6$  proves the following implication: if the equation  $x(x+1) = y!$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(1, 2), (2, 3)\}$ .

*Proof.* We leave the proof to the reader. □

The question of solving the equation  $x(x+1) = y!$  was posed by P. Erdős, see [\[2\]](#). F. Luca proved that the *abc* conjecture implies that the equation  $x(x+1) = y!$  has only finitely many solutions in positive integers, see [\[12\]](#).

**Theorem 15.** The statement  $\Sigma_6$  proves the following implication: if the equation  $x! + 1 = y^2$  has only finitely many solutions in positive integers  $x$  and  $y$ , then each such solution  $(x, y)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ .

*Proof.* We leave the proof to the reader. □

## 11 Hypothetical statements $\Omega_3, \dots, \Omega_{16}$ and their consequences

For an integer  $n \in \{3, \dots, 16\}$ , let  $\Omega_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  has a solution in integers  $x_1, \dots, x_n$  greater than  $2^{2^{n-2}}$ , then  $\mathcal{S}$  has infinitely many solutions in positive integers  $x_1, \dots, x_n$ .* For every  $n \in \{3, \dots, 16\}$ , the statement  $\Sigma_n$  implies the statement  $\Omega_n$ .

**Lemma 25.** *The number  $(65!)^2 + 1$  is prime and  $65! > 2^{2^{9-2}}$ .*

*Proof.* The following PARI/GP ([16]) command

```
(04:04) gp > isprime((65!)^2+1, {flag=2})
%1 = 1
```

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([23] p. 226). It rigorously shows that the number  $(65!)^2 + 1$  is prime.  $\square$

**Lemma 26.** *If positive integers  $x_1, \dots, x_9$  solve the system  $\mathcal{Z}_9$  and  $x_1 > 2^{2^{9-2}}$ , then  $x_1 = \min(x_1, \dots, x_9)$ .*

**Theorem 16.** *The statement  $\Omega_9$  implies the infinitude of primes of the form  $n^2 + 1$ .*

*Proof.* It follows from Lemmas [16] and [25-26].  $\square$

**Lemma 27.** *If positive integers  $x_1, \dots, x_{14}$  solve the system  $\mathcal{Z}_{14}$  and  $x_1 > 2^{2^{14-2}}$ , then  $x_1 = \min(x_1, \dots, x_{14})$ .*

**Theorem 17.** *The statement  $\Omega_{14}$  implies the infinitude of twin primes.*

*Proof.* It follows from Lemmas [19-21] and [27].  $\square$

## 12 Are there infinitely many composite Fermat numbers?

Integers of the form  $2^{2^n} + 1$  are called Fermat numbers. Primes of the form  $2^{2^n} + 1$  are called Fermat primes, as Fermat conjectured that every integer of the form  $2^{2^n} + 1$  is prime, see [11] p. 1]. Fermat correctly remarked that  $2^{2^0} + 1 = 3$ ,  $2^{2^1} + 1 = 5$ ,  $2^{2^2} + 1 = 17$ ,  $2^{2^3} + 1 = 257$ , and  $2^{2^4} + 1 = 65537$  are all prime, see [11] p. 1].

**Open Problem.** ([11] p. 159]. *Are there infinitely many composite numbers of the form  $2^{2^n} + 1$ ? Most mathematicians believe that  $2^{2^n} + 1$  is composite for every integer  $n \geq 5$ , see [10] p. 23]. Let*

$$H_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Let  $h(1) = 1$ , and let  $h(n+1) = 2^{2^{h(n)}}$  for every positive integer  $n$ .

**Lemma 28.** *The following subsystem of  $H_n$*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

*has exactly one solution  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ , namely  $(h(1), \dots, h(n))$ .*

For a positive integer  $n$ , let  $\xi_n$  denote the following statement: *if a system of equations  $S \subseteq H_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq h(n)$* . The statement  $\xi_n$  says that for subsystems of  $H_n$  the largest known solution is indeed the largest possible.

**Hypothesis 4.** *The statements  $\xi_1, \dots, \xi_{13}$  are true.*

**Proposition 5.** *Every statement  $\xi_n$  is true with an unknown integer bound that depends on  $n$ .*

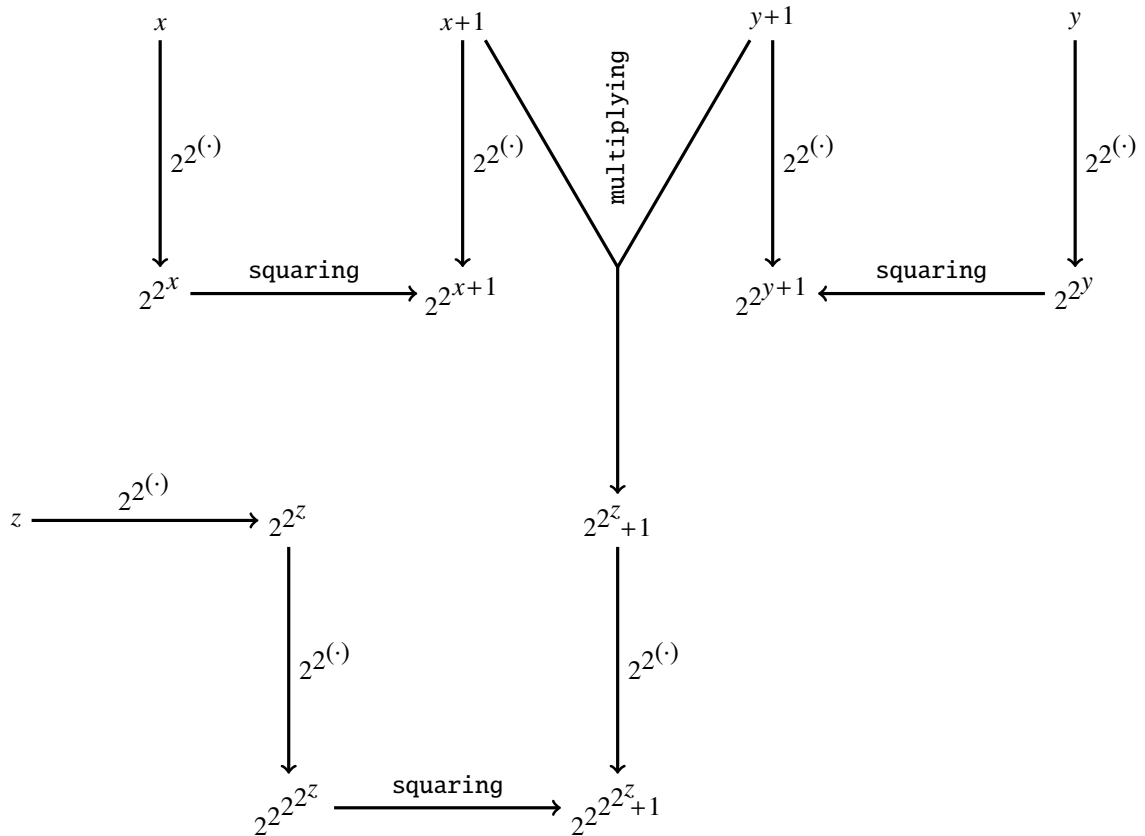
*Proof.* For every positive integer  $n$ , the system  $H_n$  has a finite number of subsystems. □

**Theorem 18.** *The statement  $\xi_{13}$  proves the following implication: if  $z \in \mathbb{N} \setminus \{0\}$  and  $2^{2^z} + 1$  is composite and greater than  $h(12)$ , then  $2^{2^z} + 1$  is composite for infinitely many positive integers  $z$ .*

*Proof.* Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{2}$$

in positive integers. By Lemma 4, we can transform equation (2) into an equivalent system of equations  $\mathcal{G}$  which has 13 variables ( $x, y, z$ , and 10 other variables) and which consists of equations of the forms  $\alpha \cdot \beta = \gamma$  and  $2^{2^\alpha} = \gamma$ , see the diagram in Figure 9.



**Fig. 9** Construction of the system  $\mathcal{G}$

Since  $2^{2^z} + 1 > h(12)$ , we obtain that  $2^{2^{2^z}+1} > h(13)$ . By this, the statement  $\xi_{13}$  implies that the system  $\mathcal{G}$  has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

**Corollary 7.** Let  $\mathcal{W}_{13}$  denote the set of composite Fermat numbers. The statement  $\xi_{13}$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{W}_{13}$ , and this number equals  $\max(\mathcal{W}_{13})$ , if  $\mathcal{W}_{13}$  is finite. Assuming the statement  $\xi_{13}$ , a single query to an oracle for the halting problem decides the infinity of  $\mathcal{W}_{13}$ . Assuming the statement  $\xi_{13}$ , the infinity of  $\mathcal{W}_{13}$  is decidable in the limit.

*Proof.* We consider an algorithm which computes  $\max(\mathcal{W}_{13} \cap [1, h(12)])$ . □

## References

- [1] C. H. Bennett, *Chaitin's Omega*, in: *Fractal music, hypercards, and more ...* (M. Gardner, ed.), W. H. Freeman, New York, 1992, 307–319.
- [2] D. Berend and J. E. Harmse, *On polynomial-factorial Diophantine equations*, *Trans. Amer. Math. Soc.* 358 (2006), no. 4, 1741–1779.
- [3] C. K. Caldwell and Y. Gallot, *On the primality of  $n! \pm 1$  and  $2 \times 3 \times 5 \times \dots \times p \pm 1$* , *Math. Comp.* 71 (2002), no. 237, 441–448, <http://doi.org/10.1090/S0025-5718-01-01315-1>.
- [4] C. S. Calude, H. Jürgensen, S. Legg, *Solving problems with finite test sets*, in: *Finite versus Infinite: Contributions to an Eternal Dilemma* (C. Calude and G. Păun, eds.), 39–52, Springer, London, 2000.
- [5] N. C. A. da Costa and F. A. Doria, *On the foundations of science (LIVRO): essays, first series*, E-papers Serviços Editoriais Ltda, Rio de Janeiro, 2013.
- [6] F. G. Dorais, *Can the twin prime problem be solved with a single use of a halting oracle?* July 23, 2011, <http://mathoverflow.net/questions/71050>.
- [7] W. B. Easton, *Powers of regular cardinals*, *Ann. Math. Logic* 1 (1970), 139–178.
- [8] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [9] T. Jech, *Set theory*, Springer, Berlin, 2003.
- [10] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers*, American Mathematical Society, Providence, RI, 2012.
- [11] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry*, Springer, New York, 2001.
- [12] F. Luca, *The Diophantine equation  $P(x) = n!$  and a result of M. Overholt*, *Glas. Mat. Ser. III* 37 (57) (2002), no. 2, 269–273
- [13] M. Mignotte and A. Pethő, *On the Diophantine equation  $x^p - x = y^q - y$* , *Publ. Mat.* 43 (1999), no. 1, 207–216.
- [14] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT*, Springer, London, 2012.
- [15] M. Overholt, *The Diophantine equation  $n! + 1 = m^2$* , *Bull. London Math. Soc.* 25 (1993), no. 2, 104.
- [16] PARI/GP online documentation, [http://pari.math.u-bordeaux.fr/dochtm/html/Arithmetic\\_functions.html](http://pari.math.u-bordeaux.fr/dochtm/html/Arithmetic_functions.html).
- [17] P. Ribenboim, *The new book of prime number records*, Springer, New York, 1996, <http://doi.org/10.1007/978-1-4612-0759-7>.



- [18] S. Siksek, *Chabauty and the Mordell–Weil Sieve*, in: *Advances on Superelliptic Curves and Their Applications* (eds. L. Beshaj, T. Shaska, E. Zhupa), 194–224, IOS Press, Amsterdam, 2015, <http://dx.doi.org/10.3233/978-1-61499-520-3-194>.
- [19] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, *Smallest prime factor of  $A020549(n) = (n!)^2 + 1$* , <http://oeis.org/A282706>.
- [20] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [21] Wolfram MathWorld, *Perfect Cuboid*, <http://mathworld.wolfram.com/PerfectCuboid.html>.
- [22] Wolfram MathWorld, *Sophie Germain prime*, <http://mathworld.wolfram.com/SophieGermainPrime.html>.
- [23] S. Y. Yan, *Number theory for computing*, 2nd ed., Springer, Berlin, 2002.
- [24] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity*, Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, <http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm>.
- [25] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer*, in: J. Cachro and K. Kijania-Placek (eds.), *Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science*, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

Apoloniusz Tyska  
 University of Agriculture  
 Faculty of Production and Power Engineering  
 Balicka 116B, 30-149 Kraków, Poland  
 E-mail: [rttyszka@cyf-kr.edu.pl](mailto:rttyszka@cyf-kr.edu.pl)