

The physical limits of computation inspire an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory ZFC as it refers to our current knowledge on \mathcal{X}

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ : $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\{x_i! = x_k : i, k \in \{1, \dots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, 9\}\}$. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(9))$. Let Ψ denote the statement: *if a system $S \subseteq B$ has at most finitely many solutions in positive integers x_1, \dots, x_9 , then each such solution (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \leq f(9)$* . We write down a system $\mathcal{A} \subseteq B$ of 8 equations and prove that the statement Ψ restricted to the system \mathcal{A} is equivalent to the statement Φ . Open Problem: Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)? (1) *There are many elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite.* (2) *No known algorithm decides the finiteness/infiniteness of \mathcal{X} .* (3) *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* (4) *There is a known algorithm that computes an integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.* (5) *There is a simple condition C , which can be formalized in ZFC, such that for almost all $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in \mathcal{X}$. The simplest known such condition C defines in \mathbb{N} the set \mathcal{X} .* We prove: (i) the set $\mathcal{X} = \{k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7), k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4), (ii) the set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3) and (5), (iii) the statement Φ implies that the set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies condition (4), (iv) no set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers whose sizes are not bounded apriori, at some future day, a computer will be able to execute this algorithm in 1 second or less.

2020 Mathematics Subject Classification: 03D20.

Key words and phrases: algorithm with no inputs that operates on integers whose sizes are not bounded apriori, computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on \mathcal{X} , mathematical statement about \mathcal{X} that refers to the current knowledge on \mathcal{X} , physical limits of computation, prime numbers of the form $n^2 + 1$.

1. Basic definitions and the philosophical goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

Definition 1. *Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.*

(1) *There are many elements of X and it is conjectured that X is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of X .*

(3) *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in X$.*

(4) *There is a known algorithm that computes an integer n satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.*

(5) *There is a simple condition C , which can be formalized in ZFC, such that for almost all $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in X$. The simplest known such condition C defines in \mathbb{N} the set X .*

The goal of condition (5) is to avoid non-naturally defined sets $X \subseteq \mathbb{N}$ like the set X in Statement 2.

Definition 2. *We say that an integer n is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, cf. [7] and [8].*

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of X . If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $[\max(X), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [4]–[6].

Definition 3. *Let Φ denote the following unproven statement:*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)]$$

Landau's conjecture implies the statement Φ . In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *No known algorithm computes an integer k such that*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$$

Proving the statement Φ will falsify Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Statements 2–3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $X \subseteq \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1

Definition 4. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. \square

Statement 2. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\mathcal{X} \supseteq \{0, \dots, \beta\}$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

Open Problem 1. Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)?

Theorem 1. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers whose sizes are not bounded a priori, at some future day, a computer will be able to execute this algorithm in 1 second or less.

Proof. The proof goes by contradiction. Since conditions (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots \quad (\text{T})$$

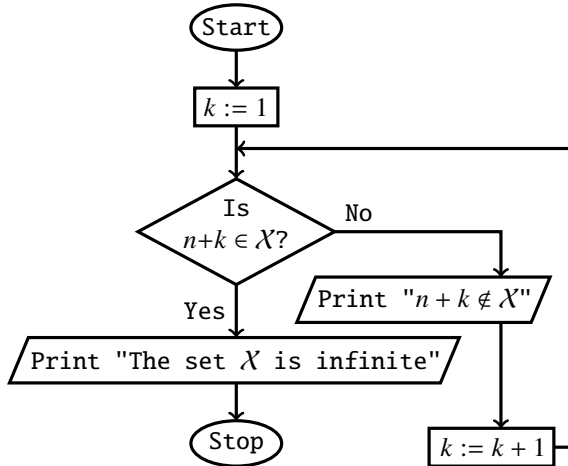


Fig. 1 An algorithm whose execution never terminates if the set \mathcal{X} is finite

The sentences from the sequence (T) and our assumptions imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will

be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set \mathcal{X} is finite, contrary to the conjecture in condition (1). \square

Theorem 1, whose assumptions contradict the current paradigm of physics, explains the title of the article.

3. Number-theoretic statements Ψ_n

Let $f(1) = 2, f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n - 1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system \mathcal{U}_n .

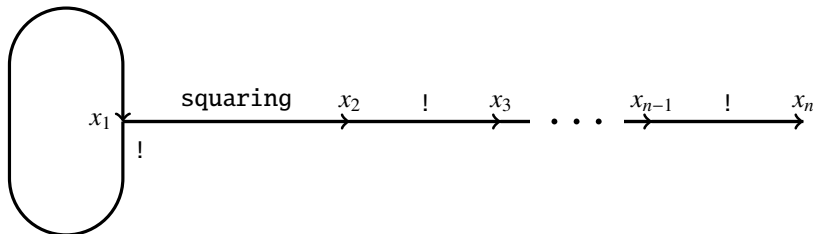


Fig. 2 Construction of the system \mathcal{U}_n

Lemma 2. For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.

Let B_n denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Ψ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$. The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible.

Theorem 2. For every statement Ψ_n , the bound $f(n)$ cannot be decreased.

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. \square

Theorem 3. For every integer $n \geq 2$, the statement Ψ_{n+1} implies the statement Ψ_n .

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then for every integer $i \in \{1, \dots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers x_1, \dots, x_{n+1} . The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leq f(n + 1) = f(n)!$. Hence, $x_i \leq f(n)$. \square

Theorem 4. Every statement Ψ_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , the system B_n has a finite number of subsystems. \square

4. A conjectural solution to Open Problem 1

Lemma 3. For every positive integers x and y , $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. (Wilson's theorem, [1, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x - 1)! + 1$.

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system \mathcal{A} .

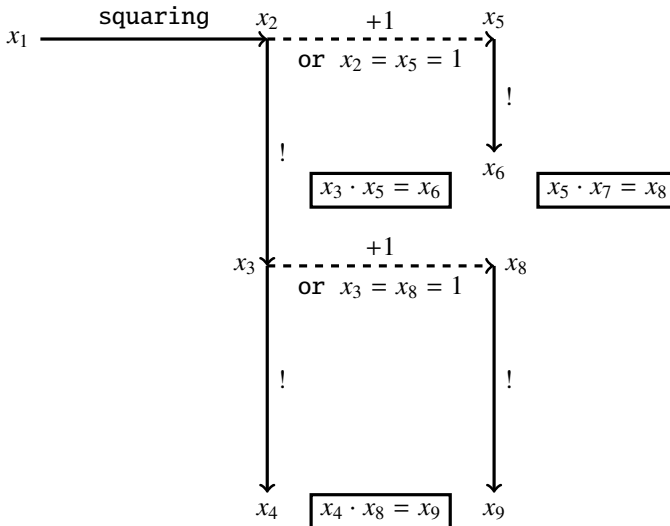


Fig. 3 Construction of the system \mathcal{A}

Lemma 5. For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 3, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \square

Lemma 6. There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$. This is true as every such tuple (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \in \{1, 2\}$.

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2} \right\} \cap \mathbb{N} = \{1, 2\}$. \square

Conjecture 1. The statement Ψ_9 is true when is restricted to the system \mathcal{A} .

Theorem 5. Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set \mathcal{P}_{n^2+1} is infinite. \square

Theorem 6. Conjecture 1 implies the statement Φ .

Proof. It follows from Theorem 5 and the equality $f(7) = (((24)!)!)!$. \square

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers x_1, \dots, x_9 solve the system \mathcal{A} , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leq (((24!)!)!) = f(7)$ holds when the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 . Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. \square

Statement 3. *The set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3) and (5). The statement Φ implies that the set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies condition (4).*

Proof. The set \mathcal{P}_{n^2+1} is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [5]. These two facts imply condition (1). Condition (3) holds trivially. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!) = \beta$, see [2]. Thus condition (2) holds. The following condition:

$k - 1$ is a square and k has no divisors greater than 1 and smaller than k

defines in \mathbb{N} the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition (5). Suppose that the statement Φ is true. This means that β is a threshold number of $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition (4) holds. \square

Acknowledgment. Sławomir Kurpaska prepared three diagrams in *TikZ*. Apoloniusz Tyszka wrote the article.

References

- [1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [2] S. Lloyd, *Ultimate physical limits to computation*, Nature 406 (2000), 1047–1054, <http://doi.org/10.1038/35023282>.
- [3] W. Marciszewski, *Logic, modern, history of*, in: *Dictionary of logic as applied in the study of language* (ed. W. Marciszewski), pp. 183–200, Springer, Dordrecht, 1981.
- [4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A002496, *Primes of the form $n^2 + 1$* , <http://oeis.org/A002496>.
- [5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A083844, *Number of primes of the form $x^2 + 1 < 10^n$* , <http://oeis.org/A083844>.
- [6] Wolfram MathWorld, *Landau's Problems*, <http://mathworld.wolfram.com/LandausProblems.html>.
- [7] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity*, Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, <http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm>.

- [8] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer*, in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

Sławomir Kurpaska
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rtkurpas@cyf-kr.edu.pl

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl