On sets $\mathcal{W} \subseteq \mathbb{N}$ such that the infinity of $\mathcal{W}$ is equivalent to the existence in $\mathcal{W}$ of an element that is greater than a threshold number computed with using the definition of $\mathcal{W}$

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***Abstract*—Let $f(1)=2$, $f(2)=4$, and let $f(n+1)=f(n)!$ for every integer $n \geqslant 2$. For a positive integer n, let Θ_n denote the statement: if a system $\mathcal{S} \subseteq \left\{x_i!=x_k:\ i,k \in \{1,\ldots,n\}\right\} \cup \left\{x_i \cdot x_j = x_k:\ i,j,k \in \{1,\ldots,n\}\right\}$ has only finitely many solutions in integers $x_1,\ldots,x_n$ greater than 1, then each such solution $(x_1,\ldots,x_n)$ satisfies $\min(x_1,\ldots,x_n) \leqslant f(n)$. The statement Θ_9 proves that if there exists an integer $x > f(9)$ such that x^2+1 (alternatively, $x!+1$) is prime, then there are infinitely many primes of the form n^2+1 (respectively, $n!+1$). The statement Θ_{16} proves that if there exists a twin prime greater than $f(16)+3$, then there are infinitely many twin primes. We formulate the statements Φ_n and prove: Φ_4 equivalently expresses that there are infinitely many primes of the form $n!+1$, Φ_6 implies that for infinitely many primes p the number $p!+1$ is prime, Φ_6 implies that there are infinitely many primes of the form $n!-1$, Φ_7 implies that there are infinitely many twin primes.**

***Index Terms*—composite Fermat numbers, prime numbers of the form $n!+1$, prime numbers of the form $n!-1$, prime numbers of the form n^2+1, prime numbers p such that $p!+1$ is prime, single query to an oracle for the halting problem, twin prime conjecture.**

I. Spectra of sentences and their threshold numbers

THE following observation concerns the theme described in the title of the article.

Observation 1. *If $\mathcal{W}$ is a subset of $\{0,\ldots,n\}$ where n is a non-negative integer, then we take any integer $m \geqslant n$ as a threshold number for $\mathcal{W}$. If $\mathcal{W}$ is an infinite subset of $\mathbb{N}$, then we take any non-negative integer m as a threshold number for $\mathcal{W}$.*

We define the set $\mathcal{U} \subseteq \mathbb{N}$ by declaring that a non-negative integer n belongs to $\mathcal{U}$ if and only if $\sin\left(10^{10^{10^{10}}}\right) > 0$. This inequality is practically undecidable, see [4].

Corollary 1. *The set $\mathcal{U}$ equals $\emptyset$ or $\mathbb{N}$. The statement "$\mathcal{U}=\emptyset$" remains unproven and the statement "$\mathcal{U}=\mathbb{N}$" remains unproven. Every non-negative integer m is a threshold number for $\mathcal{U}$. For every non-negative integer k, the sentence "$k \in \mathcal{U}$" is only theoretically decidable.*

The first-order language of graph theory contains two relation symbols of arity 2: $\sim$ and $=$, respectively for adjacency and equality of vertices. The term first-order imposes the condition that the variables represent vertices and hence the quantifiers apply to vertices only. For a first-order sentence Λ about graphs, let Spectrum(Λ) denote the set of all positive integers n such that there is a graph on n vertices satisfying Λ.

Theorem 1. *([10, p. 171]). If a sentence Λ in the language of graph theory has the form $\exists x_1 \ldots x_k\ \forall y_1 \ldots y_l\ \Upsilon(x_1,\ldots,x_k,y_1,\ldots,y_l)$, where $\Upsilon(x_1,\ldots,x_k,y_1,\ldots,y_l)$ is quantifier-free, then either* Spectrum(Λ) $\subseteq [1,(2^k \cdot 4^l)-1]$ *or* Spectrum(Λ) $\supseteq [k+l,\infty) \cap \mathbb{N}$.

Corollary 2. *The number $\left(2^k \cdot 4^l\right)-1$ is a threshold number for* Spectrum(Λ).

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [8, p. 234].

Corollary 3. *If an algorithm $\mathtt{Alg}_1$ for every recursive set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_1(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Alg}_1(\mathcal{M})+1,\infty) \neq \emptyset$.*

Corollary 4. *If an algorithm $\mathtt{Alg}_2$ for every recursively enumerable set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_2(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Alg}_2(\mathcal{M})+1,\infty) \neq \emptyset$.*

II. Basic lemmas

Let $f(1)=2$, $f(2)=4$, and let $f(n+1)=f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{V}_1$ denote the system of equations $\{x_1!=x_1\}$, and let $\mathcal{V}_2$ denote the system of equations $\{x_1!=x_1,\ x_1 \cdot x_1 = x_2\}$. For an integer $n \geqslant 3$, let $\mathcal{V}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2,\ldots,n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{V}_n$.

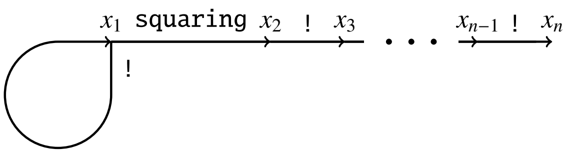

Fig. 1 Construction of the system $\mathcal{V}_n$

Lemma 1. *For every positive integer n, the system $\mathcal{V}_n$ has exactly one solution in integers greater than* 1*, namely* $\big(f(1), \ldots, f(n)\big)$.

Let

$$H_n = \Big\{x_i! = x_k : \ i,k \in \{1,\ldots,n\}\Big\} \cup$$

$$\Big\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\ldots,n\}\Big\}$$

For a positive integer n, let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers $x_1, \ldots, x_n$ greater than 1, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leqslant f(n)$. The assumption $\min(x_1, \ldots, x_n) \leqslant f(n)$ is weaker than the assumption $\max(x_1, \ldots, x_n) \leqslant f(n)$ suggested by Lemma 1.

Lemma 2. *For every positive integer n, the system H_n has a finite number of subsystems.*

Theorem 2. *Every statement Θ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 2. □

Lemma 3. *For every integers x and y greater than* 1*,* $x! \cdot y = y!$ *if and only if* $x + 1 = y$.

Lemma 4. *If* $x \geqslant 4$*, then* $\dfrac{(x-1)!+1}{x} > 1$.

Lemma 5. *(Wilson's theorem, [3, p. 89]). For every integer* $x \geqslant 2$*, x is prime if and only if x divides* $(x-1)!+1$.

III. Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [9]. It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [16, p. 297].

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

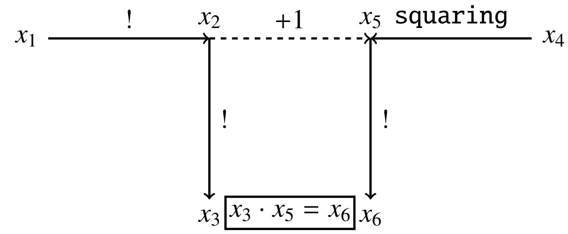

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 6. *For every integers x_1 and x_4 greater than* 1*, the system $\mathcal{A}$ is solvable in integers x_2, x_3, x_5, x_6 greater than* 1 *if and only if* $x_1! + 1 = x_4^2$*. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

and $x_1 = \min(x_1, \ldots, x_6)$.

Proof. It follows from Lemma 3. □

Theorem 3. *The statement Θ_6 proves the following implication: if the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then each such solution (x_1, x_4) satisfies $x_1 \leqslant f(6)$.*

Proof. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 6, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0,1\})^4$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Lemma 6 guarantees that $x_1 = \min(x_1, \ldots, x_6)$. By the antecedent and Lemma 6, the system $\mathcal{A}$ has only finitely many solutions in integers $x_1, \ldots, x_6$ greater than 1. Therefore, the statement Θ_6 implies that $x_1 = \min(x_1, \ldots, x_6) \leqslant f(6)$. □

Hypothesis 1. *The implication in Theorem 3 is true.*

Corollary 5. *Assuming Hypothesis 1, a single query to an oracle for the halting problem decides the problem of the infinitude of the solutions of the equation* $x! + 1 = y^2$.

IV. Are there infinitely many prime numbers of the form $n^2 + 1$?

Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [7, pp. 37–38].

Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

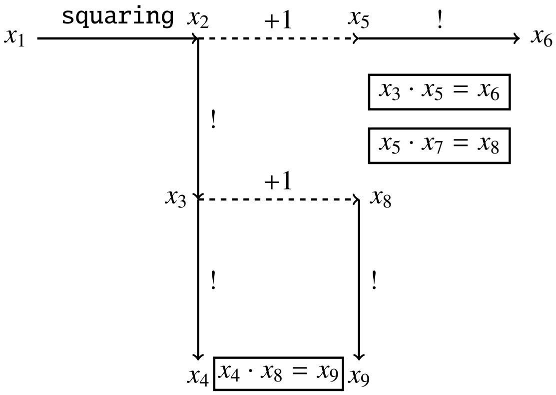

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 7. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. □

Theorem 4. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Lemma 7 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. □

Hypothesis 2. *The implication in Theorem 4 is true.*

Corollary 6. *Assuming Hypothesis 2, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n^2 + 1$.*

V. Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [1, p. 443] and [12]. Let $\mathcal{G}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system $\mathcal{G}$.

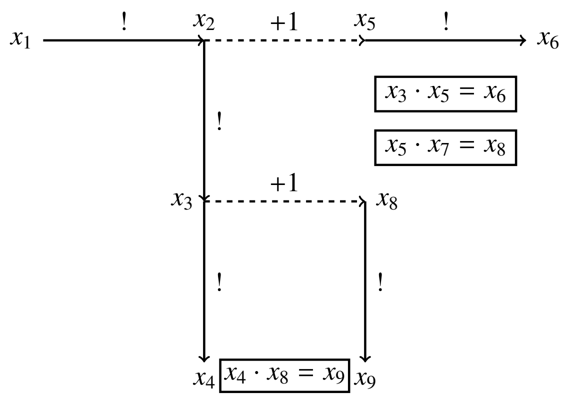

Fig. 4 Construction of the system $\mathcal{G}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_4 &= ((x_1!)!)! \\ x_5 &= x_1^{!} + 1 \\ x_6 &= (x_1! + 1)! \\ x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\ x_8 &= (x_1!)! + 1 \\ x_9 &= ((x_1!)! + 1)! \end{aligned}$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 5. □

Theorem 5. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1! + 1$ is prime. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0,1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{G}$. Lemma 8 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{G} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{G}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0,1\})^9$. According to Lemma 8, there are infinitely many primes of the form $n! + 1$. □

Hypothesis 3. *The implication in Theorem 5 is true.*

Corollary 7. *Assuming Hypothesis 3, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n! + 1$.*

VI. The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [7, p. 39].

Let C denote the following system of equations:

$$\left\{\begin{array}{rcl} x_1! &=& x_2 \\ x_2! &=& x_3 \\ x_4! &=& x_5 \\ x_6! &=& x_7 \\ x_7! &=& x_8 \\ x_9! &=& x_{10} \\ x_{12}! &=& x_{13} \\ x_{15}! &=& x_{16} \\ x_2 \cdot x_4 &=& x_5 \\ x_5 \cdot x_6 &=& x_7 \\ x_7 \cdot x_9 &=& x_{10} \\ x_4 \cdot x_{11} &=& x_{12} \\ x_3 \cdot x_{12} &=& x_{13} \\ x_9 \cdot x_{14} &=& x_{15} \\ x_8 \cdot x_{15} &=& x_{16} \end{array}\right.$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system C.

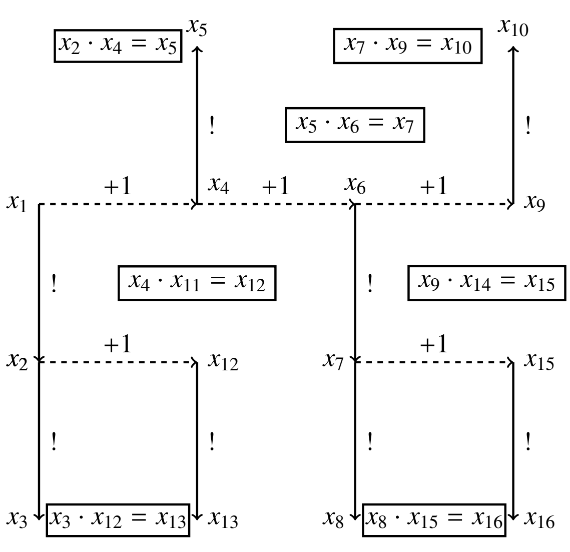

Fig. 5 Construction of the system C

Lemma 9. *If $x_4 = 2$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. □

Lemma 10. *If $x_4 = 3$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. □

Lemma 11. *For every $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0,1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} are uniquely determined by the following equalities:*

$$\begin{array}{rcl} x_1 &=& x_4 - 1 \\ x_2 &=& (x_4 - 1)! \\ x_3 &=& ((x_4 - 1)!)! \\ x_5 &=& x_4! \\ x_6 &=& x_9 - 1 \\ x_7 &=& (x_9 - 1)! \\ x_8 &=& ((x_9 - 1)!)! \\ x_{10} &=& x_9! \\ x_{11} &=& \dfrac{(x_4 - 1)! + 1}{x_4} \\ x_{12} &=& (x_4 - 1)! + 1 \\ x_{13} &=& ((x_4 - 1)! + 1)! \\ x_{14} &=& \dfrac{(x_9 - 1)! + 1}{x_9} \\ x_{15} &=& (x_9 - 1)! + 1 \\ x_{16} &=& ((x_9 - 1)! + 1)! \end{array}$$

and $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3$.

Proof. By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0,1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 11 follows from Lemma 5. □

Theorem 6. *The statement Θ_{16} proves the following implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.*

Proof. Assume the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$. By Lemma 11, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0,1\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Lemma 11 guarantees that $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement Θ_{16} and the inequality $\min(x_1, \ldots, x_{16}) > f(16)$ imply that the system C has infinitely many solutions in integers $x_1, \ldots, x_{16}$ greater than 1. According to Lemmas 9–11, there are infinitely many twin primes. □

Hypothesis 4. *The implication in Theorem 6 is true.*

Corollary 8. *(cf. [2]). Assuming Hypothesis 4, a single query to an oracle for the halting problem decides the twin prime problem.*

VII. Are there infinitely many composite Fermat numbers?

Primes of the form $2^{2^n}+1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n}+1$ is prime, see [6, p. 1]. Fermat correctly remarked that $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$, and $2^{2^4}+1=65537$ are all prime, see [6, p. 1].

Open Problem. ([6, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n}+1$?*

Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [5, p. 23].

Lemma 12. *([6, p. 38]). For every positive integer n, if a prime number p divides $2^{2^n}+1$, then there exists a positive integer k such that $p = k \cdot 2^{n+1}+1$.*

Corollary 9. *Since $k \cdot 2^{n+1}+1 \geqslant 2^{n+1}+1 \geqslant n+3$, for every positive integers x, y, and n, the equality $(x+1)(y+1) = 2^{2^n}+1$ implies that $\min(n,x,x+1,y,y+1) = n$.*

Let $g(1)=1$, and let $g(n+1) = 2^{2^{g(n)}}$ for every positive integer n. Let

$$G_n = \left\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\dots,n\}\right\} \cup \left\{2^{2^{x_i}} = x_k : \ i,k \in \{1,\dots,n\}\right\}$$

Lemma 13. *The following subsystem of G_n*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1,\dots,n-1\} \ 2^{2^{x_i}} = x_{i+1} \end{cases}$$

has exactly one solution $(x_1,\dots,x_n) \in (\mathbb{N}\setminus\{0\})^n$, namely $(g(1),\dots,g(n))$.

For a positive integer n, let Ψ_n denote the following statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers $x_1,\dots,x_n$, then each such solution $(x_1,\dots,x_n)$ satisfies $\min(x_1,\dots,x_n) \leqslant g(n)$. The assumption $\min(x_1,\dots,x_n) \leqslant g(n)$ is weaker than the assumption $\max(x_1,\dots,x_n) \leqslant g(n)$ suggested by Lemma 13.

Lemma 14. *For every positive integer n, the system G_n has a finite number of subsystems.*

Theorem 7. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 14. □

Lemma 15. *For every non-negative integers b and c, $b+1=c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Theorem 8. *The statement Ψ_{13} proves the following implication: if $2^{2^n}+1$ is composite for some integer $n > g(13)$, then $2^{2^n}+1$ is composite for infinitely many positive integers n.*

Proof. Let us consider the equation

$$(x+1)(y+1) = 2^{2^z}+1 \tag{1}$$

in positive integers. By Lemma 15, we can transform equation (1) into an equivalent system $\mathcal{F}$ which has 13 variables (x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 6.

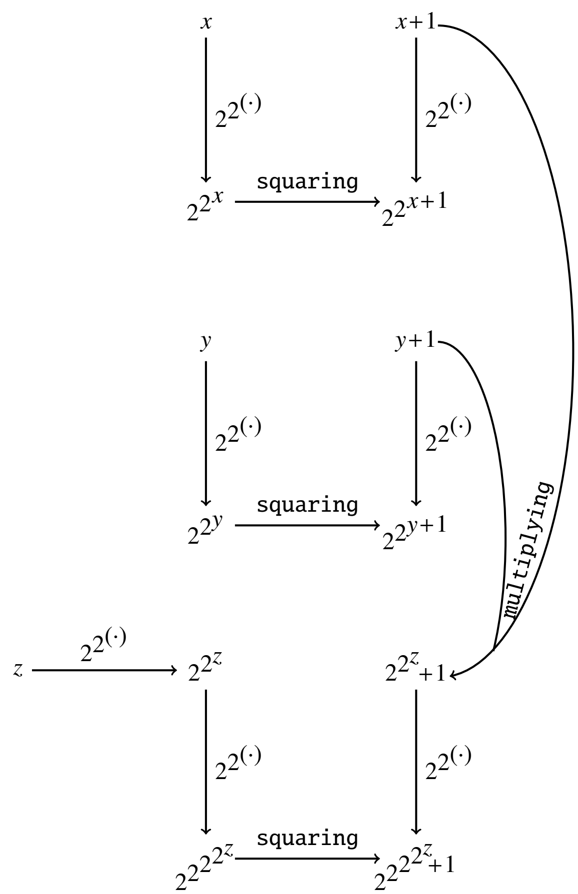

Fig. 6 Construction of the system $\mathcal{F}$

Assume that $2^{2^n}+1$ is composite for some integer $n > g(13)$. By this and Corollary 9, equation (1) has a solution $(x,y,z) \in (\mathbb{N}\setminus\{0\})^3$ such that $z=n$ and $z = \min(z,x,x+1,y,y+1)$. Hence, the system $\mathcal{F}$ has a solution in positive integers such that $z=n$ and n is the smallest number in the solution sequence. Since $n > g(13)$, the statement Ψ_{13} implies that the system $\mathcal{F}$ has infinitely many solutions in positive integers. Therefore, there are infinitely many positive integers n such that $2^{2^n}+1$ is composite. □

Hypothesis 5. *The implication in Theorem 8 is true.*

Corollary 10. *Assuming Hypothesis 5, a single query to an oracle for the halting problem decides whether or not the set of composite Fermat numbers is infinite.*

VIII. Computations of length n and the statements Φ_n

For a positive integer x, let $\Gamma(x)$ denote $(x-1)!$. Let $\mathrm{fact}^{-1}\colon \{1,2,6,24,\ldots\} \to \mathbb{N}\setminus\{0\}$ denote the inverse function to the factorial function. For positive integers x and y, let $\mathrm{rem}(x,y)$ denote the remainder from dividing x by y.

Definition. *For a positive integer n, by a computation of length n we understand any sequence of terms $x_1,\ldots,x_n$ such that x_1 is defined as the variable x, and for every integer $i \in \{2,\ldots,n\}$, x_i is defined as $\Gamma(x_{i-1})$, or $\mathrm{fact}^{-1}(x_{i-1})$, or $\mathrm{rem}(x_{i-1},x_{i-2})$ (only if $i \geqslant 3$ and x_{i-1} is defined as $\Gamma(x_{i-2})$).*

Let $\mathcal{P}$ denote the set of prime numbers.

Lemma 16. *([11, pp. 214–215]) . For every positive integer x, $\mathrm{rem}(\Gamma(x),x) \in \mathbb{N}\setminus\{0\}$ if and only if $x \in \{4\}\cup\mathcal{P}$.*

Let $h(4)=3$, and let $h(n+1)=h(n)!$ for every integer $n \geqslant 4$.

Theorem 9. *For every integer $n \geqslant 4$ and for every positive integer x, the following computation $\mathcal{H}_n$*

$$\begin{cases} x_1 &:= x \\ \forall i \in \{2,\ldots,n-3\}\ x_i &:= \mathrm{fact}^{-1}(x_{i-1}) \\ x_{n-2} &:= \Gamma(x_{n-3}) \\ x_{n-1} &:= \Gamma(x_{n-2}) \\ x_n &:= \mathrm{rem}(x_{n-1},x_{n-2}) \end{cases}$$

returns positive integers $x_1,\ldots,x_n$ if and only if $x=h(n)$.

Proof. We make three observations.

Observation 2. *If $x_{n-3}=3$, then $x_1,\ldots,x_{n-3} \in \mathbb{N}\setminus\{0\}$ and $x=x_1=h(n)$. If $x=h(n)$, then $x_1,\ldots,x_{n-3}\in\mathbb{N}\setminus\{0\}$ and $x_{n-3}=3$. Hence, $x_{n-2}=\Gamma(x_{n-3})=2$ and $x_{n-1}=\Gamma(x_{n-2})=1$. Therefore, $x_n=\mathrm{rem}(x_{n-1},x_{n-2})=1$.*

Observation 3. *If $x_{n-3}=2$, then $x=x_1=\ldots=x_{n-3}=2$. If $x=2$, then $x_1=\ldots=x_{n-3}=2$. Hence, $x_{n-2}=\Gamma(x_{n-3})=1$ and $x_{n-1}=\Gamma(x_{n-2})=1$. Therefore, $x_n=\mathrm{rem}(x_{n-1},x_{n-2})=0\notin\mathbb{N}\setminus\{0\}$.*

Observation 4. *If $x_{n-3}=1$, then $x_{n-2}=\Gamma(x_{n-3})=1$. Hence, $x_{n-1}=\Gamma(x_{n-2})=1$. Therefore, $x_n=\mathrm{rem}(x_{n-1},x_{n-2})=0\notin\mathbb{N}\setminus\{0\}$.*

Observations 2–4 cover the case when $x_{n-3}\in\{1,2,3\}$. If $x_{n-3}\geqslant 4$, then $x_{n-2}=\Gamma(x_{n-3})$ is greater than 4 and composite. By Lemma 16, $x_n=\mathrm{rem}(x_{n-1},x_{n-2})=\mathrm{rem}(\Gamma(x_{n-2}),x_{n-2})=0\notin\mathbb{N}\setminus\{0\}$. □

For an integer $n\geqslant 4$, let Φ_n denote the following statement: if a computation of length n returns positive integers $x_1,\ldots,x_n$ for at most finitely many positive integers x, then every such x does not exceed $h(n)$.

Theorem 10. *For every integer $n\geqslant 4$, the bound $h(n)$ in the statement Φ_n cannot be decreased.*

Proof. It follows from Theorem 9. □

Lemma 17. *For every positive integer n, there are only finitely many computations of length n.*

Theorem 11. *For every integer $n\geqslant 4$, the statement Φ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 17. □

IX. Consequences of the statements $\Phi_4,\ldots,\Phi_7$

Lemma 18. *If $x\in\mathcal{P}$, then $\mathrm{rem}(\Gamma(x),x)=x-1$.*

Proof. It follows from Lemma 5. □

Lemma 19. *For every positive integer x, the following computation $\mathcal{T}$*

$$\begin{cases} x_1 &:= x \\ x_2 &:= \Gamma(x_1) \\ x_3 &:= \mathrm{rem}(x_2,x_1) \\ x_4 &:= \mathrm{fact}^{-1}(x_3) \end{cases}$$

returns positive integers $x_1,\ldots,x_4$ if and only if $x=4$ or x is a prime number of the form $n!+1$.

Proof. For an integer $i\in\{1,\ldots,4\}$, let T_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{T}$ returns positive integers $x_1,\ldots,x_i$. We show that

$$T_4=\{4\}\cup(\{n!+1:\ n\in\mathbb{N}\setminus\{0\}\}\cap\mathcal{P}) \tag{2}$$

For every positive integer x, the terms x_1 and x_2 belong to $\mathbb{N}\setminus\{0\}$. By Lemma 16, the term x_3 (which equals $\mathrm{rem}(\Gamma(x),x)$) belongs to $\mathbb{N}\setminus\{0\}$ if and only if $x\in\{4\}\cup\mathcal{P}$. Hence, $T_3=\{4\}\cup\mathcal{P}$. If $x=4$, then $x_1,\ldots,x_4\in\mathbb{N}\setminus\{0\}$. Hence, $4\in T_4$. If $x\in\mathcal{P}$, then Lemma 18 implies that $x_3=\mathrm{rem}(\Gamma(x),x)=x-1\in\mathbb{N}\setminus\{0\}$. Therefore, for every $x\in\mathcal{P}$, the term $x_4=\mathrm{fact}^{-1}(x_3)$ belongs to $\mathbb{N}\setminus\{0\}$ if and only if $x\in\{n!+1:\ n\in\mathbb{N}\setminus\{0\}\}$. This proves equality (2). □

Theorem 12. *The statement Φ_4 implies that the set of primes of the form $n!+1$ is infinite.*

Proof. The number $3!+1=7$ is prime. By Lemma 19, for $x=7$ the computation $\mathcal{T}$ returns positive integers $x_1,\ldots,x_4$. Since $x=7>3=h(4)$, the statement Φ_4 guarantees that the computation $\mathcal{T}$ returns positive integers $x_1,\ldots,x_4$ for infinitely many positive integers x. By Lemma 19, there are infinitely many primes of the form $n!+1$. □

Lemma 20. *If $x\in\mathbb{N}\setminus\{0,1\}$, then $\mathrm{fact}^{-1}(\Gamma(x))=x-1$.*

Theorem 13. *If the set of primes of the form $n!+1$ is infinite, then the statement Φ_4 is true.*

Proof. There exist exactly 10 computations of length 4 that differ from $\mathcal{H}_4$ and $\mathcal{T}$, see Table 1. For every such computation $\mathcal{F}_i$, we determine the set S_i of all positive integers x such that the computation $\mathcal{F}_i$ outputs positive integers $x_1,\ldots,x_4$ on input x. We omit 10 easy proofs which use Lemmas 16 and 20. The sets S_i are infinite, see Table 1.

$\mathcal{F}_1$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \mathbb{N} \setminus \{0\} = S_1$
$\mathcal{F}_2$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \mathbb{N} \setminus \{0\} = S_2$
$\mathcal{H}_4$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{rem}(x_3, x_2)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x = 3$
$\mathcal{F}_3$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \mathbb{N} \setminus \{0\} = S_3$
$\mathcal{F}_4$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{1\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} = S_4$
$\mathcal{F}_5$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{rem}(x_2, x_1)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{4\} \cup \mathcal{P} = S_5$
$\mathcal{T}$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{rem}(x_2, x_1)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$
$\mathcal{F}_6$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{n! : n \in \mathbb{N} \setminus \{0\}\} = S_6$
$\mathcal{F}_7$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{n! : n \in \mathbb{N} \setminus \{0\}\} = S_7$
$\mathcal{F}_8$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{rem}(x_3, x_2)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{4!\} \cup \{p! : p \in \mathcal{P}\} = S_8$
$\mathcal{F}_9$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{(n!)! : n \in \mathbb{N} \setminus \{0\}\} = S_9$
$\mathcal{F}_{10}$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow$ $x \in \{((n!)!)! : n \in \mathbb{N} \setminus \{0\}\} = S_{10}$

Tab. 1 12 computations of length 4, $x \in \mathbb{N} \setminus \{0\}$

This completes the proof. □

Hypothesis 6. *The statements $\Phi_4, \ldots, \Phi_7$ are true.*

Lemma 21. *For every positive integer x, the following computation $\mathcal{Y}$*

$$\begin{cases} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \mathrm{rem}(x_2, x_1) \\ x_4 := \mathrm{fact}^{-1}(x_3) \\ x_5 := \Gamma(x_4) \\ x_6 := \mathrm{rem}(x_5, x_4) \end{cases}$$

returns positive integers $x_1, \ldots, x_6$ if and only if $x \in \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}$.

Proof. For an integer $i \in \{1, \ldots, 6\}$, let Y_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_i$. Since the computations $\mathcal{T}$ and $\mathcal{Y}$ have the same first four instructions, the equality $Y_i = T_i$ holds for every $i \in \{1, \ldots, 4\}$. In particular,

$$Y_4 = \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$$

We show that

$$Y_6 = \{4\} \cup (\{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}) \tag{3}$$

If $x = 4$, then $x_1, \ldots, x_6 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in Y_6$. Let $x \in \mathcal{P}$, and let $x = n! + 1$, where $n \in \mathbb{N} \setminus \{0\}$. Hence, $n \neq 4$. Lemma 18 implies that $x_3 = \mathrm{rem}(\Gamma(x), x) = x - 1 = n!$. Hence, $x_4 = \mathrm{fact}^{-1}(x_3) = n$ and $x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\}$. By Lemma 16, the term x_6 (which equals $\mathrm{rem}(\Gamma(n), n)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $n \in \{4\} \cup \mathcal{P}$. This proves equality (3) as $n \neq 4$. □

Theorem 14. *The statement Φ_6 implies that for infinitely many primes p the number $p! + 1$ is prime.*

Proof. The numbers 11 and $11! + 1$ are prime, see [1, p. 441] and [14]. By Lemma 21, for $x = 11! + 1$ the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_6$. Since $x = 11! + 1 > 6! = h(6)$, the statement Φ_6 guarantees that the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 21, for infinitely many primes p the number $p! + 1$ is prime. □

Lemma 22. *For every positive integer x, the following computation $\mathcal{L}$*

$$\begin{cases} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \Gamma(x_2) \\ x_4 := \mathrm{fact}^{-1}(x_3) \\ x_5 := \Gamma(x_4) \\ x_6 := \mathrm{rem}(x_5, x_4) \end{cases}$$

returns positive integers $x_1, \ldots, x_6$ if and only if $(x - 1)! - 1$ is prime.

Proof. For an integer $i \in \{1, \ldots, 6\}$, let L_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_i$. If $x \in \{1, 2, 3\}$, then $x_6 = 0$. Therefore, $L_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 20, for every integer $x \geqslant 4$, $x_4 = (x - 1)! - 1$, $x_5 = \Gamma((x - 1)! - 1)$, and $x_1, \ldots, x_5 \in \mathbb{N} \setminus \{0\}$. By Lemma 16, for every integer $x \geqslant 4$,

$$x_6 = \mathrm{rem}(\Gamma((x - 1)! - 1), (x - 1)! - 1)$$

belongs to $\mathbb{N} \setminus \{0\}$ if and only if $(x-1)! - 1 \in \{4\} \cup \mathcal{P}$. The last condition equivalently expresses that $(x-1)! - 1$ is prime as $(x-1)! - 1 \geqslant 5$ for every integer $x \geqslant 4$. Hence,

$$L_6 = (\mathbb{N} \setminus \{0,1,2,3\}) \cap \{x \in \mathbb{N} \setminus \{0,1,2,3\} : \ (x-1)! - 1 \in \mathcal{P}\} =$$
$$\{x \in \mathbb{N} \setminus \{0\} : \ (x-1)! - 1 \in \mathcal{P}\}$$

□

It is conjectured that there are infinitely many primes of the form $n! - 1$, see [1, p. 443] and [13].

Theorem 15. *The statement Φ_6 implies that there are infinitely many primes of the form $x! - 1$.*

Proof. The number $(975-1)! - 1$ is prime, see [1, p. 441] and [13]. By Lemma 22, for $x = 975$ the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_6$. Since $x = 975 > 720 = h(6)$, the statement Φ_6 guarantees that the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 22, the set $\{x \in \mathbb{N} \setminus \{0\} : \ (x-1)! - 1 \in \mathcal{P}\}$ is infinite. □

Lemma 23. *For every positive integer x, the following computation $\mathcal{D}$*

$$\begin{cases} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \mathrm{rem}(x_2, x_1) \\ x_4 := \Gamma(x_3) \\ x_5 := \mathrm{fact}^{-1}(x_4) \\ x_6 := \Gamma(x_5) \\ x_7 := \mathrm{rem}(x_6, x_5) \end{cases}$$

returns positive integers $x_1, \ldots, x_7$ if and only if both x and $x-2$ are prime.

Proof. For an integer $i \in \{1, \ldots, 7\}$, let D_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_i$. If $x = 1$, then $x_3 = 0$. Hence, $D_7 \subseteq D_3 \subseteq \mathbb{N} \setminus \{0,1\}$. If $x \in \{2,3,4\}$, then $x_7 = 0$. Therefore,

$$D_7 \subseteq (\mathbb{N} \setminus \{0,1\}) \cap (\mathbb{N} \setminus \{0,2,3,4\}) = \mathbb{N} \setminus \{0,1,2,3,4\}$$

By Lemma 16, for every integer $x \geqslant 5$, the term x_3 (which equals $\mathrm{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \mathcal{P} \setminus \{2,3\}$. By Lemma 18, for every $x \in \mathcal{P} \setminus \{2,3\}$, $x_3 = x - 1 \in \mathbb{N} \setminus \{0,1,2,3\}$. By Lemma 20, for every $x \in \mathcal{P} \setminus \{2,3\}$, the terms x_4 and x_5 belong to $\mathbb{N} \setminus \{0\}$ and $x_5 = x_3 - 1 = x - 2$. By Lemma 16, for every $x \in \mathcal{P} \setminus \{2,3\}$, the term x_7 (which equals $\mathrm{rem}(\Gamma(x_5), x_5)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x_5 = x - 2 \in \{4\} \cup \mathcal{P}$. From these facts, we obtain that

$$D_7 = (\mathbb{N} \setminus \{0,1,2,3,4\}) \cap (\mathcal{P} \setminus \{2,3\}) \cap (\{6\} \cup \{p+2 : \ p \in \mathcal{P}\}) =$$
$$\{p \in \mathcal{P} : \ p - 2 \in \mathcal{P}\}$$

□

Theorem 16. *The statement Φ_7 implies that there are infinitely many twin primes.*

Proof. Harvey Dubner proved that the numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime, see [15, p. 87]. By Lemma 23, for $x = 459 \cdot 2^{8529} + 1$ the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_7$. Since $x > 720! = h(7)$, the statement Φ_7 guarantees that the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_7$ for infinitely many positive integers x. By Lemma 23, there are infinitely many twin primes. □

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