The physical limits of computation inspire an open problem that concerns abstract computable sets \( X \subseteq \mathbb{N} \) and cannot be formalized in the set theory ZFC as it refers to our current knowledge on \( X \).

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Abstract. Let \( f(1) = 2, \ f(2) = 4, \) and let \( f(n+1) = f(n)! \) for every integer \( n \geq 2 \). Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2+1} \) of primes of the form \( n^2 + 1 \) is infinite. Landau’s conjecture implies the following unproven statement \( \Phi \): 

\[
\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)].
\]

Let \( B \) denote the system of equations:

\[
\{x_i! = x_k : i, k \in \{1, \ldots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\}.
\]

We write down a system \( \mathcal{U} \subseteq B \) of 9 equations which has exactly two solutions in positive integers, namely \((1, \ldots, 1)\) and \((f(1), \ldots, f(9))\). Let \( \Psi \) denote the statement: *if a system \( S \subseteq B \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_9 \), then each such solution \((x_1, \ldots, x_9)\) satisfies \( x_1, \ldots, x_9 \leq f(9) \).* We write down a system \( \mathcal{A} \subseteq B \) of 8 equations. Theorem 1. *The statement \( \Psi \) restricted to the system \( \mathcal{A} \) is equivalent to the statement \( \Phi \).*

Open Problem. *Is there a set \( X \subseteq \mathbb{N} \) that satisfies conditions (1)–(5)? (1) There are many elements of \( X \) and it is conjectured that \( X \) is infinite. (2) No known algorithm decides the finiteness/infiniteness of \( X \). (3) There is a known algorithm that for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \). (4) There is a known algorithm that computes an integer \( n \) satisfying \( \text{card}(X) < \omega \Rightarrow X \subseteq (\infty, n] \). (5) There is a naturally defined condition \( C \), which can be formalized in ZFC, such that for almost all \( k \in \mathbb{N} \), \( k \) satisfies the condition \( C \) if and only if \( k \in X \). The simplest known such condition \( C \) defines in \( \mathbb{N} \) the set \( X \). We define a set \( X \subseteq \mathbb{N} \). We prove: (i) the set \( X \) satisfies conditions (1)–(5) except the requirement that \( X \) is naturally defined; (ii) the statement \( \Phi \) implies that the set \( X = \{1\} \cup \mathcal{P}_{n^2+1} \) satisfies conditions (1)–(5). Proving Landau’s conjecture will disprove the statements (i) and (ii). Theorem 2. *No set \( X \subseteq \mathbb{N} \) will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less. Physics disproves the assumption of Theorem 2.*

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1. Basic definitions and the philosophical goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

**Definition 1.** Conditions (1)–(5) concern sets \( X \subseteq \mathbb{N} \).

1. There are many elements of \( X \) and it is conjectured that \( X \) is infinite.
2. No known algorithm decides the finiteness/infiniteness of \( X \).
3. There is a known algorithm that for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \).
4. There is a known algorithm that computes an integer \( n \) satisfying \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \).
5. There is a naturally defined condition \( C \), which can be formalized in ZFC, such that for almost all \( k \in \mathbb{N} \), \( k \) satisfies the condition \( C \) if and only if \( k \in X \). The simplest known such condition \( C \) defines in \( \mathbb{N} \) the set \( X \).

**Definition 2.** We say that an integer \( n \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \), cf. [7] and [8].

If a set \( X \subseteq \mathbb{N} \) is empty or infinite, then any integer \( n \) is a threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is non-empty and finite, then the all threshold numbers of \( X \) form the set \( [\max(X), \infty) \cap \mathbb{N} \).

Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2+1} \) of primes of the form \( n^2 + 1 \) is infinite, see [4]–[6].

**Definition 3.** Let \( \Phi \) denote the following unproven statement:

\[
\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]
\]

Landau’s conjecture implies the statement \( \Phi \). In Section 4, we heuristically justify the statement \( \Phi \) without invoking Landau’s conjecture.

**Statement 1.** No known algorithm computes an integer \( k \) such that

\[
\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]
\]

Proving the statement \( \Phi \) will disprove Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Statements 2–4 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets \( X \subseteq \mathbb{N} \).
2. The physical limits of computation inspire Open Problem 1

Definition 4. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta)))))))))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at http://wolframalpha.com $\square$.

Statement 2. The set $X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

Proof. Condition (1) holds as $X \supseteq \{0, \ldots, \beta\}$ and the set $P_{n^2+1}$ is conjecturally infinite. By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of $P_{n^2+1}$ is greater than $\beta$, see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set $\{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$ is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus condition (4) holds. $\square$

Let $[\cdot]$ denote the integer part function. For a non-negative integer $n$, let $g(n)$ denote the number of positive integers $k$ such that $n \geq \beta$ and $2^k$ divides $2^\beta \cdot \left\lfloor \frac{n}{\beta} \right\rfloor$.

Lemma 2. The function $g : \mathbb{N} \rightarrow \mathbb{N}$ satisfies $g(0) = \ldots = g(\beta - 1) = 0$ and maps $\mathbb{N} \cap [\beta, \infty)$ onto itself taking every value in $\mathbb{N} \cap [\beta, \infty)$ infinitely many times.

Statement 3. The set $X = \{n \in \mathbb{N} : g(n)^2 + 1 \text{ has no divisors greater than 1 and smaller than } g(n)^2 + 1\}$ satisfies conditions (1)–(5) except the requirement that $X$ is naturally defined.

Proof. We use Lemma 2 and argue as in the proof of Statement 2 $\square$.

Proving Landau’s conjecture will disprove Statements 2 and 3.

Open Problem 1. Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)?

Theorem 1. No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less.

Proof. The proof goes by contradiction. We fix an integer $n$ that satisfies condition (4). Since conditions (2)–(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin X, \ n + 2 \notin X, \ n + 3 \notin X, \ldots \ (T)$$
The sentences from the sequence (T) and our assumption imply that for every integer \( m > n \) computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that \( (n, m] \cap X = \emptyset \). Thus, at some future day, numerical evidence will support the conjecture that the set \( X \) is finite, contrary to the conjecture in condition (1). \( \square \)

Physics disproves the assumption of Theorem 1.

### 3. Number-theoretic statements \( \Psi_n \)

Let \( f(1) = 2, f(2) = 4 \), and let \( f(n + 1) = f(n)! \) for every integer \( n \geq 2 \). Let \( \mathcal{U}_1 \) denote the system of equations which consists of the equation \( x_1! = x_1 \). For an integer \( n \geq 2 \), let \( \mathcal{U}_n \) denote the following system of equations:

\[
\begin{align*}
    x_1! & = x_1 \\
    x_1 \cdot x_1 & = x_2 \\
    \forall i \in \{2, \ldots, n-1\} \quad x_i! & = x_{i+1}
\end{align*}
\]

The diagram in Figure 2 illustrates the construction of the system \( \mathcal{U}_n \).

\[ \text{Fig. 2} \quad \text{Construction of the system } \mathcal{U}_n \]

**Lemma 3.** For every positive integer \( n \), the system \( \mathcal{U}_n \) has exactly two solutions in positive integers, namely \((1, \ldots, 1)\) and \((f(1), \ldots, f(n))\).
Let $B_n$ denote the following system of equations:
\[
\left\{ x_i! = x_k : i, k \in \{1, \ldots, n\} \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\}
\]
For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

**Theorem 2.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

*Proof.* It follows from Lemma 3 because $U_n \subseteq B_n$.

**Theorem 3.** For every integer $n \geq 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.

*Proof.* If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement $\Psi_{n+1}$ implies that $x_i! = x_{n+1} \leq f(n + 1) = f(n)!$. Hence, $x_i \leq f(n)$.

**Theorem 4.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

*Proof.* For every positive integer $n$, the system $B_n$ has a finite number of subsystems.

4. A conjectural solution to Open Problem

**Lemma 4.** For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if
\[
(x + 1 = y) \lor (x = y = 1)
\]

**Lemma 5.** (Wilson’s theorem, [1] p. 89). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.

Let $\mathcal{A}$ denote the following system of equations:
\[
\begin{align*}
    x_2! &= x_3 \\
    x_3! &= x_4 \\
    x_4! &= x_6 \\
    x_8! &= x_9 \\
    x_1 \cdot x_1 &= x_2 \\
    x_3 \cdot x_5 &= x_6 \\
    x_4 \cdot x_8 &= x_9 \\
    x_5 \cdot x_7 &= x_8
\end{align*}
\]

Lemma 4 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.
Lemma 6. For every integer \( x_1 \geq 2 \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) is prime. In this case, the integers \( x_2, \ldots, x_9 \) are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_2^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{1}{x_1^4 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{align*}
\]

Proof. By Lemma 4, for every integer \( x_1 \geq 2 \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) divides \((x_1^2)! + 1\). Hence, the claim of Lemma 6 follows from Lemma 5.

Lemma 7. There are only finitely many tuples \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\), which solve the system \( \mathcal{A} \) and satisfy \( x_1 = 1 \). This is true as every such tuple \((x_1, \ldots, x_9)\) satisfies \( x_1, \ldots, x_9 \in \{1, 2\} \).

Proof. The equality \( x_1 = 1 \) implies that \( x_2 = x_1 \cdot x_1 = 1 \). Hence, \( x_3 = x_2^2 = 1 \). Therefore, \( x_4 = x_3^2 = 1 \). The equalities \( x_5 = x_6 \) and \( x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6 \) imply that \( x_5, x_6 \in \{1, 2\} \). The equalities \( x_8 = x_9 \) and \( x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9 \) imply that \( x_8, x_9 \in \{1, 2\} \). The equality \( x_5 \cdot x_7 = x_8 \) implies that \( x_7 = \frac{x_8}{x_5} = \frac{x_8}{2} \in \left\{ \frac{1}{2}, \frac{1}{2}, \frac{2}{2} \right\} \cap \mathbb{N} = \{1, 2\} \).
**Conjecture 1.** The statement $\Psi_9$ is true when restricted to the system $\mathcal{A}$.

**Theorem 5.** Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.

**Proof.** Suppose that the antecedent holds. By Lemma 6 there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently, $x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$.

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 6 and 7, the set $\mathcal{P}_{n^2+1}$ is infinite. \hfill $\square$

**Theorem 6.** Conjecture 1 implies the statement $\Phi$.

**Proof.** It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. \hfill $\square$

**Theorem 7.** The statement $\Phi$ implies Conjecture 1.

**Proof.** By Lemmas 6 and 7 if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 6 and the statement $\Phi$ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every $x_i$ does not exceed $f(9)$. \hfill $\square$

**Statement 4.** The statement $\Phi$ implies that the set $X = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(5).

**Proof.** The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28}]$, see [5]. These two facts imply condition (1). By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds. Condition (3) holds trivially. The statement $\Phi$ implies that $\beta$ is a threshold number of $X = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition (4) holds. The following condition:

$k - 1$ is a square and $k$ has no divisors greater than 1 and smaller than $k$

defines in $\mathbb{N}$ the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition (5). \hfill $\square$

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