The physical limits of computation inspire an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory ZFC as it refers to our current knowledge on $\mathcal{X}$

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Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ: $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\left\{x_i! = x_k :\ i,k \in \{1,\ldots,9\}\right\} \cup \left\{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,9\}\right\}$. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers, namely $(1,\ldots,1)$ and $(f(1),\ldots,f(9))$. Let Ψ denote the statement: *if a system $\mathcal{S} \subseteq B$ has at most finitely many solutions in positive integers $x_1,\ldots,x_9$, then each such solution $(x_1,\ldots,x_9)$ satisfies $x_1,\ldots,x_9 \leqslant f(9)$*. We write down a system $\mathcal{A} \subseteq B$ of 8 equations. The statement Ψ restricted to the system $\mathcal{A}$ is equivalent to the statement Φ. Open Problem: *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1)-(5)`*?* `(1)` *There are many elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.* `(2)` *No known algorithm decides the finiteness/infiniteness of $\mathcal{X}$.* `(3)` *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* `(4)` *There is a known algorithm that computes an integer n satisfying* $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(5)` *There is a naturally defined condition C, which can be formalized in ZFC, such that for almost all $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in \mathcal{X}$. The simplest known such condition C defines in $\mathbb{N}$ the set $\mathcal{X}$.* We define a set $\mathcal{X} \subseteq \mathbb{N}$. The set $\mathcal{X}$ satisfies conditions `(1)-(5)` except the requirement that $\mathcal{X}$ is naturally defined. The statement Φ implies that the set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions `(1)-(5)`. Proving Landau's conjecture will disprove the last two statements. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions `(1)-(4)` forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less. Physics disproves this assumption.

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1. Basic definitions and the aim of the article

Definition 1. *Conditions* (1)-(5) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1) *There are many elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of* $\mathcal{X}$.

(3) *There is a known algorithm that for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.

(4) *There is a known algorithm that computes an integer* n *satisfying* $\operatorname{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) *There is a naturally defined condition* $\mathcal{C}$*, which can be formalized in ZFC, such that for almost all* $k \in \mathbb{N}$*,* k *satisfies the condition* $\mathcal{C}$ *if and only if* $k \in \mathcal{X}$*. The simplest known such condition* $\mathcal{C}$ *defines in* $\mathbb{N}$ *the set* $\mathcal{X}$.

Definition 2. *We say that an integer* n *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$*, if* $\operatorname{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$*, cf.* [6] *and* [7].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite, see [3]–[5].

Definition 3. *Let* Φ *denote the following unproven statement:*

$$\operatorname{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement Φ. In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *No known algorithm computes an integer* k *such that*

$$\operatorname{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$$

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2–4 and for Open Problem 1 on abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$, see the next sections. Summarizing, Open Problem 1 is a new open problem in computability theory that cannot be formalized in *ZFC* and is related to fundamental concepts of mathematics.

2. The physical limits of computation inspire Open Problem 1

Definition 4. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions `(1)-(4)`.

Proof. Condition `(1)` holds as $\mathcal{X} \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β, see [2]. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

Let $[\cdot]$ denote the integer part function. For a non-negative integer n, let $g(n)$ denote the number of positive integers k such that $n \geqslant \beta$ and 2^k divides $2^\beta \cdot \left[\frac{n}{\beta}\right]$.

Lemma 2. *The function* $g : \mathbb{N} \to \mathbb{N}$ *satisfies* $g(0) = \ldots = g(\beta - 1) = 0$ *and maps* $\mathbb{N} \cap [\beta, \infty)$ *onto itself taking every value in* $\mathbb{N} \cap [\beta, \infty)$ *infinitely many times.*

Statement 3. *The set*

$$\mathcal{X} = \left\{n \in \mathbb{N} : g(n)^2 + 1 \text{ has no divisors greater than 1 and smaller than } g(n)^2 + 1\right\}$$

satisfies conditions `(1)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.*

Proof. We use Lemma 2 and argue as in the proof of Statement 2. □

Proving Landau's conjecture will disprove Statements 2 and 3.

Open Problem 1. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1)-(5)`*?*

Theorem 1. *No set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy conditions* `(1)-(4)` *forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies condition `(4)`. Since conditons `(2)-(4)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X},\ n + 2 \notin \mathcal{X},\ n + 3 \notin \mathcal{X},\ \ldots \qquad \texttt{(T)}$$

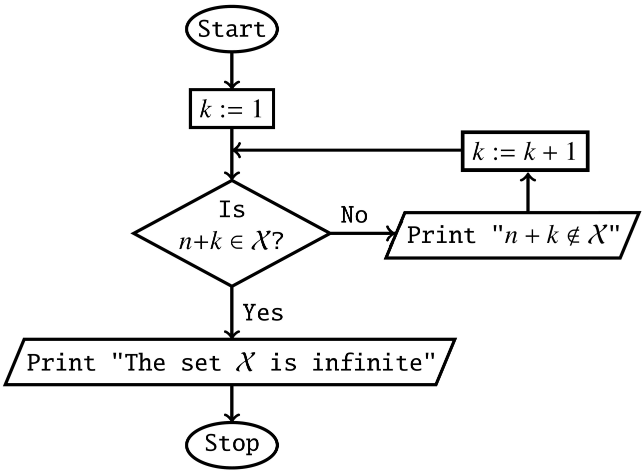

Fig. 1 A semi-algorithm that halts if and only if the set $\mathcal{X}$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in condition (1). □

Physics disproves the assumption of Theorem 1.

3. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \ldots, n-1\} \; x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

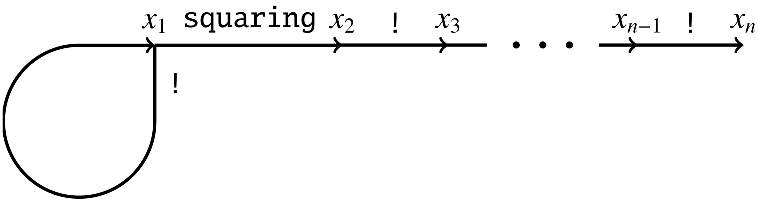

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 3. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $(f(1), \ldots, f(n))$.

Let B_n denote the following system of equations:

$$\left\{x_i! = x_k :\ i,k \in \{1,\ldots,n\}\right\} \cup \left\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\ldots,n\}\right\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 3 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$, then for every integer $i \in \{1,\ldots,n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1,\ldots,x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 4. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 5. *(Wilson's theorem,* [1, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 4 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

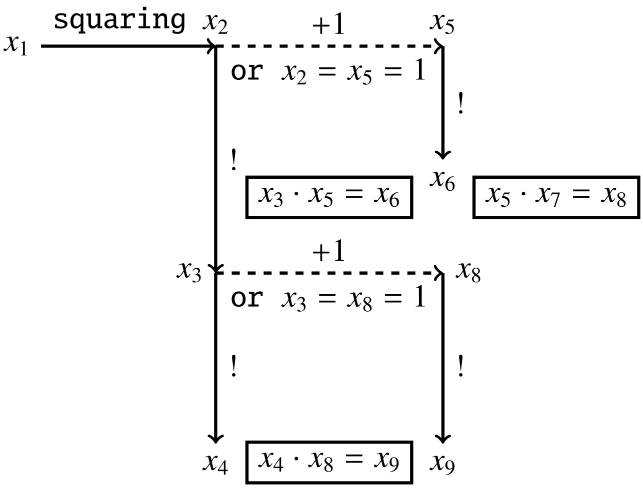

Fig. 3 Construction of the system $\mathcal{A}$

Lemma 6. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \dots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

Proof. By Lemma 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 6 follows from Lemma 5. □

Lemma 7. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \dots, x_9)$ satisfies $x_1, \dots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap \mathbb{N} = \{1, 2\}$. □

Conjecture 1. *The statement* Ψ_9 *is true when is restricted to the system* $\mathcal{A}$.

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer* $x_1 \geqslant 2$ *such that* $x_1^2 + 1$ *is prime and greater than* $f(7)$*, then the set* $\mathcal{P}_{n^2+1}$ *is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 6, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 6 and 7, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement* Φ.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement* Φ *implies Conjecture 1.*

Proof. By Lemmas 6 and 7, if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 6 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 4. *The statement* Φ *implies that the set* $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ *satisfies conditions* `(1)-(5)`.

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [4]. These two facts imply condition `(1)`. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition `(2)` holds. Condition `(3)` holds trivially. The statement Φ implies that β is a threshold number of $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition `(4)` holds. The following condition:

$k - 1$ *is a square and* k *has no divisors greater than* 1 *and smaller than* k

defines in $\mathbb{N}$ the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition `(5)`. □

Proving Landau's conjecture will disprove Statement 4.

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