THE PHYSICAL LIMITS OF COMPUTATION INSPIRE AN OPEN PROBLEM THAT CONCERNS ABSTRACT COMPUTABLE SETS $X \subseteq \mathbb{N}$ AND CANNOT BE FORMALIZED IN THE SET THEORY $ZFC$ AS IT REFERS TO OUR CURRENT KNOWLEDGE ON $X$

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Abstract. Open Problem: Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)-(5)?
(1) There are many elements of $X$ and it is conjectured that $X$ is infinite. (2) No known algorithm decides the finiteness/infiniteness of $X$. (3) There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in X$. (4) There is a known algorithm that computes an integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$. (5) There is a naturally defined condition $C$, which can be formalized in $ZFC$, such that for all except at most finitely many $k \in \mathbb{N}$, $k$ satisfies the condition $C$ if and only if $k \in X$. The simplest known such condition $C$ defines in $\mathbb{N}$ the set $X$. There is a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)-(5) except the requirement that $X$ is naturally defined. Let $P_{n^2+1}$ denote the set of primes of the form $n^2 + 1$. We heuristically prove the following statement $\Phi$: $\text{card}(P_{n^2+1}) < \omega \Rightarrow P_{n^2+1} \subseteq [2, ((24!)!)!]$. This proof does not justify that $\text{card}(P_{n^2+1}) = \omega$. The statement $\Phi$ implies that the set $X = \{1\} \cup P_{n^2+1}$ satisfies conditions (1)-(5). If we ignore the physical limits of computation, then no set $X \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever.

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1. Basic definitions and the aim of the article

Definition 1. Conditions (1)-(5) concern sets $X \subseteq \mathbb{N}$.
(1) There are many elements of $X$ and it is conjectured that $X$ is infinite.
(2) No known algorithm decides the finiteness/infiniteness of $X$.
(3) There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in X$.
(4) There is a known algorithm that computes an integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.
(5) There is a naturally defined condition $C$, which can be formalized in $ZFC$, such that for all except at most finitely many $k \in \mathbb{N}$, $k$ satisfies the condition $C$ if and only if $k \in X$. The simplest known such condition $C$ defines in $\mathbb{N}$ the set $X$.

Definition 2. We say that an integer $n$ is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, cf. [6] and [7].
If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $[\max(X), \infty) \cap \mathbb{N}$.

Edmund Landau’s conjecture states that the set $P_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [3]–[5].

**Definition 3.** Let $\Phi$ denote the following unproven statement:

$$\text{card}(P_{n^2+1}) < \omega \Rightarrow P_{n^2+1} \subseteq (\beta, \infty)$$

Landau’s conjecture implies the statement $\Phi$. In Section 4, we heuristically prove the statement $\Phi$. This proof does not justify that $\text{card}(P_{n^2+1}) = \omega$.

**Statement 1.** No known algorithm computes an integer $k$ such that $\text{card}(P_{n^2+1}) < \omega \Rightarrow P_{n^2+1} \subseteq (\beta, k]$.

Proving the statement $\Phi$ will disprove Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Statements 2–4 and for Open Problem 1 on abstract computable sets $X \subseteq \mathbb{N}$, see the next sections. Summarizing, Open Problem 1 is a new open problem in computability theory that cannot be formalized in ZFC and is related to fundamental concepts of mathematics.

2. **The physical limits of computation inspire Open Problem 1**

**Definition 4.** Let $\beta = (((24!)!)!)!$.

**Lemma 1.** $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2))))) \approx 1.42298.$

**Proof.** We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com) □

**Statement 2.** The set $X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

**Proof.** Condition (1) holds as $X \supseteq \{0, \ldots, \beta\}$ and the set $P_{n^2+1}$ is conjecturally infinite. By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of $P_{n^2+1}$ is greater than $\beta$, see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set $\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$ is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus condition (4) holds. □

Let $[\cdot]$ denote the integer part function. For a non-negative integer $n$, let $g(n)$ denote the greatest non-negative integer $k$ such that $2^k$ divides $\max\left(2^\beta \cdot \left\lfloor \frac{n}{\beta} \right\rfloor, 1\right)$.

**Lemma 2.** The function $g : \mathbb{N} \to \mathbb{N}$ satisfies $g(0) = \ldots = g(\beta - 1) = 0$ and maps $\mathbb{N} \cap [\beta, \infty)$ onto itself taking every value in $\mathbb{N} \cap [\beta, \infty)$ infinitely many times.

**Statement 3.** The set $X = \{n \in \mathbb{N} : g(n)^2 + 1 \text{ has no divisors greater than 1 and smaller than } g(n)^2 + 1\}$ satisfies conditions (1)–(5) except the requirement that $X$ is naturally defined.

**Proof.** We use Lemma 2 and argue as in the proof of Statement 2. □

Proving Landau’s conjecture will disprove Statements 2 and 3.
Open Problem 1. Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)?

Theorem 1. No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less.

Proof. The proof goes by contradiction. We fix an integer $n$ that satisfies condition (4). Since conditions (2)–(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$(T) \quad n + 1 \notin X, \ n + 2 \notin X, \ n + 3 \notin X, \ldots$$

Fig. 1 Semi-algorithm that halts if and only if the set $X$ is infinite

The sentences from the sequence $(T)$ and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $X$ is finite, contrary to the conjecture in condition (1).

Physics disproves the assumption of Theorem 1.

3. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\left\{ \begin{array}{l}
x_1! = x_1 \\
x_1 \cdot x_2! = x_2 \\
\forall i \in \{2, \ldots, n-1\} \ x_i! = x_{i+1}
\end{array} \right.$$

The diagram in Figure 2 illustrates the construction of the system $U_n$.

![Diagram](image)

**Fig. 2** Construction of the system $U_n$

**Lemma 3.** For every positive integer $n$, the system $U_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.

Let $B_n$ denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \ldots, n\} \} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n) \leq (f(n))$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \\{0\} \Psi_n$.

**Theorem 2.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

**Proof.** It follows from Lemma 3 because $U_n \subseteq B_n$. □

**Theorem 3.** For every integer $n \geq 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.

**Proof.** If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i! = x_n+1\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$. The statement $\Psi_{n+1}$ implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)!$. Hence, $x_i \leq f(n)$. □

**Theorem 4.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

## 4. A conjectural solution to Open Problem 1

**Lemma 4.** For every positive integers $x$ and $y$, $x! \cdot y! = y!$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 5.** (Wilson’s theorem, [1] p. 89). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.

Let $A$ denote the following system of equations:

$$\begin{cases}
  x_2! = x_3 \\
  x_3! = x_4 \\
  x_5! = x_6 \\
  x_8! = x_9 \\
  x_1 \cdot x_1 = x_2 \\
  x_3 \cdot x_5 = x_6 \\
  x_4 \cdot x_8 = x_9 \\
  x_5 \cdot x_7 = x_8 
\end{cases}$$

Lemma 4 and the diagram in Figure 3 explain the construction of the system $A$. 
Lemma 6. For every integer \( x_1 \geq 2 \), the system \( A \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) is prime. In this case, the integers \( x_2, \ldots, x_9 \) are uniquely determined by the following equalities:

\[
\begin{align*}
  x_2 &= x_1^2 \\
  x_3 &= (x_1^2)! \\
  x_4 &= (((x_1^2)!))! \\
  x_5 &= x_1^2 + 1 \\
  x_6 &= (x_1^2)! + 1 \\
  x_7 &= \frac{x_5 \cdot x_7}{x_6} = x_8 \\
  x_8 &= (x_1^2)! + 1 \\
  x_9 &= (((x_1^2)! + 1))!
\end{align*}
\]

Proof. By Lemma 4 for every integer \( x_1 \geq 2 \), the system \( A \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) divides \( (x_1^2)! + 1 \). Hence, the claim of Lemma 6 follows from Lemma 5.

Lemma 7. There are only finitely many tuples \( (x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 \), which solve the system \( A \) and satisfy \( x_1 = 1 \). This is true as every such tuple \( (x_1, \ldots, x_9) \) satisfies \( x_1, x_9 \in \{1, 2\} \).

Proof. The equality \( x_1 = 1 \) implies that \( x_2 = x_1 \cdot x_1 = 1 \). Hence, \( x_3 = x_2 = 1 \). Therefore, \( x_4 = x_3 = 1 \). The equalities \( x_5 = x_6 \) and \( x_7 = x_8 = x_9 = 1 \) imply that \( x_5, x_6, x_7, x_8, x_9 \in \{1, 2\} \). The equalities \( x_8 = x_9 \) and \( x_8 = 1 \) imply that \( x_8, x_9 \in \{1, 2\} \). The equality \( x_5 \cdot x_7 = x_8 \) implies that \( x_7 = \frac{x_8}{x_5} \in \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} \cap \mathbb{N} = \{1, 2\} \). □

Conjecture 1. The statement \( \Psi_9 \) is true when is restricted to the system \( A \).
Theorem 5. Conjecture \([\text{7}]\) proves the following implication: if there exists an integer \(x_1 \geq 2\) such that \(x_1^2 + 1\) is prime and greater than \(f(7)\), then the set \(P_{n+1}\) is infinite.

Proof. Suppose that the antecedent holds. By Lemma \([\text{6}]\) there exists a unique tuple \((x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8\) such that the tuple \((x_1, x_2, \ldots, x_9)\) solves the system \(\mathcal{A}\). Since \(x_1^2 + 1 > f(7)\), we obtain that \(x_1^2 \geq f(7)\). Hence, \((x_1^2)! \geq f(7)! = f(8)\). Consequently,
\[
x_9 = ((x_1^2)! + 1)! > (f(8) + 1)! > f(8)! = f(9)
\]
Conjecture \([\text{1}]\) and the inequality \(x_9 > f(9)\) imply that the system \(\mathcal{A}\) has infinitely many solutions \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\). According to Lemmas \([\text{6}]\) and \([\text{7}]\) the set \(P_{n+1}\) is infinite. \(\square\)

Theorem 6. Conjecture \([\text{7}]\) implies the statement \(\Phi\).

Proof. It follows from Theorem \([\text{5}]\) and the equality \(f(7) = (((24!)!)!)!\). \(\square\)

Theorem 7. The statement \(\Phi\) implies Conjecture \([\text{7}]\)

Proof. By Lemmas \([\text{6}]\) and \([\text{7}]\) if positive integers \(x_1, \ldots, x_9\) solve the system \(\mathcal{A}\), then
\[
(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})
\]
or \(x_1, \ldots, x_9 \in \{1, 2\}\). In the first case, Lemma \([\text{9}]\) and the statement \(\Phi\) imply that the inequality \(x_5 \leq (((24!)!)!)! = f(7)\) holds when the system \(\mathcal{A}\) has at most finitely many solutions in positive integers \(x_1, \ldots, x_9\). Hence, \(x_2 = x_5 - 1 < f(7)\) and \(x_3 = x_2! < f(7)! = f(8)\). Continuing this reasoning in the same manner, we can show that every \(x_i\) does not exceed \(f(9)\). \(\square\)

Statement 4. The statement \(\Phi\) implies that the set \(X = \{1\} \cup P_{n+1}\) satisfies conditions (1)–(5).

Proof. The set \(P_{n+1}\) is conjecturally infinite. There are 2199894223892 primes of the form \(n^2 + 1\) in the interval \([2, 10^{28}]\), see \([\text{4}]\). These two facts imply condition (1). By Lemma \([\text{1}]\) due to known physics we are not able to confirm by a direct computation that some element of \(\{1\} \cup P_{n+1}\) is greater than \(f(7) = (((24!)!)!)! = \beta\), see \([\text{2}]\). Thus condition (2) holds. Condition (3) holds trivially. The statement \(\Phi\) implies that \(\beta\) is a threshold number of \(X = \{1\} \cup P_{n+1}\). Thus condition (4) holds. The following condition:
\[
k - 1 \text{ is a square and } k \text{ has no divisors greater than 1 and smaller than } k
\]
defines in \(\mathbb{N}\) the set \(\{1\} \cup P_{n+1}\). This proves condition (5). \(\square\)

Proving Landau’s conjecture will disprove Statement \([\text{4}]\)

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References

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