

**THE PHYSICAL LIMITS OF COMPUTATION INSPIRE AN OPEN PROBLEM
THAT CONCERNS ABSTRACT COMPUTABLE SETS $\mathcal{X} \subseteq \mathbb{N}$ AND CANNOT
BE FORMALIZED IN ZFC AS IT REFERS TO OUR CURRENT
KNOWLEDGE ON \mathcal{X}**

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ABSTRACT. Open Problem: Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)-(5)?
(1) *There are many elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite.* (2) *No known algorithm decides the finiteness/infiniteness of \mathcal{X} .* (3) *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* (4) *There is a known algorithm that computes an integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.* (5) *There is a naturally defined condition C , which can be formalized in ZFC, such that for all except at most finitely many $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in \mathcal{X}$. The simplest known such condition C defines in \mathbb{N} the set \mathcal{X} .* There is a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)-(5) except the requirement that \mathcal{X} is naturally defined. Let \mathcal{P}_{n^2+1} denote the set of primes of the form $n^2 + 1$. We heuristically prove the following statement Φ : $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, (((24!)!)!)]$. This proof does not argue that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$. The statement Φ implies that the set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)-(5). If we ignore the physical limits of computation, then no set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm AL that returns $n = \begin{cases} 0 & \text{if } \text{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}) & \text{if } \text{card}(\mathcal{X}) \notin \{0, \omega\} \end{cases}$. This n satisfies the implication in condition (4). If $\mathcal{X} = \mathcal{P}_{n^2+1}$, then the algorithm AL is unknown for us because its definition is ineffective. This clarifies the distinction between known algorithms and existing algorithms.

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1. BASIC DEFINITIONS

Definition 1. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$, cf. [6] and [7].*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of \mathcal{X} form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [3]–[5].

Definition 2. *Let Φ denote the following unproven statement:*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)]$$

Landau's conjecture implies the statement Φ . In Section 4, we heuristically prove the statement Φ . This proof does not argue that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. *No known algorithm computes an integer k such that*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$$

There exists an algorithm ALG that returns $k = \begin{cases} 0 & \text{if } \text{card}(\mathcal{P}_{n^2+1}) = \omega \\ \max(\mathcal{P}_{n^2+1}) & \text{if } \text{card}(\mathcal{P}_{n^2+1}) < \omega \end{cases}$.

This k satisfies the implication in Statement 1. The algorithm ALG is unknown for us because its definition is ineffective. This clarifies the distinction between known algorithms and existing algorithms. Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2–4 in the next sections.

Definition 3. *Conditions (1)–(5) concern sets $\mathcal{X} \subseteq \mathbb{N}$.*

(1) *There are many elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of \mathcal{X} .*

(3) *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*

(4) *There is a known algorithm that computes an integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.*

(5) *There is a naturally defined condition C , which can be formalized in *ZFC*, such that for all except at most finitely many $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in \mathcal{X}$. The simplest known such condition C defines in \mathbb{N} the set \mathcal{X} .*

2. THE PHYSICAL LIMITS OF COMPUTATION INSPIRE OPEN PROBLEM 1

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)?*

Open Problem 1, which concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$, represents an open problem in computability theory that cannot be formalized in *ZFC* and is related to fundamental concepts of arithmetic.

Definition 4. *Let $\beta = (((24!)!)!)!$.*

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. □

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\mathcal{X} \supseteq \{0, \dots, \beta\}$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. □

Let $[\cdot]$ denote the integer part function. For a non-negative integer n , let $g(n)$ denote the greatest non-negative integer k such that 2^k divides $\max\left(2^\beta \cdot \left[\frac{n}{\beta}\right], 1\right)$.

Lemma 2. *The function $g : \mathbb{N} \rightarrow \mathbb{N}$ satisfies $g(0) = \dots = g(\beta - 1) = 0$ and maps $\mathbb{N} \cap [\beta, \infty)$ onto itself taking every value in $\mathbb{N} \cap [\beta, \infty)$ infinitely many times.*

Statement 3. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : g(n)^2 + 1 \text{ has no divisors greater than } 1 \text{ and smaller than } g(n)^2 + 1\}$$

satisfies conditions (1)-(5) except the requirement that \mathcal{X} is naturally defined.

Proof. We use Lemma 2 and argue as in the proof of Statement 2. □

Proving Landau's conjecture will disprove Statements 2 and 3.

Theorem 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies condition (4). Since conditions (2)-(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$(T) \quad n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots$$

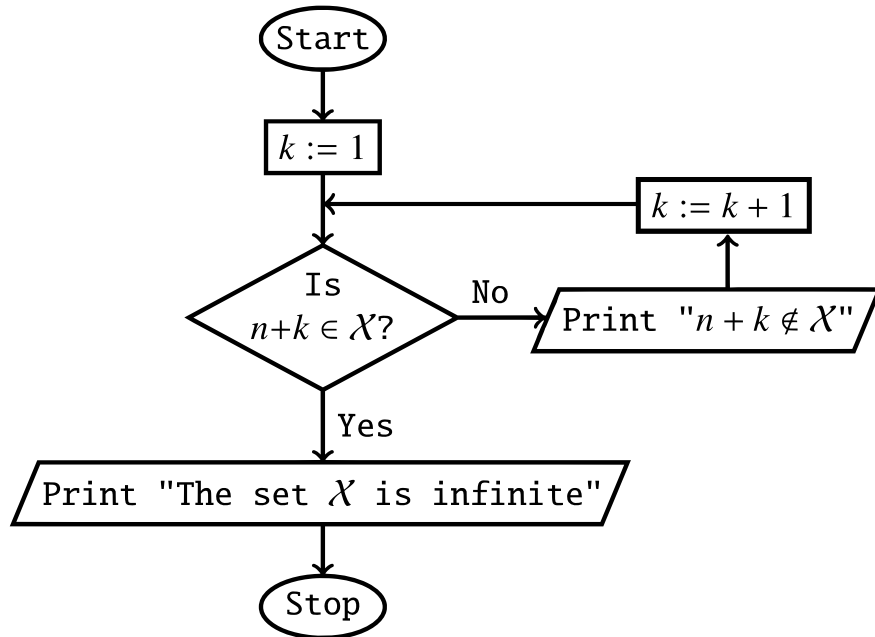


Fig. 1 Semi-algorithm that halts if and only if the set \mathcal{X} is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set \mathcal{X} is finite, contrary to the conjecture in condition (1). □

Physics disproves the assumption of Theorem 1.

3. NUMBER-THEORETIC STATEMENTS Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system \mathcal{U}_n .

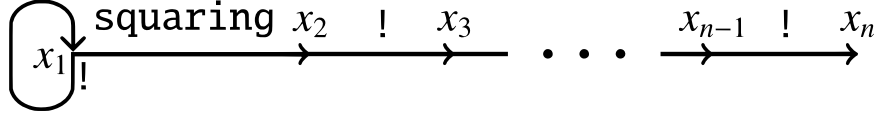


Fig. 2 Construction of the system \mathcal{U}_n

Lemma 3. *For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.*

Let B_n denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Ψ_n denote the following statement: *if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

Theorem 2. *For every statement Ψ_n , the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 3 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geq 2$, the statement Ψ_{n+1} implies the statement Ψ_n .*

Proof. If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then for every integer $i \in \{1, \dots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers x_1, \dots, x_{n+1} . The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)!$. Hence, $x_i \leq f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

4. A CONJECTURAL SOLUTION TO OPEN PROBLEM 1

Lemma 4. *For every positive integers x and y , $x! \cdot y = y!$ if and only if*

$$(x+1 = y) \vee (x = y = 1)$$

Lemma 5. *(Wilson's theorem, [1, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x-1)! + 1$.*

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 4 and the diagram in Figure 3 explain the construction of the system \mathcal{A} .

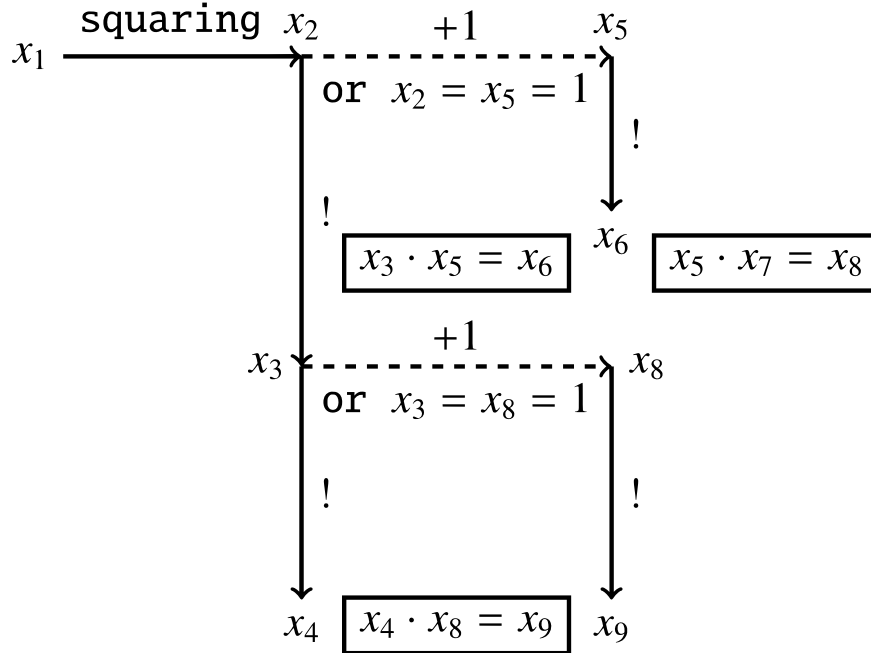


Fig. 3 Construction of the system \mathcal{A}

Lemma 6. For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 4, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 6 follows from Lemma 5. \square

Lemma 7. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$. This is true as every such tuple (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap \mathbb{N} = \{1, 2\}$. \square

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system \mathcal{A} .*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 6, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 6 and 7, the set \mathcal{P}_{n^2+1} is infinite. \square

Theorem 6. *Conjecture 1 implies the statement Φ .*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. \square

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 6 and 7, if positive integers x_1, \dots, x_9 solve the system \mathcal{A} , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 6 and the statement Φ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 . Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. \square

Statement 4. *The statement Φ implies that the set $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(5).*

Proof. The set \mathcal{P}_{n^2+1} is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [4]. These two facts imply condition (1). By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds. Condition (3) holds trivially. The statement Φ implies that β is a threshold number of $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition (4) holds. The following condition:

$k - 1$ is a square and k has no divisors greater than 1 and smaller than k

defines in \mathbb{N} the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition (5). \square

Proving Landau's conjecture will disprove Statement 4.

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REFERENCES

- [1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [2] S. Lloyd, *Ultimate physical limits to computation*, *Nature* 406 (2000), 1047–1054, <http://doi.org/10.1038/35023282>.
- [3] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A002496, *Primes of the form $n^2 + 1$* , <http://oeis.org/A002496>.
- [4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A083844, *Number of primes of the form $x^2 + 1 < 10^n$* , <http://oeis.org/A083844>.
- [5] Wolfram MathWorld, *Landau's Problems*, <http://mathworld.wolfram.com/LandausProblems.html>.
- [6] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity*, Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, <http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm>.
- [7] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer*, in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

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