On sets $\mathcal{W} \subseteq \mathbb{N}$ such that the infinity of $\mathcal{W}$ is equivalent to the existence in $\mathcal{W}$ of an element that is greater than a threshold number computed with using the definition of $\mathcal{W}$

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Abstract—Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geqslant 2$. For a positive integer n, let Θ_n denote the statement: if a system $\mathcal{S} \subseteq \left\{x_i! = x_k : i, k \in \{1, \dots, n\}\right\} \cup \left\{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\right\}$ has only finitely many solutions in integers $x_1, \dots, x_n$ greater than 1, then each such solution $(x_1, \dots, x_n)$ satisfies $\min(x_1, \dots, x_n) \leqslant f(n)$. The statement Θ_9 proves that if there exists an integer $x > f(9)$ such that $x^2 + 1$ (alternatively, $x! + 1$) is prime, then there are infinitely many primes of the form $n^2 + 1$ (respectively, $n! + 1$). The statement Θ_{16} proves that if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes. We formulate the statements Φ_n and prove: Φ_4 equivalently expresses that there are infinitely many primes of the form $n! + 1$, Φ_6 implies that for infinitely many primes p the number $p! + 1$ is prime, Φ_6 implies that there are infinitely many primes of the form $n! - 1$, Φ_7 implies that there are infinitely many twin primes.

Index Terms—composite Fermat numbers, prime numbers of the form $n! + 1$, prime numbers of the form $n! - 1$, prime numbers of the form $n^2 + 1$, prime numbers p such that $p! + 1$ is prime, single query to an oracle for the halting problem, twin prime conjecture.

I. Spectra of sentences and their threshold numbers

THE following observation concerns the theme described in the title of the article.

Observation 1. *If $\mathcal{W}$ is a subset of $\{0, \dots, n\}$ where n is a non-negative integer, then we take any integer $m \geqslant n$ as a threshold number for $\mathcal{W}$. If $\mathcal{W}$ is an infinite subset of $\mathbb{N}$, then we take any non-negative integer m as a threshold number for $\mathcal{W}$.*

We define the set $\mathcal{U} \subseteq \mathbb{N}$ by declaring that a non-negative integer n belongs to $\mathcal{U}$ if and only if $\sin\left(10^{10^{10^{10^{10}}}}\right) > 0$. This inequality is practically undecidable, see [5].

Corollary 1. *The set $\mathcal{U}$ equals $\emptyset$ or $\mathbb{N}$. The statement "$\mathcal{U} = \emptyset$" remains unproven and the statement "$\mathcal{U} = \mathbb{N}$" remains unproven. Every non-negative integer m is a threshold number for $\mathcal{U}$. For every non-negative integer k, the sentence "$k \in \mathcal{U}$" is only theoretically decidable.*

The first-order language of graph theory contains two relation symbols of arity 2: $\sim$ and $=$, respectively for adjacency and equality of vertices. The term first-order imposes the condition that the variables represent vertices and hence the quantifiers apply to vertices only. For a first-order sentence Λ about graphs, let Spectrum(Λ) denote the set of all positive integers n such that there is a graph on n vertices satisfying Λ. By a graph on n vertices we understand a set of n elements with a binary relation which is symmetric and irreflexive.

Theorem 1. *([12, p. 171]). If a sentence Λ in the language of graph theory has the form $\exists x_1 \dots x_k \; \forall y_1 \dots y_l \; \Upsilon(x_1, \dots, x_k, y_1, \dots, y_l)$, where $\Upsilon(x_1, \dots, x_k, y_1, \dots, y_l)$ is quantifier-free, then either* Spectrum(Λ) $\subseteq [1, (2^k \cdot 4^l) - 1]$ *or* Spectrum(Λ) $\supseteq [k + l, \infty) \cap \mathbb{N}$.

Corollary 2. *The number $\left(2^k \cdot 4^l\right) - 1$ is a threshold number for* Spectrum(Λ).

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [10, p. 234].

Corollary 3. *If an algorithm $\mathtt{Alg}_1$ for every recursive set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_1(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Alg}_1(\mathcal{M}) + 1, \infty) \neq \emptyset$.*

Corollary 4. *If an algorithm $\mathtt{Alg}_2$ for every recursively enumerable set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_2(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Alg}_2(\mathcal{M}) + 1, \infty) \neq \emptyset$.*

Let $K = \{j \in \mathbb{N} : 2^{\aleph_j} = \aleph_{j+1}\}$.

Theorem 2. *If ZFC is consistent, then for every non-negative integer n the sentence*

$$\text{"n is a threshold number for K"}$$

is not provable in ZFC.

Proof. There exists a model $\mathcal{E}$ of ZFC such that

$$\forall i \in \{0, \ldots, n+1\} \; \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+1}$$

and

$$\forall i \in \{n+2, n+3, n+4, \ldots\} \; \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+2}$$

see [3] and [6, p. 232]. In the model $\mathcal{E}$, $K = \{0, \ldots, n+1\}$ and n is not a threshold number for K. $\square$

Theorem 3. *If ZFC is consistent, then for every non-negative integer n the sentence*

$$\text{"}n \text{ is not a threshold number for } K\text{"}$$

is not provable in ZFC.

Proof. The Generalized Continuum Hypothesis (GCH) is consistent with ZFC, see [6, p. 188] and [6, p. 190]. GCH implies that $K = \mathbb{N}$. Consequently, GCH implies that every non-negative integer n is a threshold number for K. $\square$

II. Basic lemmas

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{V}_1$ denote the system of equations $\{x_1! = x_1\}$, and let $\mathcal{V}_2$ denote the system of equations $\{x_1! = x_1, \; x_1 \cdot x_1 = x_2\}$. For an integer $n \geq 3$, let $\mathcal{V}_n$ denote the following system of equations:

$$\begin{cases} x_1! & = & x_1 \\ x_1 \cdot x_1 & = & x_2 \\ \forall i \in \{2, \ldots, n-1\} \; x_i! & = & x_{i+1} \end{cases}$$

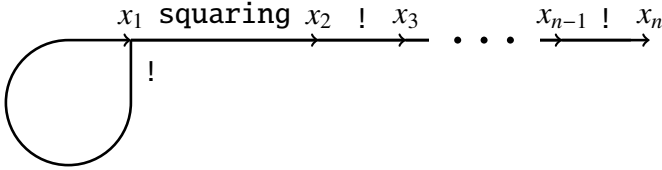The diagram in Figure 1 illustrates the construction of the system $\mathcal{V}_n$.

Fig. 1 Construction of the system $\mathcal{V}_n$

Lemma 1. *For every positive integer n, the system $\mathcal{V}_n$ has exactly one solution in integers greater than 1, namely $\big(f(1), \ldots, f(n)\big)$.*

Let

$$H_n = \Big\{x_i! = x_k : \; i, k \in \{1, \ldots, n\}\Big\} \cup$$

$$\Big\{x_i \cdot x_j = x_k : \; i, j, k \in \{1, \ldots, n\}\Big\}$$

For a positive integer n, let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers $x_1, \ldots, x_n$ greater than 1, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leq f(n)$. The assumption $\min(x_1, \ldots, x_n) \leq f(n)$ is weaker than the assumption $\max(x_1, \ldots, x_n) \leq f(n)$ suggested by Lemma 1.

Lemma 2. *For every positive integer n, the system H_n has a finite number of subsystems.*

Theorem 4. *Every statement Θ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 2. $\square$

Lemma 3. *For every integers x and y greater than 1, $x! \cdot y = y!$ if and only if $x + 1 = y$.*

Lemma 4. *If $x \geq 4$, then $\dfrac{(x-1)! + 1}{x} > 1$.*

Lemma 5. *(Wilson's theorem, [4, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x-1)! + 1$.*

III. Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [11]. It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [18, p. 297].

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! & = & x_2 \\ x_2! & = & x_3 \\ x_5! & = & x_6 \\ x_4 \cdot x_4 & = & x_5 \\ x_3 \cdot x_5 & = & x_6 \end{cases}$$

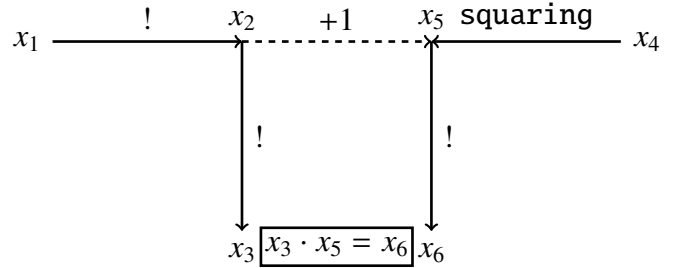Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 6. *For every integers x_1 and x_4 greater than 1, the system $\mathcal{A}$ is solvable in integers x_2, x_3, x_5, x_6 greater than 1 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 & = & x_1! \\ x_3 & = & (x_1!)! \\ x_5 & = & x_1! + 1 \\ x_6 & = & (x_1! + 1)! \end{aligned}$$

and $x_1 = \min(x_1, \ldots, x_6)$.

Proof. It follows from Lemma 3. $\square$

Theorem 5. *The statement Θ_6 proves the following implication: if the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then each such solution (x_1, x_4) satisfies $x_1 \leq f(6)$.*

Proof. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 6, there exists a

unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0, 1\})^4$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Lemma 6 guarantees that $x_1 = \min(x_1, \ldots, x_6)$. By the antecedent and Lemma 6, the system $\mathcal{A}$ has only finitely many solutions in integers $x_1, \ldots, x_6$ greater than 1. Therefore, the statement Θ_6 implies that $x_1 = \min(x_1, \ldots, x_6) \leqslant f(6)$. $\qquad\square$

Hypothesis 1. *The implication in Theorem 5 is true.*

Corollary 5. *Assuming Hypothesis 1, a single query to an oracle for the halting problem decides the problem of the infinitude of the solutions of the equation $x! + 1 = y^2$.*

IV. Are there infinitely many prime numbers of the form $n^2 + 1$?

Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [9, pp. 37–38].

Let $\mathcal{B}$ denote the following system of equations:

$$
\begin{cases}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{cases}
$$

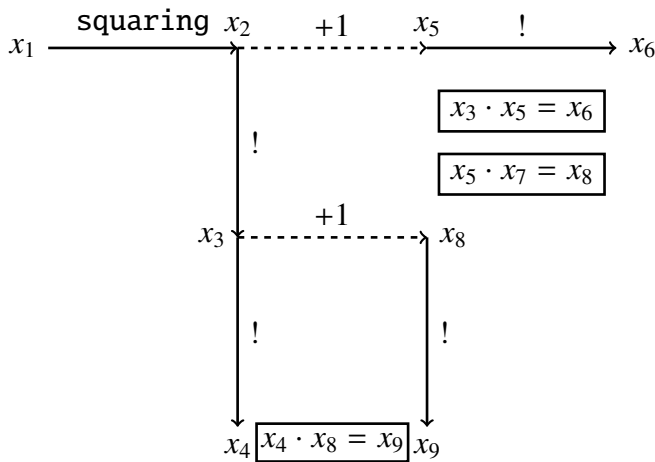Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 7. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$*

is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

$$
\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}
$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. $\qquad\square$

Theorem 6. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Lemma 7 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. $\qquad\square$

Hypothesis 2. *The implication in Theorem 6 is true.*

Corollary 6. *Assuming Hypothesis 2, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n^2 + 1$.*

V. Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [1, p. 443] and [14]. Let $\mathcal{G}$ denote the following system of equations:

$$
\begin{cases}
x_1! &= x_2 \\
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{cases}
$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system $\mathcal{G}$.

Fig. 4 Construction of the system $\mathcal{G}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_2 &= x_1! \\
x_3 &= (x_1!)! \\
x_4 &= ((x_1!)!)! \\
x_5 &= x_1^! + 1 \\
x_6 &= (x_1! + 1)! \\
x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\
x_8 &= (x_1!)! + 1 \\
x_9 &= ((x_1!)! + 1)!
\end{aligned}
$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 5. □

Theorem 7. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1! + 1$ is prime. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{G}$. Lemma 8 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{G} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{G}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 8, there are infinitely many primes of the form $n! + 1$. □

Hypothesis 3. *The implication in Theorem 7 is true.*

Corollary 7. *Assuming Hypothesis 3, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n! + 1$.*

VI. THE TWIN PRIME CONJECTURE

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [9, p. 39].

Let C denote the following system of equations:

$$
\begin{cases}
x_1! &= x_2 \\
x_2! &= x_3 \\
x_4! &= x_5 \\
x_6! &= x_7 \\
x_7! &= x_8 \\
x_9! &= x_{10} \\
x_{12}! &= x_{13} \\
x_{15}! &= x_{16} \\
x_2 \cdot x_4 &= x_5 \\
x_5 \cdot x_6 &= x_7 \\
x_7 \cdot x_9 &= x_{10} \\
x_4 \cdot x_{11} &= x_{12} \\
x_3 \cdot x_{12} &= x_{13} \\
x_9 \cdot x_{14} &= x_{15} \\
x_8 \cdot x_{15} &= x_{16}
\end{cases}
$$

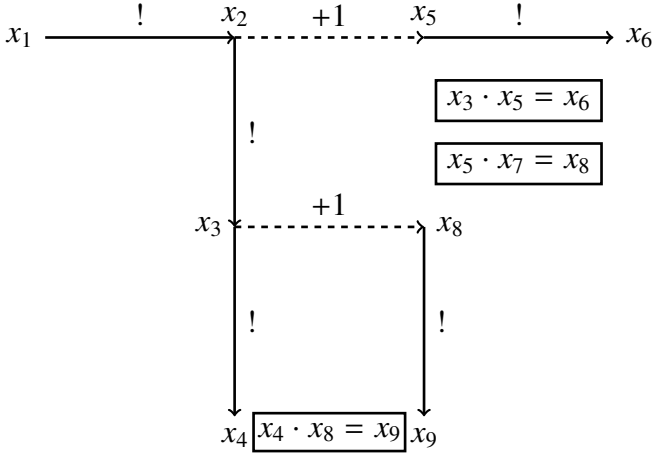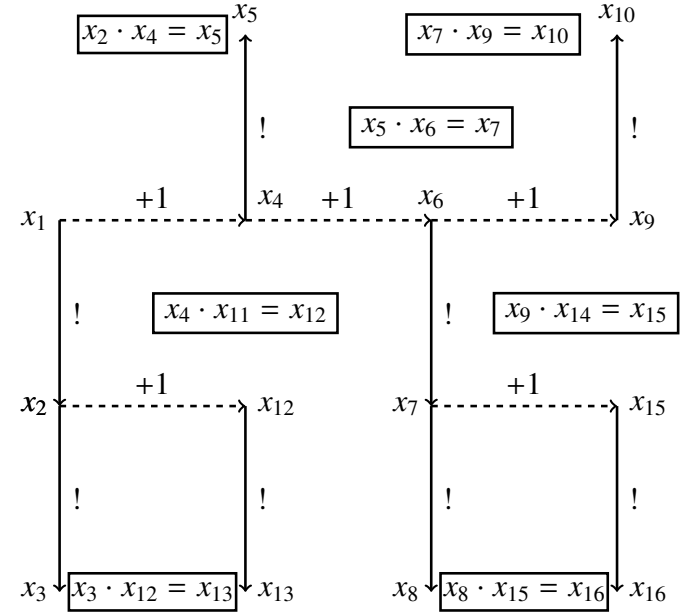Lemma 3 and the diagram in Figure 5 explain the construction of the system C.

Fig. 5 Construction of the system C

Lemma 9. *If $x_4 = 2$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. □

Lemma 10. *If $x_4 = 3$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. □

Lemma 11. *For every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}
$$

and $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3$.

Proof. By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if

$$
\left(x_4 + 2 = x_9\right) \wedge \left(x_4 | (x_4 - 1)! + 1\right) \wedge \left(x_9 | (x_9 - 1)! + 1\right)
$$

Hence, the claim of Lemma 11 follows from Lemma 5. $\qquad\square$

Theorem 8. *The statement Θ_{16} proves the following implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.*

Proof. Assume the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 11, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0, 1\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Lemma 11 guarantees that $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement Θ_{16} and the inequality $\min(x_1, \ldots, x_{16}) > f(16)$ imply that the system C has infinitely many solutions in integers $x_1, \ldots, x_{16}$ greater than 1. According to Lemmas 9–11, there are infinitely many twin primes. $\qquad\square$

Hypothesis 4. *The implication in Theorem 8 is true.*

Corollary 8. *(cf. [2]). Assuming Hypothesis 4, a single query to an oracle for the halting problem decides the twin prime problem.*

VII. ARE THERE INFINITELY MANY COMPOSITE FERMAT NUMBERS?

Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [8, p. 1]. Fermat correctly remarked that

$2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [8, p. 1].

Open Problem. ([8, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [7, p. 23].

Lemma 12. *([8, p. 38]). For every positive integer n, if a prime number p divides $2^{2^n} + 1$, then there exists a positive integer k such that $p = k \cdot 2^{n+1} + 1$.*

Corollary 9. *Since $k \cdot 2^{n+1} + 1 \geqslant 2^{n+1} + 1 \geqslant n + 3$, for every positive integers x, y, and n, the equality $(x+1)(y+1) = 2^{2^n} + 1$ implies that $\min(n, x, x+1, y, y+1) = n$.*

Let $g(1) = 1$, and let $g(n+1) = 2^{2^{g(n)}}$ for every positive integer n. Let

$$
G_n = \left\{ x_i \cdot x_j = x_k : \; i, j, k \in \{1, \ldots, n\} \right\} \cup
$$

$$
\left\{ 2^{2^{x_i}} = x_k : \; i, k \in \{1, \ldots, n\} \right\}
$$

Lemma 13. *The following subsystem of G_n*

$$
\begin{cases}
x_1 \cdot x_1 = x_1 \\
\forall i \in \{1, \ldots, n-1\} \; 2^{2^{x_i}} = x_{i+1}
\end{cases}
$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(g(1), \ldots, g(n))$.

For a positive integer n, let Ψ_n denote the following statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leqslant g(n)$. The assumption $\min(x_1, \ldots, x_n) \leqslant g(n)$ is weaker than the assumption $\max(x_1, \ldots, x_n) \leqslant g(n)$ suggested by Lemma 13.

Lemma 14. *For every positive integer n, the system G_n has a finite number of subsystems.*

Theorem 9. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 14. $\qquad\square$

Lemma 15. *For every non-negative integers b and c, $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Theorem 10. *The statement Ψ_{13} proves the following implication: if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n.*

Proof. Let us consider the equation

$$
(x+1)(y+1) = 2^{2^z} + 1 \tag{1}
$$

in positive integers. By Lemma 15, we can transform equation (1) into an equivalent system $\mathcal{F}$ which has 13 variables

(x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^{\alpha}} = \gamma$, see the diagram in Figure 6.
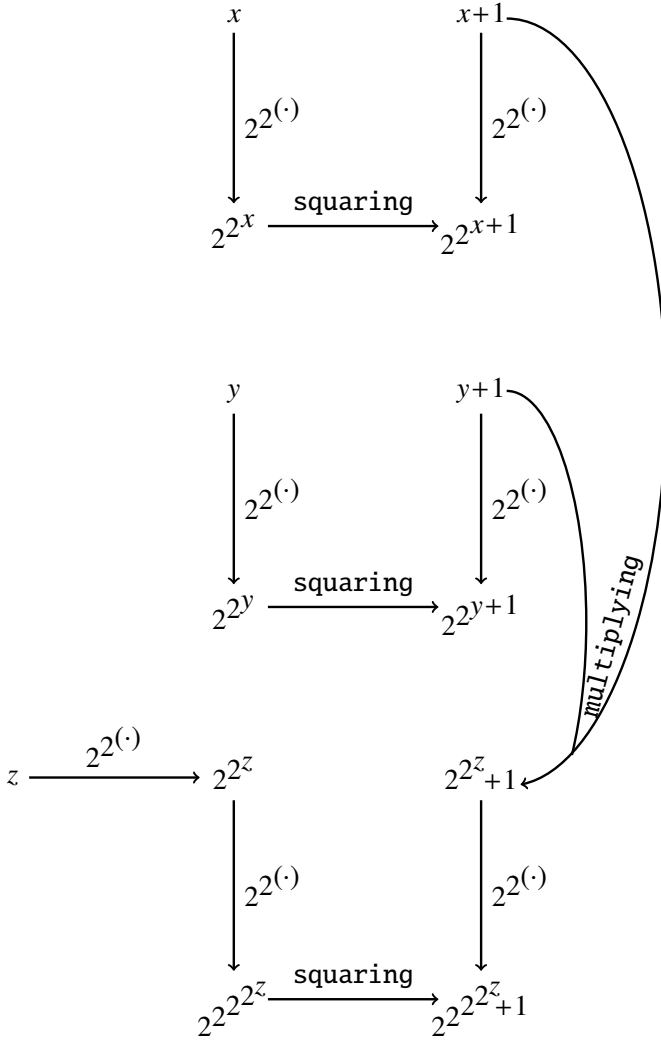

Fig. 6 Construction of the system $\mathcal{F}$

Assume that $2^{2^n} + 1$ is composite for some integer $n > g(13)$. By this and Corollary 9, equation (1) has a solution $(x, y, z) \in (\mathbb{N} \setminus \{0\})^3$ such that $z = n$ and $z = \min(z, x, x + 1, y, y + 1)$. Hence, the system $\mathcal{F}$ has a solution in positive integers such that $z = n$ and n is the smallest number in the solution sequence. Since $n > g(13)$, the statement Ψ_{13} implies that the system $\mathcal{F}$ has infinitely many solutions in positive integers. Therefore, there are infinitely many positive integers n such that $2^{2^n} + 1$ is composite. $\square$

Hypothesis 5. *The implication in Theorem 10 is true.*

Corollary 10. *Assuming Hypothesis 5, a single query to an oracle for the halting problem decides whether or not the set of composite Fermat numbers is infinite.*

VIII. Computations of length n and the statements Φ_n

For a positive integer x, let $\Gamma(x)$ denote $(x - 1)!$. Let $\mathrm{fact}^{-1} \colon \{1, 2, 6, 24, \ldots\} \to \mathbb{N} \setminus \{0\}$ denote the inverse function to the factorial function. For positive integers x and y, let $\mathrm{rem}(x, y)$ denote the remainder from dividing x by y.

Definition. *For a positive integer n, by a computation of length n we understand any sequence of terms $x_1, \ldots, x_n$ such that x_1 is defined as the variable x, and for every integer $i \in \{2, \ldots, n\}$, x_i is defined as $\Gamma(x_{i-1})$, or $\mathrm{fact}^{-1}(x_{i-1})$, or $\mathrm{rem}(x_{i-1}, x_{i-2})$ (only if $i \geqslant 3$ and x_{i-1} is defined as $\Gamma(x_{i-2})$).*

Let $\mathcal{P}$ denote the set of prime numbers.

Lemma 16. *([13, pp. 214–215]). For every positive integer x, $\mathrm{rem}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$.*

Let $h(4) = 3$, and let $h(n + 1) = h(n)!$ for every integer $n \geqslant 4$.

Theorem 11. *For every integer $n \geqslant 4$ and for every positive integer x, the following computation $\mathcal{H}_n$*

$$
\begin{cases}
& x_1 & := & x \\
\forall i \in \{2, \ldots, n - 3\}\ & x_i & := & \mathrm{fact}^{-1}(x_{i-1}) \\
& x_{n-2} & := & \Gamma(x_{n-3}) \\
& x_{n-1} & := & \Gamma(x_{n-2}) \\
& x_n & := & \mathrm{rem}(x_{n-1}, x_{n-2})
\end{cases}
$$

returns positive integers $x_1, \ldots, x_n$ if and only if $x = h(n)$.

Proof. We make three observations.

Observation 2. *If $x_{n-3} = 3$, then $x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x = x_1 = h(n)$. If $x = h(n)$, then $x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x_{n-3} = 3$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 2$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = 1$.*

Observation 3. *If $x_{n-3} = 2$, then $x = x_1 = \ldots = x_{n-3} = 2$. If $x = 2$, then $x_1 = \ldots = x_{n-3} = 2$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 1$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.*

Observation 4. *If $x_{n-3} = 1$, then $x_{n-2} = \Gamma(x_{n-3}) = 1$. Hence, $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.*

Observations 2–4 cover the case when $x_{n-3} \in \{1, 2, 3\}$. If $x_{n-3} \geqslant 4$, then $x_{n-2} = \Gamma(x_{n-3})$ is greater than 4 and composite. By Lemma 16, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = \mathrm{rem}(\Gamma(x_{n-2}), x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$. $\square$

For an integer $n \geqslant 4$, let Φ_n denote the following statement: if a computation of length n returns positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers x, then every such x does not exceed $h(n)$.

Theorem 12. *For every integer $n \geqslant 4$, the bound $h(n)$ in the statement Φ_n cannot be decreased.*

Proof. It follows from Theorem 11. $\square$

Lemma 17. *For every positive integer n, there are only finitely many computations of length n.*

Theorem 13. *For every integer $n \geqslant 4$, the statement Φ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 17. $\qquad\square$

IX. Consequences of the statements $\Phi_4, \ldots, \Phi_7$

Lemma 18. *If $x \in \mathcal{P}$, then $\mathrm{rem}(\Gamma(x), x) = x - 1$.*

Proof. It follows from Lemma 5. $\qquad\square$

Lemma 19. *For every positive integer x, the following computation $\mathcal{T}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \mathrm{rem}(x_2, x_1) \\ x_4 & := & \mathrm{fact}^{-1}(x_3) \end{cases}$$

returns positive integers $x_1, \ldots, x_4$ if and only if $x = 4$ or x is a prime number of the form $n! + 1$.

Proof. For an integer $i \in \{1, \ldots, 4\}$, let T_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{T}$ returns positive integers $x_1, \ldots, x_i$. We show that

$$T_4 = \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P}) \qquad (2)$$

For every positive integer x, the terms x_1 and x_2 belong to $\mathbb{N} \setminus \{0\}$. By Lemma 16, the term x_3 (which equals $\mathrm{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$. Hence, $T_3 = \{4\} \cup \mathcal{P}$. If $x = 4$, then $x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in T_4$. If $x \in \mathcal{P}$, then Lemma 18 implies that $x_3 = \mathrm{rem}(\Gamma(x), x) = x - 1 \in \mathbb{N} \setminus \{0\}$. Therefore, for every $x \in \mathcal{P}$, the term $x_4 = \mathrm{fact}^{-1}(x_3)$ belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\}$. This proves equality (2). $\qquad\square$

Theorem 14. *The statement Φ_4 implies that the set of primes of the form $n! + 1$ is infinite.*

Proof. The number $3! + 1 = 7$ is prime. By Lemma 19, for $x = 7$ the computation $\mathcal{T}$ returns positive integers $x_1, \ldots, x_4$. Since $x = 7 > 3 = h(4)$, the statement Φ_4 guarantees that the computation $\mathcal{T}$ returns positive integers $x_1, \ldots, x_4$ for infinitely many positive integers x. By Lemma 19, there are infinitely many primes of the form $n! + 1$. $\qquad\square$

Lemma 20. *If $x \in \mathbb{N} \setminus \{0, 1\}$, then $\mathrm{fact}^{-1}(\Gamma(x)) = x - 1$.*

Theorem 15. *If the set of primes of the form $n! + 1$ is infinite, then the statement Φ_4 is true.*

Proof. There exist exactly 10 computations of length 4 that differ from $\mathcal{H}_4$ and $\mathcal{T}$, see Table 1. For every such computation $\mathcal{F}_i$, we determine the set S_i of all positive integers x such that the computation $\mathcal{F}_i$ outputs positive integers $x_1, \ldots, x_4$ on input x. We omit 10 easy proofs which use Lemmas 16 and 20. The sets S_i are infinite, see Table 1.

	x_1	x_2	x_3	x_4	condition
$\mathcal{F}_1$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \mathbb{N} \setminus \{0\} = S_1$
$\mathcal{F}_2$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \mathbb{N} \setminus \{0\} = S_2$
$\mathcal{H}_4$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{rem}(x_3, x_2)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x = 3$
$\mathcal{F}_3$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \mathbb{N} \setminus \{0\} = S_3$
$\mathcal{F}_4$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{1\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} = S_4$
$\mathcal{F}_5$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{rem}(x_2, x_1)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{4\} \cup \mathcal{P} = S_5$
$\mathcal{T}$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{rem}(x_2, x_1)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$
$\mathcal{F}_6$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{n! : n \in \mathbb{N} \setminus \{0\}\} = S_6$
$\mathcal{F}_7$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{rem}(x_3, x_2)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{4!\} \cup \{p! : p \in \mathcal{P}\} = S_7$
$\mathcal{F}_8$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{n! : n \in \mathbb{N} \setminus \{0\}\} = S_8$
$\mathcal{F}_9$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{(n!)! : n \in \mathbb{N} \setminus \{0\}\} = S_9$
$\mathcal{F}_{10}$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \iff x \in \{((n!)!)! : n \in \mathbb{N} \setminus \{0\}\} = S_{10}$

Tab. 1 12 computations of length 4, $x \in \mathbb{N} \setminus \{0\}$

This completes the proof. $\qquad\square$

Hypothesis 6. *The statements $\Phi_4, \ldots, \Phi_7$ are true.*

Lemma 21. *For every positive integer x, the following computation $\mathcal{Y}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \mathrm{rem}(x_2, x_1) \\ x_4 & := & \mathrm{fact}^{-1}(x_3) \\ x_5 & := & \Gamma(x_4) \\ x_6 & := & \mathrm{rem}(x_5, x_4) \end{cases}$$

returns positive integers $x_1, \ldots, x_6$ if and only if $x \in \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}$.

Proof. For an integer $i \in \{1, \ldots, 6\}$, let Y_i denote the set of positive integers x such that the first i instructions of the

computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_i$. Since the computations $\mathcal{T}$ and $\mathcal{Y}$ have the same first four instructions, the equality $Y_i = T_i$ holds for every $i \in \{1, \ldots, 4\}$. In particular,

$$Y_4 = \{4\} \cup (\{n! + 1 : \ n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$$

We show that

$$Y_6 = \{4\} \cup (\{p! + 1 : \ p \in \mathcal{P}\} \cap \mathcal{P}) \qquad (3)$$

If $x = 4$, then $x_1, \ldots, x_6 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in Y_6$. Let $x \in \mathcal{P}$, and let $x = n! + 1$, where $n \in \mathbb{N} \setminus \{0\}$. Hence, $n \neq 4$. Lemma 18 implies that $x_3 = \text{rem}(\Gamma(x), x) = x - 1 = n!$. Hence, $x_4 = \text{fact}^{-1}(x_3) = n$ and $x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\}$. By Lemma 16, the term x_6 (which equals $\text{rem}(\Gamma(n), n)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $n \in \{4\} \cup \mathcal{P}$. This proves equality (3) as $n \neq 4$. $\qquad\square$

Theorem 16. *The statement Φ_6 implies that for infinitely many primes p the number $p! + 1$ is prime.*

Proof. The numbers 11 and $11! + 1$ are prime, see [1, p. 441] and [16]. By Lemma 21, for $x = 11! + 1$ the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_6$. Since $x = 11! + 1 > 6! = h(6)$, the statement Φ_6 guarantees that the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 21, for infinitely many primes p the number $p! + 1$ is prime. $\qquad\square$

Lemma 22. *For every positive integer x, the following computation $\mathcal{L}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \Gamma(x_2) \\ x_4 & := & \text{fact}^{-1}(x_3) \\ x_5 & := & \Gamma(x_4) \\ x_6 & := & \text{rem}(x_5, x_4) \end{cases}$$

returns positive integers $x_1, \ldots, x_6$ if and only if $(x - 1)! - 1$ is prime.

Proof. For an integer $i \in \{1, \ldots, 6\}$, let L_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_i$. If $x \in \{1, 2, 3\}$, then $x_6 = 0$. Therefore, $L_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 20, for every integer $x \geqslant 4$, $x_4 = (x - 1)! - 1$, $x_5 = \Gamma((x - 1)! - 1)$, and $x_1, \ldots, x_5 \in \mathbb{N} \setminus \{0\}$. By Lemma 16, for every integer $x \geqslant 4$,

$$x_6 = \text{rem}(\Gamma((x - 1)! - 1), (x - 1)! - 1)$$

belongs to $\mathbb{N} \setminus \{0\}$ if and only if $(x - 1)! - 1 \in \{4\} \cup \mathcal{P}$. The last condition equivalently expresses that $(x - 1)! - 1$ is prime as $(x - 1)! - 1 \geqslant 5$ for every integer $x \geqslant 4$. Hence,

$$L_6 = (\mathbb{N} \setminus \{0, 1, 2, 3\}) \cap \{x \in \mathbb{N} \setminus \{0, 1, 2, 3\} : \ (x - 1)! - 1 \in \mathcal{P}\} =$$

$$\{x \in \mathbb{N} \setminus \{0\} : \ (x - 1)! - 1 \in \mathcal{P}\}$$

$\qquad\square$

It is conjectured that there are infinitely many primes of the form $n! - 1$, see [1, p. 443] and [15].

Theorem 17. *The statement Φ_6 implies that there are infinitely many primes of the form $x! - 1$.*

Proof. The number $(975 - 1)! - 1$ is prime, see [1, p. 441] and [15]. By Lemma 22, for $x = 975$ the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_6$. Since $x = 975 > 720 = h(6)$, the statement Φ_6 guarantees that the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 22, the set $\{x \in \mathbb{N} \setminus \{0\} : \ (x - 1)! - 1 \in \mathcal{P}\}$ is infinite. $\qquad\square$

Lemma 23. *For every positive integer x, the following computation $\mathcal{D}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \text{rem}(x_2, x_1) \\ x_4 & := & \Gamma(x_3) \\ x_5 & := & \text{fact}^{-1}(x_4) \\ x_6 & := & \Gamma(x_5) \\ x_7 & := & \text{rem}(x_6, x_5) \end{cases}$$

returns positive integers $x_1, \ldots, x_7$ if and only if both x and $x - 2$ are prime.

Proof. For an integer $i \in \{1, \ldots, 7\}$, let D_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_i$. If $x = 1$, then $x_3 = 0$. Hence, $D_7 \subseteq D_3 \subseteq \mathbb{N} \setminus \{0, 1\}$. If $x \in \{2, 3, 4\}$, then $x_7 = 0$. Therefore,

$$D_7 \subseteq (\mathbb{N} \setminus \{0, 1\}) \cap (\mathbb{N} \setminus \{0, 2, 3, 4\}) = \mathbb{N} \setminus \{0, 1, 2, 3, 4\}$$

By Lemma 16, for every integer $x \geqslant 5$, the term x_3 (which equals $\text{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \mathcal{P} \setminus \{2, 3\}$. By Lemma 18, for every $x \in \mathcal{P} \setminus \{2, 3\}$, $x_3 = x - 1 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 20, for every $x \in \mathcal{P} \setminus \{2, 3\}$, the terms x_4 and x_5 belong to $\mathbb{N} \setminus \{0\}$ and $x_5 = x_3 - 1 = x - 2$. By Lemma 16, for every $x \in \mathcal{P} \setminus \{2, 3\}$, the term x_7 (which equals $\text{rem}(\Gamma(x_5), x_5)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x_5 = x - 2 \in \{4\} \cup \mathcal{P}$. From these facts, we obtain that

$$D_7 = (\mathbb{N} \setminus \{0, 1, 2, 3, 4\}) \cap (\mathcal{P} \setminus \{2, 3\}) \cap (\{6\} \cup \{p + 2 : \ p \in \mathcal{P}\}) =$$

$$\{p \in \mathcal{P} : \ p - 2 \in \mathcal{P}\}$$

$\qquad\square$

Theorem 18. *The statement Φ_7 implies that there are infinitely many twin primes.*

Proof. Harvey Dubner proved that the numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime, see [17, p. 87]. By Lemma 23, for $x = 459 \cdot 2^{8529} + 1$ the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_7$. Since $x > 720! = h(7)$, the statement Φ_7 guarantees that the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_7$ for infinitely many positive integers x. By Lemma 23, there are infinitely many twin primes. $\qquad\square$

X. The number of computations of length n

For a positive integer n, let $c(n)$ denote the number of computations of length n. Then, $c(1) = 1$, $c(2) = 2$, and $c(n) = c(n-2) + 2 \cdot c(n-1)$ for every integer $n \geqslant 3$. Hence, $c(3) = 5$, $c(4) = 12$, $c(5) = 29$, $c(6) = 70$, $c(7) = 169$, and

$$c(1) + c(2) + c(3) + c(4) + c(5) + c(6) + c(7) = 288$$

The following *MuPAD* code for every integer $n \in \{1, \ldots, 7\}$ displays the computations of length n in the lexicographic order. We assume here that $x_1 := x$ is the first instruction, and for every integer $i \in \{2, \ldots, 7\}$ the instruction $x_i := \Gamma(x_{i-1})$ precedes the instruction $x_i := \mathrm{fact}^{-1}(x_{i-1})$, and for every integer $i \in \{3, \ldots, 7\}$ the instruction $x_i := \mathrm{fact}^{-1}(x_{i-1})$ precedes the instruction $x_i := \mathrm{rem}(x_{i-1}, x_{i-2})$. The output of the code is available at `https://www.cyf-kr.edu.pl/~rttyszka/288computations.txt`.

```
print(Unquoted,"------------------------"):
for n from 1 to 7 do
k:=1:
print(Unquoted,"Computations of length ".n):
print(Unquoted,"------------------------"):
for i1 from 1 to 1+sign(n-1) do
for i2 from 1 to 2+sign(n-(5/2)) do
for i3 from 1 to 2+sign(n-(7/2)) do
for i4 from 1 to 2+sign(n-(9/2)) do
for i5 from 1 to 2+sign(n-(11/2)) do
for i6 from 1 to 2+sign(n-(13/2)) do
if (i1=1 or i2<>3) and (i2=1 or i3<>3) and
(i3=1 or i4<>3) and (i4=1 or i5<>3) and
(i5=1 or i6<>3) then
print(Unquoted,"Computation No. ".k):
print(Unquoted,"x_1:=x"):
if (n>1 and i1=1) then
print(Unquoted,"x_2:=Gamma(x_1)") end_if:
if (n>1 and i1=2) then
print(Unquoted,"x_2:=fact^{-1}(x_1)") end_if:
if (n>2 and i2=1) then
print(Unquoted,"x_3:=Gamma(x_2)") end_if:
if (n>2 and i2=2) then
print(Unquoted,"x_3:=fact^{-1}(x_2)") end_if:
if (n>2 and i2=3) then
print(Unquoted,"x_3:=rem(x_2,x_1)") end_if:
if (n>3 and i3=1) then
print(Unquoted,"x_4:=Gamma(x_3)") end_if:
if (n>3 and i3=2) then
print(Unquoted,"x_4:=fact^{-1}(x_3)") end_if:
if (n>3 and i3=3) then
print(Unquoted,"x_4:=rem(x_3,x_2)") end_if:
if (n>4 and i4=1)
then print(Unquoted,"x_5:=Gamma(x_4)") end_if:
if (n>4 and i4=2) then
print(Unquoted,"x_5:=fact^{-1}(x_4)") end_if:
if (n>4 and i4=3) then
print(Unquoted,"x_5:=rem(x_4,x_3)") end_if:
if (n>5 and i5=1) then
print(Unquoted,"x_6:=Gamma(x_5)") end_if:
if (n>5 and i5=2) then
print(Unquoted,"x_6:=fact^{-1}(x_5)") end_if:
if (n>5 and i5=3) then
print(Unquoted,"x_6:=rem(x_5,x_4)") end_if:
if (n>6 and i6=1) then
print(Unquoted,"x_7:=Gamma(x_6)") end_if:
if (n>6 and i6=2) then
print(Unquoted,"x_7:=fact^{-1}(x_6)") end_if:
if (n>6 and i6=3) then
print(Unquoted,"x_7:=rem(x_6,x_5)") end_if:
print(Unquoted,"------------------------"):
k:=k+1:
end_if:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
```

MuPAD is a general-purpose computer algebra system. The commercial version of *MuPAD* is no longer available as a stand-alone product, but only as the *Symbolic Math Toolbox* of *MATLAB*. Fortunately, the above code can be executed by *MuPAD Light*, which was offered for free for research and education until autumn 2005.

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