A condition $\Gamma(X)$ (which conjecturally holds for $X = \{1\} \cup \mathcal{P}_{n^2+1}$) expresses the current knowledge on a set $X \subseteq \mathbb{N}$ and strengthens the conjunction (the infiniteness of X is conjectured and unproven) $\wedge$ (card$(X) < \omega \Rightarrow X \subseteq [0, (((24!)!)!)!])$, whereas the statement $\exists\, X \subseteq \mathbb{N}\ \Gamma(X)$ (which is proven) fails for everyone who for every terminating algorithm with no input knows its output

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ABSTRACT. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geqslant 2$. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ: card$(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\left\{x_i! = x_k : i, k \in \{1, \ldots, 9\}\right\} \cup \left\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\right\}$. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. Let Ψ denote the statement: *if a system $\mathcal{S} \subseteq B$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \leqslant f(9)$.* We write down a system $\mathcal{A} \subseteq B$ of 8 equations. The statement Ψ restricted to the system $\mathcal{A}$ is equivalent to the statement Φ. It heuristically proves the statement Φ. This proof does not yield that card$(\mathcal{P}_{n^2+1}) = \omega$. Algorithms always terminate. We explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose existence is constructive and currently known to us). Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$. (1) *There are many elements of X and it is conjectured that X is infinite.* (2) *No known algorithm with no input returns the logical value of the statement* card$(X) = \omega$. (3) *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in X$.* (4) *There is a known algorithm with no input that computes an integer n satisfying* card$(X) < \omega \Rightarrow X \subseteq (-\infty, n]$. (5) *There is a known condition C, which can be formalized in ZFC, such that for all except at most finitely many $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in X$. The simplest known such condition C defines in $\mathbb{N}$ the set X.* The set $X = \{k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7), k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4). A more complicated set $X \subseteq \mathbb{N}$ satisfies conditions (1)–(5). No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. The statement Φ implies that conditions (1)–(5) hold for $X = \{1\} \cup \mathcal{P}_{n^2+1}$. To define the condition $\Gamma(X)$ from the title, we formulate condition (4) for $n = (((24!)!)!)!$ and take the conjunction of conditions (1)–(4) or the conjunction of conditions (1)–(5).

1. Definitions and the distinction between existing algorithms and known algorithms

Algorithms always terminate. Semi-algorithms may not terminate.

Definition 1. *Conditions* (1)-(5) *concern sets* $X \subseteq \mathbb{N}$.

(1) *There are many elements of X and it is conjectured that X is infinite.*

(2) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(X) = \omega$.

(3) *There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in X$.*

(4) *There is a known algorithm with no input that computes an integer n satisfying* $\mathrm{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.

(5) *There is a known condition C, which can be formalized in ZFC, such that for all except at most finitely many $k \in \mathbb{N}$, k satisfies the condition C if and only if $k \in X$. The simplest known such condition C defines in $\mathbb{N}$ the set X.*

Condition (2) implies that no known proof shows the finiteness/infiniteness of X.

Definition 2. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. $\square$

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [6]–[8]. Let $[\cdot]$ denote the integer part function.

Example 1. *The set $X = \mathcal{P}_{n^2+1}$ satifies condition* (2).

Example 2. *The set* $X = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases}$ *does not satisfy condition* (2) *because we know an algorithm with no input that computes* $[\frac{\beta}{\pi}]$.

Example 3. *(*[1]*,* [4]*,* [5, p. 9]*). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1, \ k+2, \ k+3, \ \ldots\} \times \{0\})$$

No known algorithm computes the function h.

Examples 1–3 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose existence is constructive and currently known to us).

Definition 3. *Let* Φ *denote the following unproven statement:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ. In Section 4, we heuristically prove the statement Φ. This proof does not yield that $\mathrm{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. *Condition* (4) *fails for* $X = \mathcal{P}_{n^2+1}$.

Proof. For every set $X \subseteq \mathbb{N}$, there exists an algorithm $\mathtt{Alg}(X)$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(X) \in \{0, \omega\} \\ \max(X), & \text{otherwise} \end{cases}$$

This n satisfies the implication in condition (4), but the algorithm $\mathtt{Alg}(\mathcal{P}_{n^2+1})$ is unknown for us because its definition is ineffective. □

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in mathematics because it refers to the current mathematical knowledge. The same is true for Statements 2–4 in the next sections.

Definition 4. *We say that an integer n is a threshold number of a set $X \subseteq \mathbb{N}$, if* $\mathrm{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of X. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $[\max(X), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problems 1 and 2

Open Problem 1. *Is there a set $X \subseteq \mathbb{N}$, which is well-known in number theory and satisfies conditions* (1)-(4)*?*

Open Problem 2. *Is there a set $X \subseteq \mathbb{N}$, which is well-known in number theory and satisfies conditions* (1)-(5)*?*

Statement 2. *The set*

$$X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions (1)-(4).

Proof. Condition (1) holds as $X \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β, see [3]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of X. Thus condition (4) holds. □

For a non-negative integer n, let $g(n)$ denote the greatest non-negative integer k such that 2^k divides $\max\left(2^\beta \cdot [\frac{n}{\beta}], 1\right)$.

Lemma 2. *The function $g : \mathbb{N} \to \mathbb{N}$ satisfies $g(0) = \ldots = g(\beta - 1) = 0$ and maps $\mathbb{N} \cap [\beta, \infty)$ onto itself taking every value in $\mathbb{N} \cap [\beta, \infty)$ infinitely many times.*

Statement 3. *The set*

$$X = \left\{n \in \mathbb{N} : g(n)^2 + 1 \text{ has no divisors greater than } 1 \text{ and smaller than } g(n)^2 + 1\right\}$$

satisfies conditions (1)-(5).

Proof. We use Lemma 2 and argue as in the proof of Statement 2. □

Proving Landau's conjecture will disprove Statements 2 and 3.

Theorem 1. *No set $X \subseteq \mathbb{N}$ will satisfy conditions* `(1)-(4)` *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies condition `(4)`. Since conditons `(2)-(4)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

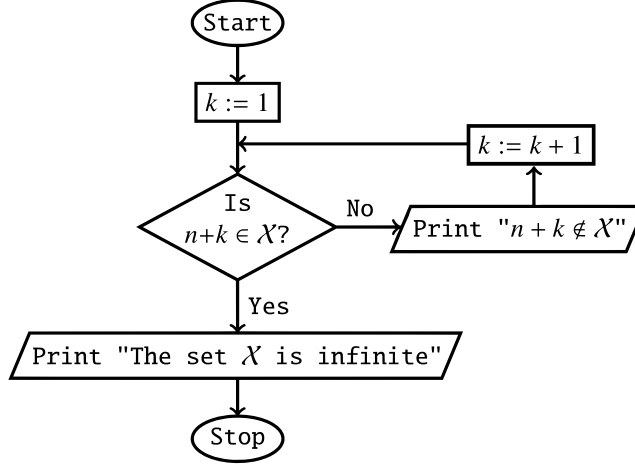`(T)` $\qquad\qquad\qquad\qquad n+1 \notin X, \; n+2 \notin X, \; n+3 \notin X, \; \ldots$

Fig. 1 Semi-algorithm that terminates if and only if the set X is infinite

The sentences from the sequence `(T)` and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set X is finite, contrary to the conjecture in condition `(1)`. $\hfill\square$

The physical limits of computation ([3]) disprove the assumption of Theorem 1.

3. NUMBER-THEORETIC STATEMENTS Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\} \; x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.
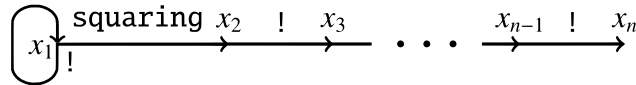

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 3. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $(f(1), \ldots, f(n))$.

Let B_n denote the following system of equations:

$$\left\{ x_i! = x_k : i, k \in \{1, \ldots, n\} \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 3 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n + 1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A CONJECTURAL SOLUTION TO OPEN PROBLEMS 1 AND 2

Lemma 4. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \lor (x = y = 1)$$

Lemma 5. *(Wilson's theorem, [2, p. 89]). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x - 1)! + 1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! & = & x_3 \\ x_3! & = & x_4 \\ x_5! & = & x_6 \\ x_8! & = & x_9 \\ x_1 \cdot x_1 & = & x_2 \\ x_3 \cdot x_5 & = & x_6 \\ x_4 \cdot x_8 & = & x_9 \\ x_5 \cdot x_7 & = & x_8 \end{cases}$$

Lemma 4 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

Fig. 3 Construction of the system $\mathcal{A}$

Lemma 6. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}
$$

Proof. By Lemma 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 6 follows from Lemma 5. □

Lemma 7. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap \mathbb{N} = \{1, 2\}$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system $\mathcal{A}$.*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 6, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 6 and 7, the set $\mathcal{P}_{n^2+1}$ is infinite.   □

Theorem 6. *Conjecture 1 implies the statement Φ.*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$.   □

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 6 and 7, if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 6 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$.   □

Statement 4. *The statement Φ implies that conditions (1)-(5) hold for $X = \{1\} \cup \mathcal{P}_{n^2+1}$.*

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [7]. These two facts imply condition (1). By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [3]. Thus condition (2) holds. Condition (3) holds trivially. The statement Φ implies that β is a threshold number of $X = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition (4) holds. The following condition:

$k - 1$ *is a square and k has no divisors greater than* 1 *and smaller than k*

defines in $\mathbb{N}$ the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition (5).   □

Proving Landau's conjecture will disprove Statement 4. By Statement 4, to define the condition $\Gamma(X)$ from the article title, we formulate condition (4) for $n = (((24!)!)!)!$ and take the conjunction of conditions (1)-(4) or the conjunction of conditions (1)-(5).

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