On sets $\mathcal{W} \subseteq \mathbb{N}$ whose infinity follows from the existence in $\mathcal{W}$ of an element which is greater than a threshold number computed for $\mathcal{W}$

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***Abstract*—Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. For a positive integer n, let Θ_n denote the statement: if a system $\mathcal{S} \subseteq \left\{x_i! = x_k : \ i,k \in \{1,\ldots,n\}\right\} \cup \left\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\ldots,n\}\right\}$ has only finitely many solutions in integers $x_1,\ldots,x_n$ greater than 1, then each such solution $(x_1,\ldots,x_n)$ satisfies $\min(x_1,\ldots,x_n) \leqslant f(n)$. The statement Θ_9 proves that if there exists an integer $x > f(9)$ such that x^2+1 (alternatively, $x!+1$) is prime, then there are infinitely many primes of the form n^2+1 (respectively, $n!+1$). The statement Θ_{16} proves that if there exists a twin prime greater than $f(16)+3$, then there are infinitely many twin primes. We formulate the statements Φ_n and prove: Φ_4 equivalently expresses that there are infinitely many primes of the form $n!+1$, Φ_6 implies that for infinitely many primes p the number $p!+1$ is prime, Φ_6 implies that there are infinitely many primes of the form $n!-1$, Φ_7 implies that there are infinitely many twin primes.**

***Index Terms*—composite Fermat numbers, prime numbers of the form $n!+1$, prime numbers of the form $n!-1$, prime numbers of the form n^2+1, prime numbers p such that $p!+1$ is prime, single query to an oracle for the halting problem, twin prime conjecture.**

I. Spectra of sentences and their threshold numbers

THE following observation concerns the theme described in the title of the article.

Observation 1. *If $\mathcal{W}$ is a subset of $\{0,\ldots,n\}$ where n is a non-negative integer, then we take any integer $m \geqslant n$ as a threshold number for $\mathcal{W}$. If $\mathcal{W}$ is an infinite subset of $\mathbb{N}$, then we take any non-negative integer m as a threshold number for $\mathcal{W}$.*

We define the set $\mathcal{U} \subseteq \mathbb{N}$ by declaring that a non-negative integer n belongs to $\mathcal{U}$ if and only if $\sin\left(10^{10^{10^{10}}}\right) > 0$. This inequality is practically undecidable, see [5].

Corollary 1. *The set $\mathcal{U}$ equals $\emptyset$ or $\mathbb{N}$. The statement "$\mathcal{U} = \emptyset$" remains unproven and the statement "$\mathcal{U} = \mathbb{N}$" remains unproven. Every non-negative integer m is a threshold number for $\mathcal{U}$. For every non-negative integer k, the sentence "$k \in \mathcal{U}$" is only theoretically decidable.*

The first-order language of graph theory contains two relation symbols of arity 2: $\sim$ and $=$, respectively for adjacency and equality of vertices. The term first-order imposes the condition that the variables represent vertices and hence the quantifiers apply to vertices only. For a first-order sentence Λ about graphs, let Spectrum(Λ) denote the set of all positive integers n such that there is a graph on n vertices satisfying Λ. By a graph on n vertices we understand a set of n elements with a binary relation which is symmetric and irreflexive.

Theorem 1. *([12, p. 171]). If a sentence Λ in the language of graph theory has the form $\exists x_1 \ldots x_k \ \forall y_1 \ldots y_l$ $\Upsilon(x_1,\ldots,x_k,y_1,\ldots,y_l)$, where $\Upsilon(x_1,\ldots,x_k,y_1,\ldots,y_l)$ is quantifier-free, then either* $\mathrm{Spectrum}(\Lambda) \subseteq [1,(2^k \cdot 4^l)-1]$ *or* $\mathrm{Spectrum}(\Lambda) \supseteq [k+l,\infty) \cap \mathbb{N}$.

Corollary 2. *The number $\left(2^k \cdot 4^l\right) - 1$ is a threshold number for* Spectrum(Λ).

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [10, p. 234].

Corollary 3. *If an algorithm $\mathtt{Alg}_1$ for every recursive set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_1(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Alg}_1(\mathcal{M})+1,\infty) \neq \emptyset$.*

Corollary 4. *If an algorithm $\mathtt{Alg}_2$ for every recursively enumerable set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_2(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Alg}_2(\mathcal{M})+1,\infty) \neq \emptyset$.*

Let $K = \{j \in \mathbb{N} : \ 2^{\aleph_j} = \aleph_{j+1}\}$.

Theorem 2. *If ZFC is consistent, then for every non-negative integer n the sentence*

"n is a threshold number for K"

is not provable in ZFC.

Proof. There exists a model $\mathcal{E}$ of ZFC such that

$$\forall i \in \{0,\ldots,n+1\}\ \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+1}$$

and

$$\forall i \in \{n+2, n+3, n+4, \ldots\}\ \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+2}$$

see [3] and [6, p. 232]. In the model $\mathcal{E}$, $K = \{0,\ldots,n+1\}$ and n is not a threshold number for K. □

Theorem 3. *If ZFC is consistent, then for every non-negative integer n the sentence*

"n is not a threshold number for K"

is not provable in ZFC.

Proof. The Generalized Continuum Hypothesis (GCH) is consistent with ZFC, see [6, p. 188] and [6, p. 190]. GCH implies that $K = \mathbb{N}$. Consequently, GCH implies that every non-negative integer n is a threshold number for K. □

II. Basic lemmas

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{V}_1$ denote the system of equations $\{x_1! = x_1\}$, and let $\mathcal{V}_2$ denote the system of equations $\{x_1! = x_1,\ x_1 \cdot x_1 = x_2\}$. For an integer $n \geqslant 3$, let $\mathcal{V}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2,\ldots,n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{V}_n$.

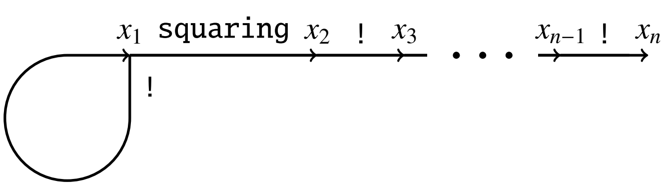

Fig. 1 Construction of the system $\mathcal{V}_n$

Lemma 1. *For every positive integer n, the system* $\mathcal{V}_n$ *has exactly one solution in integers greater than* 1*, namely* $\big(f(1),\ldots,f(n)\big)$.

Let

$$H_n = \Big\{x_i! = x_k :\ i,k \in \{1,\ldots,n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\ldots,n\}\Big\}$$

For a positive integer n, let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers $x_1,\ldots,x_n$ greater than 1, then each such solution $(x_1,\ldots,x_n)$ satisfies $\min(x_1,\ldots,x_n) \leqslant f(n)$. The assumption $\min(x_1,\ldots,x_n) \leqslant f(n)$ is weaker than the assumption $\max(x_1,\ldots,x_n) \leqslant f(n)$ suggested by Lemma 1.

Lemma 2. *For every positive integer n, the system* H_n *has a finite number of subsystems.*

Theorem 4. *Every statement* Θ_n *is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 2. □

Lemma 3. *For every integers x and y greater than* 1*,* $x! \cdot y = y!$ *if and only if* $x + 1 = y$.

Lemma 4. *If* $x \geqslant 4$*, then* $\dfrac{(x-1)!+1}{x} > 1$.

Lemma 5. *(Wilson's theorem, [4, p. 89]). For every integer* $x \geqslant 2$*, x is prime if and only if x divides* $(x-1)!+1$.

III. Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [11]. It is conjectured that $x! + 1$ is a square only for $x \in \{4,5,7\}$, see [18, p. 297].

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

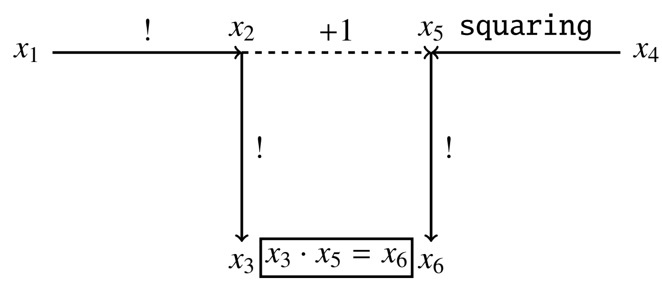

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 6. *For every integers* x_1 *and* x_4 *greater than* 1*, the system* $\mathcal{A}$ *is solvable in integers* x_2, x_3, x_5, x_6 *greater than* 1 *if and only if* $x_1! + 1 = x_4^2$*. In this case, the integers* x_2, x_3, x_5, x_6 *are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

and $x_1 = \min(x_1,\ldots,x_6)$.

Proof. It follows from Lemma 3. □

Theorem 5. *The statement* Θ_6 *proves the following implication: if the equation* $x_1! + 1 = x_4^2$ *has only finitely many solutions in positive integers, then each such solution* (x_1, x_4) *satisfies* $x_1 \leqslant f(6)$.

Proof. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0,1\}$. By Lemma 6, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0,1\})^4$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Lemma 6 guarantees that $x_1 = \min(x_1, \ldots, x_6)$. By the antecedent and Lemma 6, the system $\mathcal{A}$ has only finitely many solutions in integers $x_1, \ldots, x_6$ greater than 1. Therefore, the statement Θ_6 implies that $x_1 = \min(x_1, \ldots, x_6) \leqslant f(6)$. $\square$

Hypothesis 1. *The implication in Theorem 5 is true.*

Corollary 5. *Assuming Hypothesis 1, a single query to an oracle for the halting problem decides the problem of the infinitude of the solutions of the equation* $x! + 1 = y^2$.

IV. Are there infinitely many prime numbers of the form $n^2 + 1$?

Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [9, pp. 37–38].

Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

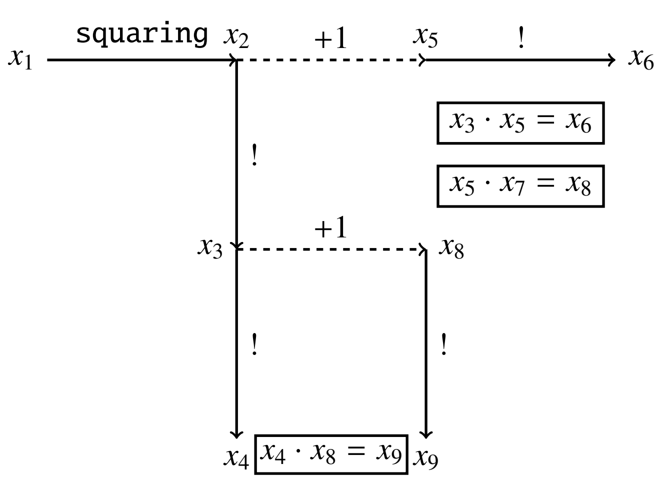

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 7. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. $\square$

Theorem 6. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0,1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Lemma 7 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0,1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. $\square$

Hypothesis 2. *The implication in Theorem 6 is true.*

Corollary 6. *Assuming Hypothesis 2, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n^2 + 1$.*

V. Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [1, p. 443] and [14]. Let $\mathcal{G}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system $\mathcal{G}$.

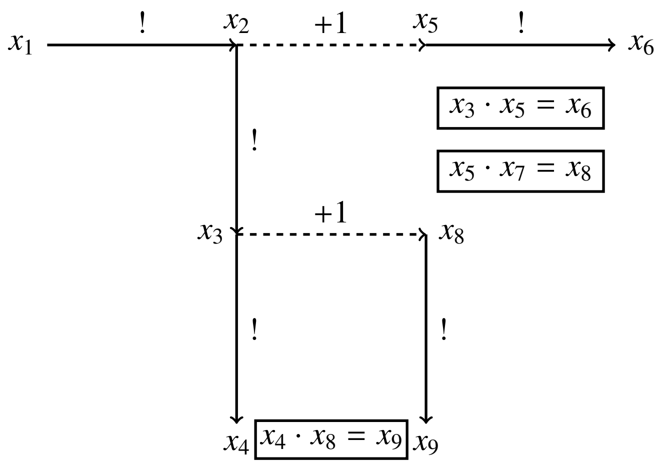

Fig. 4 Construction of the system $\mathcal{G}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{array}{rcl}
x_2 & = & x_1! \\
x_3 & = & (x_1!)! \\
x_4 & = & ((x_1!)!)! \\
x_5 & = & x_1^{!} + 1 \\
x_6 & = & (x_1! + 1)! \\
x_7 & = & \dfrac{(x_1!)! + 1}{x_1! + 1} \\
x_8 & = & (x_1!)! + 1 \\
x_9 & = & ((x_1!)! + 1)!
\end{array}$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 5. □

Theorem 7. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1! + 1$ is prime. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{G}$. Lemma 8 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{G} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{G}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 8, there are infinitely many primes of the form $n! + 1$. □

Hypothesis 3. *The implication in Theorem 7 is true.*

Corollary 7. *Assuming Hypothesis 3, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n! + 1$.*

VI. The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [9, p. 39].

Let C denote the following system of equations:

$$\left\{\begin{array}{rcl}
x_1! & = & x_2 \\
x_2! & = & x_3 \\
x_4! & = & x_5 \\
x_6! & = & x_7 \\
x_7! & = & x_8 \\
x_9! & = & x_{10} \\
x_{12}! & = & x_{13} \\
x_{15}! & = & x_{16} \\
x_2 \cdot x_4 & = & x_5 \\
x_5 \cdot x_6 & = & x_7 \\
x_7 \cdot x_9 & = & x_{10} \\
x_4 \cdot x_{11} & = & x_{12} \\
x_3 \cdot x_{12} & = & x_{13} \\
x_9 \cdot x_{14} & = & x_{15} \\
x_8 \cdot x_{15} & = & x_{16}
\end{array}\right.$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system C.

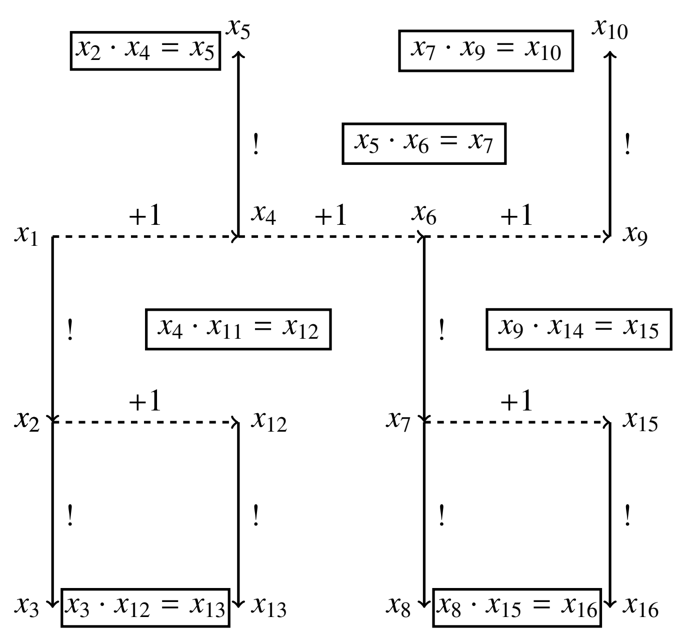

Fig. 5 Construction of the system C

Lemma 9. *If $x_4 = 2$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. □

Lemma 10. *If $x_4 = 3$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. □

Lemma 11. *For every $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0,1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} are uniquely determined by the following equalities:*

$$\begin{array}{rcl}
x_1 &=& x_4 - 1 \\
x_2 &=& (x_4 - 1)! \\
x_3 &=& ((x_4 - 1)!)! \\
x_5 &=& x_4! \\
x_6 &=& x_9 - 1 \\
x_7 &=& (x_9 - 1)! \\
x_8 &=& ((x_9 - 1)!)! \\
x_{10} &=& x_9! \\
x_{11} &=& \dfrac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &=& (x_4 - 1)! + 1 \\
x_{13} &=& ((x_4 - 1)! + 1)! \\
x_{14} &=& \dfrac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &=& (x_9 - 1)! + 1 \\
x_{16} &=& ((x_9 - 1)! + 1)!
\end{array}$$

and $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3$.

Proof. By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0,1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 11 follows from Lemma 5. □

Theorem 8. *The statement Θ_{16} proves the following implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.*

Proof. Assume the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$. By Lemma 11, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0,1\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Lemma 11 guarantees that $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement Θ_{16} and the inequality $\min(x_1, \ldots, x_{16}) > f(16)$ imply that the system C has infinitely many solutions in integers $x_1, \ldots, x_{16}$ greater than 1. According to Lemmas 9–11, there are infinitely many twin primes. □

Hypothesis 4. *The implication in Theorem 8 is true.*

Corollary 8. *(cf. [2]). Assuming Hypothesis 4, a single query to an oracle for the halting problem decides the twin prime problem.*

VII. Are there infinitely many composite Fermat numbers?

Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [8, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [8, p. 1].

Open Problem. ([8, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [7, p. 23].

Lemma 12. *([8, p. 38]). For every positive integer n, if a prime number p divides $2^{2^n} + 1$, then there exists a positive integer k such that $p = k \cdot 2^{n+1} + 1$.*

Corollary 9. *Since $k \cdot 2^{n+1} + 1 \geqslant 2^{n+1} + 1 \geqslant n + 3$, for every positive integers x, y, and n, the equality $(x+1)(y+1) = 2^{2^n} + 1$ implies that $\min(n, x, x+1, y, y+1) = n$.*

Let $g(1) = 1$, and let $g(n+1) = 2^{2^{g(n)}}$ for every positive integer n. Let

$$G_n = \Big\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\Big\} \cup \Big\{2^{2^{x_i}} = x_k : \ i, k \in \{1, \ldots, n\}\Big\}$$

Lemma 13. *The following subsystem of G_n*

$$\begin{cases} \qquad\qquad\qquad x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \ldots, n-1\} \ 2^{2^{x_i}} = x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(g(1), \ldots, g(n))$.

For a positive integer n, let Ψ_n denote the following statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leqslant g(n)$. The assumption $\min(x_1, \ldots, x_n) \leqslant g(n)$ is weaker than the assumption $\max(x_1, \ldots, x_n) \leqslant g(n)$ suggested by Lemma 13.

Lemma 14. *For every positive integer n, the system G_n has a finite number of subsystems.*

Theorem 9. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 14. □

Lemma 15. *For every non-negative integers b and c, $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Theorem 10. *The statement Ψ_{13} proves the following implication: if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n.*

Proof. Let us consider the equation

$$(x+1)(y+1) = 2^{2^z} + 1 \qquad (1)$$

in positive integers. By Lemma 15, we can transform equation (1) into an equivalent system $\mathcal{F}$ which has 13 variables

(x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^{\alpha}} = \gamma$, see the diagram in Figure 6.

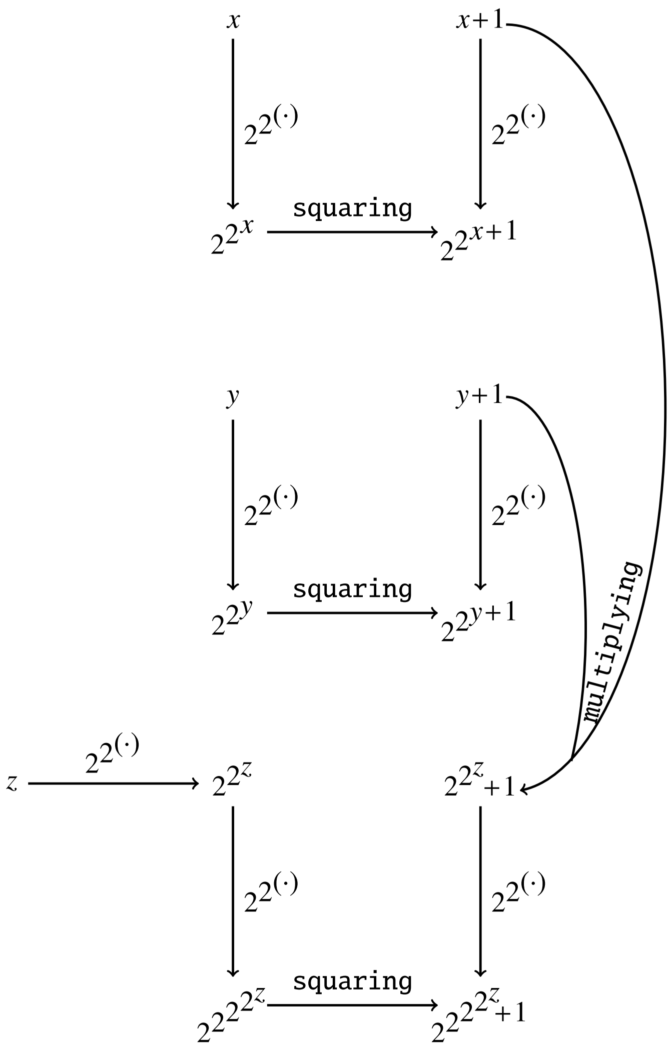

Fig. 6 Construction of the system $\mathcal{F}$

Assume that $2^{2^n}+1$ is composite for some integer $n > g(13)$. By this and Corollary 9, equation (1) has a solution $(x, y, z) \in (\mathbb{N} \setminus \{0\})^3$ such that $z = n$ and $z = \min(z, x, x+1, y, y+1)$. Hence, the system $\mathcal{F}$ has a solution in positive integers such that $z = n$ and n is the smallest number in the solution sequence. Since $n > g(13)$, the statement Ψ_{13} implies that the system $\mathcal{F}$ has infinitely many solutions in positive integers. Therefore, there are infinitely many positive integers n such that $2^{2^n}+1$ is composite. □

Hypothesis 5. *The implication in Theorem 10 is true.*

Corollary 10. *Assuming Hypothesis 5, a single query to an oracle for the halting problem decides whether or not the set of composite Fermat numbers is infinite.*

VIII. Computations of length n and the statements Φ_n

For a positive integer x, let $\Gamma(x)$ denote $(x-1)!$. Let $\mathrm{fact}^{-1}\colon \{1, 2, 6, 24, \ldots\} \to \mathbb{N} \setminus \{0\}$ denote the inverse function to the factorial function. For positive integers x and y, let $\mathrm{rem}(x, y)$ denote the remainder from dividing x by y.

Definition. *For a positive integer n, by a computation of length n we understand any sequence of terms $x_1, \ldots, x_n$ such that x_1 is defined as the variable x, and for every integer $i \in \{2, \ldots, n\}$, x_i is defined as $\Gamma(x_{i-1})$, or $\mathrm{fact}^{-1}(x_{i-1})$, or $\mathrm{rem}(x_{i-1}, x_{i-2})$ (only if $i \geqslant 3$ and x_{i-1} is defined as $\Gamma(x_{i-2})$).*

For a positive integer n, let $c(n)$ denote the number of computations of length n. Then, $c(1) = 1$, $c(2) = 2$, and $c(n) = c(n-2) + 2 \cdot c(n-1)$ for every integer $n \geqslant 3$. Hence, $c(3) = 5$, $c(4) = 12$, $c(5) = 29$, $c(6) = 70$, and $c(7) = 169$.

Let $\mathcal{P}$ denote the set of prime numbers.

Lemma 16. *([13, pp. 214–215]) . For every positive integer x, $\mathrm{rem}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$.*

Let $h(4) = 3$, and let $h(n+1) = h(n)!$ for every integer $n \geqslant 4$.

Theorem 11. *For every integer $n \geqslant 4$ and for every positive integer x, the following computation $\mathcal{H}_n$*

$$\begin{cases} x_1 &:= & x \\ \forall i \in \{2, \ldots, n-3\}\; x_i &:= & \mathrm{fact}^{-1}(x_{i-1}) \\ x_{n-2} &:= & \Gamma(x_{n-3}) \\ x_{n-1} &:= & \Gamma(x_{n-2}) \\ x_n &:= & \mathrm{rem}(x_{n-1}, x_{n-2}) \end{cases}$$

returns positive integers $x_1, \ldots, x_n$ if and only if $x = h(n)$.

Proof. We make three observations.

Observation 2. *If $x_{n-3} = 3$, then $x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x = x_1 = h(n)$. If $x = h(n)$, then $x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x_{n-3} = 3$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 2$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = 1$.*

Observation 3. *If $x_{n-3} = 2$, then $x = x_1 = \ldots = x_{n-3} = 2$. If $x = 2$, then $x_1 = \ldots = x_{n-3} = 2$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 1$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.*

Observation 4. *If $x_{n-3} = 1$, then $x_{n-2} = \Gamma(x_{n-3}) = 1$. Hence, $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.*

Observations 2–4 cover the case when $x_{n-3} \in \{1, 2, 3\}$. If $x_{n-3} \geqslant 4$, then $x_{n-2} = \Gamma(x_{n-3})$ is greater than 4 and composite. By Lemma 16, $x_n = \mathrm{rem}(x_{n-1}, x_{n-2}) = \mathrm{rem}(\Gamma(x_{n-2}), x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$. □

For an integer $n \geqslant 4$, let Φ_n denote the following statement: if a computation of length n returns positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers x, then every such x does not exceed $h(n)$.

Theorem 12. *For every integer $n \geqslant 4$, the bound $h(n)$ in the statement Φ_n cannot be decreased.*

Proof. It follows from Theorem 11. □

Lemma 17. *For every positive integer n, there are only finitely many computations of length n.*

Theorem 13. *For every integer $n \geqslant 4$, the statement Φ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 17. □

IX. Consequences of the statements $\Phi_4, \ldots, \Phi_7$

Lemma 18. *If $x \in \mathcal{P}$, then* $\mathrm{rem}(\Gamma(x), x) = x - 1$.

Proof. It follows from Lemma 5. □

Lemma 19. *For every positive integer x, the following computation $\mathcal{T}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \mathrm{rem}(x_2, x_1) \\ x_4 & := & \mathrm{fact}^{-1}(x_3) \end{cases}$$

returns positive integers $x_1, \ldots, x_4$ if and only if $x = 4$ or x is a prime number of the form $n! + 1$.

Proof. For an integer $i \in \{1, \ldots, 4\}$, let T_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{T}$ returns positive integers $x_1, \ldots, x_i$. We show that

$$T_4 = \{4\} \cup (\{n! + 1 : \; n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P}) \qquad (2)$$

For every positive integer x, the terms x_1 and x_2 belong to $\mathbb{N} \setminus \{0\}$. By Lemma 16, the term x_3 (which equals $\mathrm{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$. Hence, $T_3 = \{4\} \cup \mathcal{P}$. If $x = 4$, then $x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in T_4$. If $x \in \mathcal{P}$, then Lemma 18 implies that $x_3 = \mathrm{rem}(\Gamma(x), x) = x - 1 \in \mathbb{N} \setminus \{0\}$. Therefore, for every $x \in \mathcal{P}$, the term $x_4 = \mathrm{fact}^{-1}(x_3)$ belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{n! + 1 : \; n \in \mathbb{N} \setminus \{0\}\}$. This proves equality (2). □

Theorem 14. *The statement Φ_4 implies that the set of primes of the form $n! + 1$ is infinite.*

Proof. The number $3! + 1 = 7$ is prime. By Lemma 19, for $x = 7$ the computation $\mathcal{T}$ returns positive integers $x_1, \ldots, x_4$. Since $x = 7 > 3 = h(4)$, the statement Φ_4 guarantees that the computation $\mathcal{T}$ returns positive integers $x_1, \ldots, x_4$ for infinitely many positive integers x. By Lemma 19, there are infinitely many primes of the form $n! + 1$. □

Lemma 20. *If $x \in \mathbb{N} \setminus \{0, 1\}$, then* $\mathrm{fact}^{-1}(\Gamma(x)) = x - 1$.

Theorem 15. *If the set of primes of the form $n! + 1$ is infinite, then the statement Φ_4 is true.*

Proof. There exist exactly 10 computations of length 4 that differ from $\mathcal{H}_4$ and $\mathcal{T}$, see Table 1. For every such computation $\mathcal{F}_i$, we determine the set S_i of all positive integers x such that the computation $\mathcal{F}_i$ outputs positive integers $x_1, \ldots, x_4$ on input x. We omit 10 easy proofs which use Lemmas 16 and 20. The sets S_i are infinite, see Table 1.

$\mathcal{F}_1$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \mathbb{N} \setminus \{0\} = S_1$
$\mathcal{F}_2$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \mathbb{N} \setminus \{0\} = S_2$
$\mathcal{H}_4$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{rem}(x_3, x_2)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x = 3$
$\mathcal{F}_3$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \mathbb{N} \setminus \{0\} = S_3$
$\mathcal{F}_4$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{1\} \cup \{n! + 1 : \; n \in \mathbb{N} \setminus \{0\}\} = S_4$
$\mathcal{F}_5$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{rem}(x_2, x_1)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{4\} \cup \mathcal{P} = S_5$
$\mathcal{T}$	$x_1 := x$	$x_2 := \Gamma(x_1)$	$x_3 := \mathrm{rem}(x_2, x_1)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{4\} \cup (\{n! + 1 : \; n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$
$\mathcal{F}_6$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{n! : \; n \in \mathbb{N} \setminus \{0\}\} = S_6$
$\mathcal{F}_7$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{n! : \; n \in \mathbb{N} \setminus \{0\}\} = S_7$
$\mathcal{F}_8$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \Gamma(x_2)$	$x_4 := \mathrm{rem}(x_3, x_2)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{4!\} \cup \{p! : \; p \in \mathcal{P}\} = S_8$
$\mathcal{F}_9$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \Gamma(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{(n!)! : \; n \in \mathbb{N} \setminus \{0\}\} = S_9$
$\mathcal{F}_{10}$	$x_1 := x$	$x_2 := \mathrm{fact}^{-1}(x_1)$	$x_3 := \mathrm{fact}^{-1}(x_2)$	$x_4 := \mathrm{fact}^{-1}(x_3)$	$x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{((n!)!)! : \; n \in \mathbb{N} \setminus \{0\}\} = S_{10}$

Tab. 1 12 computations of length 4, $x \in \mathbb{N} \setminus \{0\}$

This completes the proof. □

Hypothesis 6. *The statements $\Phi_4, \ldots, \Phi_7$ are true.*

Lemma 21. *For every positive integer x, the following computation $\mathcal{Y}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \mathrm{rem}(x_2, x_1) \\ x_4 & := & \mathrm{fact}^{-1}(x_3) \\ x_5 & := & \Gamma(x_4) \\ x_6 & := & \mathrm{rem}(x_5, x_4) \end{cases}$$

returns positive integers $x_1, \ldots, x_6$ if and only if $x \in \{4\} \cup \{p! + 1 : \; p \in \mathcal{P}\} \cap \mathcal{P}$.

Proof. For an integer $i \in \{1, \ldots, 6\}$, let Y_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_i$. Since the computations $\mathcal{T}$ and $\mathcal{Y}$ have the same first four instructions, the equality $Y_i = T_i$ holds for every $i \in \{1, \ldots, 4\}$. In particular,

$$Y_4 = \{4\} \cup (\{n! + 1 : \ n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$$

We show that

$$Y_6 = \{4\} \cup (\{p! + 1 : \ p \in \mathcal{P}\} \cap \mathcal{P}) \tag{3}$$

If $x = 4$, then $x_1, \ldots, x_6 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in Y_6$. Let $x \in \mathcal{P}$, and let $x = n! + 1$, where $n \in \mathbb{N} \setminus \{0\}$. Hence, $n \neq 4$. Lemma 18 implies that $x_3 = \mathrm{rem}(\Gamma(x), x) = x - 1 = n!$. Hence, $x_4 = \mathrm{fact}^{-1}(x_3) = n$ and $x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\}$. By Lemma 16, the term x_6 (which equals $\mathrm{rem}(\Gamma(n), n)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $n \in \{4\} \cup \mathcal{P}$. This proves equality (3) as $n \neq 4$. □

Theorem 16. *The statement Φ_6 implies that for infinitely many primes p the number $p! + 1$ is prime.*

Proof. The numbers 11 and $11! + 1$ are prime, see [1, p. 441] and [16]. By Lemma 21, for $x = 11! + 1$ the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_6$. Since $x = 11! + 1 > 6! = h(6)$, the statement Φ_6 guarantees that the computation $\mathcal{Y}$ returns positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 21, for infinitely many primes p the number $p! + 1$ is prime. □

Lemma 22. *For every positive integer x, the following computation $\mathcal{L}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \Gamma(x_2) \\ x_4 & := & \mathrm{fact}^{-1}(x_3) \\ x_5 & := & \Gamma(x_4) \\ x_6 & := & \mathrm{rem}(x_5, x_4) \end{cases}$$

returns positive integers $x_1, \ldots, x_6$ if and only if $(x - 1)! - 1$ is prime.

Proof. For an integer $i \in \{1, \ldots, 6\}$, let L_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_i$. If $x \in \{1, 2, 3\}$, then $x_6 = 0$. Therefore, $L_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 20, for every integer $x \geqslant 4$, $x_4 = (x - 1)! - 1$, $x_5 = \Gamma((x - 1)! - 1)$, and $x_1, \ldots, x_5 \in \mathbb{N} \setminus \{0\}$. By Lemma 16, for every integer $x \geqslant 4$,

$$x_6 = \mathrm{rem}(\Gamma((x - 1)! - 1), (x - 1)! - 1)$$

belongs to $\mathbb{N} \setminus \{0\}$ if and only if $(x - 1)! - 1 \in \{4\} \cup \mathcal{P}$. The last condition equivalently expresses that $(x - 1)! - 1$ is prime as $(x - 1)! - 1 \geqslant 5$ for every integer $x \geqslant 4$. Hence,

$$L_6 = (\mathbb{N} \setminus \{0, 1, 2, 3\}) \cap \{x \in \mathbb{N} \setminus \{0, 1, 2, 3\} : \ (x - 1)! - 1 \in \mathcal{P}\} =$$
$$\{x \in \mathbb{N} \setminus \{0\} : \ (x - 1)! - 1 \in \mathcal{P}\}$$

□

It is conjectured that there are infinitely many primes of the form $n! - 1$, see [1, p. 443] and [15].

Theorem 17. *The statement Φ_6 implies that there are infinitely many primes of the form $x! - 1$.*

Proof. The number $(975 - 1)! - 1$ is prime, see [1, p. 441] and [15]. By Lemma 22, for $x = 975$ the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_6$. Since $x = 975 > 720 = h(6)$, the statement Φ_6 guarantees that the computation $\mathcal{L}$ returns positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 22, the set $\{x \in \mathbb{N} \setminus \{0\} : \ (x - 1)! - 1 \in \mathcal{P}\}$ is infinite. □

Lemma 23. *For every positive integer x, the following computation $\mathcal{D}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \mathrm{rem}(x_2, x_1) \\ x_4 & := & \Gamma(x_3) \\ x_5 & := & \mathrm{fact}^{-1}(x_4) \\ x_6 & := & \Gamma(x_5) \\ x_7 & := & \mathrm{rem}(x_6, x_5) \end{cases}$$

returns positive integers $x_1, \ldots, x_7$ if and only if both x and $x - 2$ are prime.

Proof. For an integer $i \in \{1, \ldots, 7\}$, let D_i denote the set of positive integers x such that the first i instructions of the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_i$. If $x = 1$, then $x_3 = 0$. Hence, $D_7 \subseteq D_3 \subseteq \mathbb{N} \setminus \{0, 1\}$. If $x \in \{2, 3, 4\}$, then $x_7 = 0$. Therefore,

$$D_7 \subseteq (\mathbb{N} \setminus \{0, 1\}) \cap (\mathbb{N} \setminus \{0, 2, 3, 4\}) = \mathbb{N} \setminus \{0, 1, 2, 3, 4\}$$

By Lemma 16, for every integer $x \geqslant 5$, the term x_3 (which equals $\mathrm{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \mathcal{P} \setminus \{2, 3\}$. By Lemma 18, for every $x \in \mathcal{P} \setminus \{2, 3\}$, $x_3 = x - 1 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 20, for every $x \in \mathcal{P} \setminus \{2, 3\}$, the terms x_4 and x_5 belong to $\mathbb{N} \setminus \{0\}$ and $x_5 = x_3 - 1 = x - 2$. By Lemma 16, for every $x \in \mathcal{P} \setminus \{2, 3\}$, the term x_7 (which equals $\mathrm{rem}(\Gamma(x_5), x_5)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x_5 = x - 2 \in \{4\} \cup \mathcal{P}$. From these facts, we obtain that

$$D_7 = (\mathbb{N} \setminus \{0, 1, 2, 3, 4\}) \cap (\mathcal{P} \setminus \{2, 3\}) \cap (\{6\} \cup \{p + 2 : \ p \in \mathcal{P}\}) =$$
$$\{p \in \mathcal{P} : \ p - 2 \in \mathcal{P}\}$$

□

Theorem 18. *The statement Φ_7 implies that there are infinitely many twin primes.*

Proof. Harvey Dubner proved that the numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime, see [17, p. 87]. By Lemma 23, for $x = 459 \cdot 2^{8529} + 1$ the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_7$. Since $x > 720! = h(7)$, the statement Φ_7 guarantees that the computation $\mathcal{D}$ returns positive integers $x_1, \ldots, x_7$ for infinitely many positive integers x. By Lemma 23, there are infinitely many twin primes. □

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