

Is there a widely known set $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness remains conjectured and unproven, where \mathcal{X} is decidable by a constructively existing algorithm and a constructively defined integer n satisfies $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$?

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ABSTRACT. Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ : $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\{x_i! = x_k : i, k \in \{1, \dots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, 9\}\}$. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers x_1, \dots, x_9 , namely $(1, \dots, 1)$ and $(f(1), \dots, f(9))$. We write down a system $\mathcal{A} \subseteq B$ of 8 equations. Let Λ denote the statement: *if the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 , then each such solution (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \leq f(9)$* . The statement Λ is equivalent to the statement Φ . It heuristically proves the statement Φ . This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$. Algorithms always terminate. We explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in ZFC) and *known algorithms* (i.e. algorithms whose existence is constructive and currently known to us). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n . Conditions (1)–(5) concern sets $\mathcal{X} \subseteq \mathbb{N}$. (1) *There are many elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite.* (2) *No known algorithm with no input returns the logical value of the statement $\text{card}(\mathcal{X}) = \omega$.* (3) *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* (4) *A known algorithm with no input returns an integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.* (5) *There is a known formula $\phi(x)$ of Peano arithmetic such that for all except at most finitely many $k \in \mathbb{N}$, $\phi(k) \Leftrightarrow k \in \mathcal{X}$. The simplest known such formula $\phi(x)$ satisfies $\{k \in \mathbb{N} : \phi(k)\} = \mathcal{X}$.* The set $\mathcal{X} = \{k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7), k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4). A more complicated set $\mathcal{X} \subseteq \mathbb{N}$ satisfies conditions (1)–(5). No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. The statement Φ implies that conditions (1)–(5) hold for $\mathcal{X} = \{1\} \cup \mathcal{P}_{n^2+1}$.

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1. DEFINITIONS AND THE DISTINCTION BETWEEN EXISTING ALGORITHMS AND KNOWN ALGORITHMS

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–3 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose existence is constructive and currently known to us). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n .

Definition 1. Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.

- (1) There are many elements of X and it is conjectured that X is infinite.
- (2) No known algorithm with no input returns the logical value of the statement $\text{card}(X) = \omega$.
- (3) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in X$.
- (4) A known algorithm with no input returns an integer n satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.
- (5) There is a known formula $\phi(x)$ of Peano arithmetic such that for all except at most finitely many $k \in \mathbb{N}$, $\phi(k) \Leftrightarrow k \in X$. The simplest known such formula $\phi(x)$ satisfies $\{k \in \mathbb{N} : \phi(k)\} = X$.

Condition (2) implies that no known proof shows the finiteness/infiniteness of X .

Definition 2. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. □

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [6]–[8]. Let $[\cdot]$ denote the integer part function.

Example 1. The set $X = \mathcal{P}_{n^2+1}$ satisfies condition (2).

Example 2. The set $X = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases}$ does not satisfy condition (2) because we know an algorithm with no input that computes $[\frac{\beta}{\pi}]$.

Example 3. ([1], [4], [5, p. 9]). The function

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \dots, k\} \times \{1\}) \cup (\{k+1, k+2, k+3, \dots\} \times \{0\})$$

No known algorithm computes the function h .

Definition 3. Let Φ denote the following unproven statement:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ . In Section 4, we heuristically prove the statement Φ . This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. *Condition (4) fails for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in condition (4), but the algorithm $\text{Alg}(\mathcal{P}_{n^2+1})$ is unknown for us because its definition is ineffective. \square

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in mathematics because it refers to the current mathematical knowledge. The same is true for Statements 2–4 in the next sections.

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of \mathcal{X} form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

2. THE PHYSICAL LIMITS OF COMPUTATION INSPIRE OPEN PROBLEMS 1 AND 2

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$, which is well-known in number theory and satisfies conditions (1)–(4)?*

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$, which is well-known in number theory and satisfies conditions (1)–(5)?*

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\mathcal{X} \supseteq \{0, \dots, \beta\}$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β , see [3]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

For a non-negative integer n , let $g(n)$ denote the greatest non-negative integer k such that 2^k divides $\max(2^\beta \cdot \lfloor \frac{n}{\beta} \rfloor, 1)$.

Lemma 2. *The function $g : \mathbb{N} \rightarrow \mathbb{N}$ satisfies $g(0) = \dots = g(\beta - 1) = 0$ and maps $\mathbb{N} \cap [\beta, \infty)$ onto itself taking every value in $\mathbb{N} \cap [\beta, \infty)$ infinitely many times.*

Statement 3. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : g(n)^2 + 1 \text{ has no divisors greater than } 1 \text{ and smaller than } g(n)^2 + 1\}$$

satisfies conditions (1)–(5).

Proof. We use Lemma 2 and argue as in the proof of Statement 2. \square

Proving Landau's conjecture will disprove Statements 2 and 3.

Theorem 1. *No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies condition (4). Since conditions (2)–(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $n + 1 \notin X, n + 2 \notin X, n + 3 \notin X, \dots$

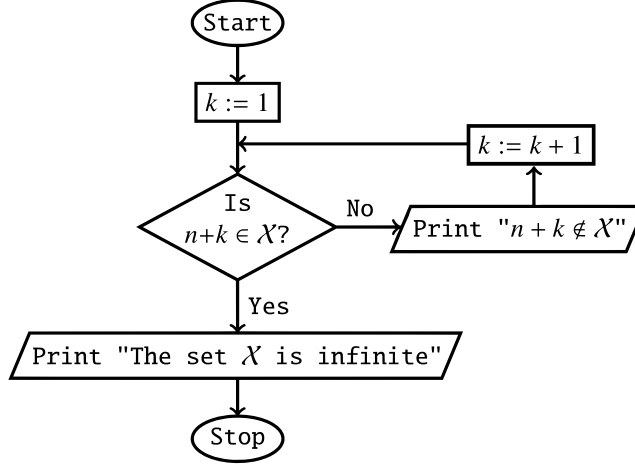


Fig. 1 Semi-algorithm that terminates if and only if the set X is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set X is finite, contrary to the conjecture in condition (1). \square

The physical limits of computation ([3]) disprove the assumption of Theorem 1.

3. NUMBER-THEORETIC STATEMENTS Ψ_n

Let $f(1) = 2, f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system \mathcal{U}_n .

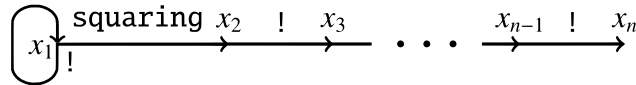


Fig. 2 Construction of the system \mathcal{U}_n

Lemma 3. *For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.*

Let B_n denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Ψ_n denote the following statement: *if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$* . The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

Theorem 2. *For every statement Ψ_n , the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 3 because $\mathcal{U}_n \subseteq B_n$. \square

Theorem 3. *For every integer $n \geq 2$, the statement Ψ_{n+1} implies the statement Ψ_n .*

Proof. If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then for every integer $i \in \{1, \dots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers x_1, \dots, x_{n+1} . The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)!$. Hence, $x_i \leq f(n)$. \square

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system B_n has a finite number of subsystems. \square

4. A CONJECTURAL SOLUTION TO OPEN PROBLEMS 1 AND 2

Lemma 4. *For every positive integers x and y , $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 5. (*Wilson's theorem*, [2, p. 89]). *For every integer $x \geq 2$, x is prime if and only if x divides $(x - 1)! + 1$.*

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 4 and the diagram in Figure 3 explain the construction of the system \mathcal{A} .



Fig. 3 Construction of the system \mathcal{A}

Lemma 6. *For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:*

$$\begin{aligned}
 x_2 &= x_1^2 \\
 x_3 &= (x_1^2)! \\
 x_4 &= ((x_1^2)!)! \\
 x_5 &= x_1^2 + 1 \\
 x_6 &= (x_1^2 + 1)! \\
 x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
 x_8 &= (x_1^2)! + 1 \\
 x_9 &= ((x_1^2)! + 1)!
 \end{aligned}$$

Proof. By Lemma 4, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 6 follows from Lemma 5. \square

Lemma 7. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$. This is true as every such tuple (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap \mathbb{N} = \{1, 2\}$. \square

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system \mathcal{A} .*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 6, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 6 and 7, the set \mathcal{P}_{n^2+1} is infinite. \square

Theorem 6. *Conjecture 1 implies the statement Φ .*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. \square

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 6 and 7, if positive integers x_1, \dots, x_9 solve the system \mathcal{A} , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 6 and the statement Φ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 . Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. \square

Statement 4. *The statement Φ implies that conditions (1)–(5) hold for $X = \{1\} \cup \mathcal{P}_{n^2+1}$.*

Proof. The set \mathcal{P}_{n^2+1} is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [7]. These two facts imply condition (1). By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [3]. Thus condition (2) holds. Condition (3) holds trivially. The statement Φ implies that β is a threshold number of $X = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition (4) holds. The following condition:

$k - 1$ is a square and k has no divisors greater than 1 and smaller than k

defines in \mathbb{N} the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition (5). \square

Proving Landau's conjecture will disprove Statement 4.

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