Are there a set $\mathcal{X} \subseteq \mathbb{N}$ and a constructively defined integer n such that $(\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]) \wedge$ (a constructively defined algorithm decides $\mathcal{X}$ and there are many elements of $\mathcal{X}$) $\wedge$ (the infiniteness of $\mathcal{X}$ is conjectured and cannot be decided by any known method) $\wedge$ ($\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements)?

AGNIESZKA KOZDĘBA and APOLONIUSZ TYSZKA

Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geqslant 2$. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\{x_i! = x_k : i, k \in \{1, \ldots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\}$. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1, \ldots, x_9$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. We write down a system $\mathcal{A} \subseteq B$ of 8 equations. Let Λ denote the statement: *if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \leqslant f(9)$.* The statement Λ is equivalent to the statement Φ. It heuristically proves the statement Φ. This proof does not yield that $\mathrm{card}(\mathcal{P}_{n^2+1}) = \omega$. Let $\mathcal{F}(\mathcal{X})$ denote the conjunction of the first three conditions from the title. The set $\mathcal{X} = \{k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7), k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies the formula $\mathcal{F}(\mathcal{X})$. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy the formula $\mathcal{F}(\mathcal{X})$ forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction of the last three conditions from the title. The statement Φ implies that the conjunction from the title holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$.

2020 Mathematics Subject Classification: 03D20.

Key words and phrases: constructively defined integer n satisfies $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$, constructively defined algorithm, current knowledge on a set $\mathcal{X} \subseteq \mathbb{N}$, decidable and conjecturally infinite set $\mathcal{X} \subseteq \mathbb{N}$, distinction between existing algorithms and known algorithms, known elements of a set $\mathcal{X} \subseteq \mathbb{N}$, physical limits of computation, primes of the form $n^2 + 1$.

1. Definitions and the distinction between existing algorithms and known algorithms

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known to us). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n.

Definition 1. *Conditions* (1)-(5) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1) *There are many elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is infinite.*

(2) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.

(3) *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.

(4) *A known algorithm with no input returns an integer* n *satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) $\mathcal{X}$ *is naturally defined i.e.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.*

Condition (2) implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$.

Definition 2. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite, see [6]–[8]. Let $[\cdot]$ denote the integer part function.

Example 1. *The set* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *satifies condition* (2).

Example 2. *The set* $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$ *does not satisfy condition* (2) *because we know an algorithm with no input that computes* $[\frac{\beta}{\pi}]$.

Example 3. *(*[1], [4], [5, p. 9]*). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \textit{if the decimal expansion of } \pi \textit{ contains } n \textit{ consecutive zeros} \\ 0, & \textit{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ *or there exists* $k \in \mathbb{N}$ *such that*

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

No known algorithm computes the function h.

Example 4. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \textit{if the continuum hypothesis is true} \\ \emptyset, & \textit{otherwise} \end{cases}$$

is decidable. No constructively existing algorithm decides $\mathcal{X}$*, which holds forever.*

Definition 3. *Let Φ denote the following unproven statement:*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ. In Section 4, we heuristically prove the statement Φ. This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. *Condition* `(4)` *remains unproven for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm `Alg`$(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in condition `(4)`, but the algorithm `Alg`$(\mathcal{P}_{n^2+1})$ is unknown for us because its definition is ineffective. □

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in mathematics because it refers to the current mathematical knowledge. The same is true for Open Problem 1 and Statements 2 and 3.

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies conditions* `(1)-(5)`*?*

Open Problem 1 asks: *Are there a set $\mathcal{X} \subseteq \mathbb{N}$ and a constructively defined integer n such that $(\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]) \wedge$ (a constructively defined algorithm decides $\mathcal{X}$ and there are many elements of $\mathcal{X}$) $\wedge$ (the infiniteness of $\mathcal{X}$ is conjectured and cannot be decided by any known method) $\wedge$ ($\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements)?*

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions `(1)-(4)`. *Condition* `(5)` *fails for* $\mathcal{X}$.

Proof. Condition `(1)` holds as $\mathcal{X} \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β, see [3]. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus $\mathcal{X}$ satisfies condition `(4)`. Condition `(5)` fails for $\mathcal{X}$ as the set of known elements of $\mathcal{X}$ equals $\{0, \ldots, \beta\}$. □

Proving Landau's conjecture will disprove Statement 2.

Theorem 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions* (1)-(4) *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies condition (4). Since conditons (2)-(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $$n + 1 \notin \mathcal{X},\ n + 2 \notin \mathcal{X},\ n + 3 \notin \mathcal{X},\ \ldots$$

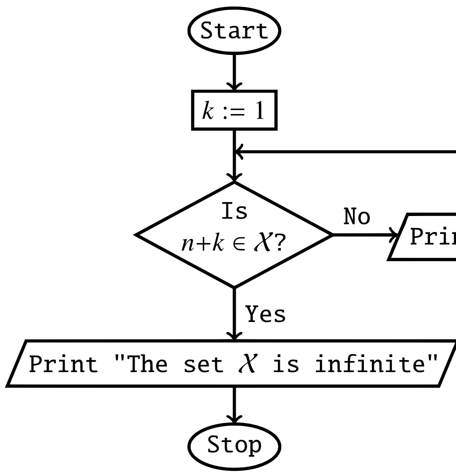

Fig. 1 Semi-algorithm that terminates if and only if the set $\mathcal{X}$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in condition (1). □

The physical limits of computation ([3]) disprove the assumption of Theorem 1.

3. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! & = & x_1 \\ x_1 \cdot x_1 & = & x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! & = & x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

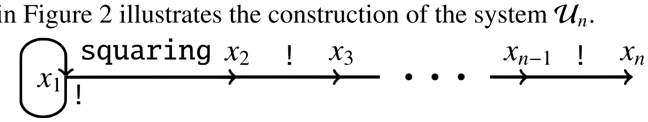

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 2. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.*

Let B_n denote the following system of equations:

$$\left\{x_i! = x_k : \ i, k \in \{1, \ldots, n\}\right\} \cup \left\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\right\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \ \Psi_n$.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. *(Wilson's theorem,* [2, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)! + 1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

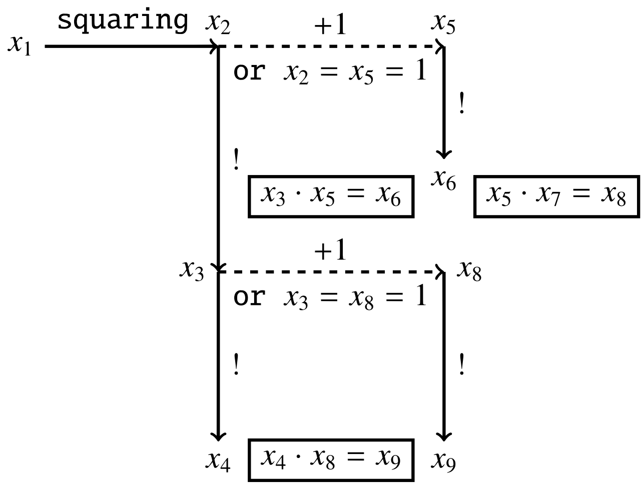

Fig. 3 Construction of the system $\mathcal{A}$

Lemma 5. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \dots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Lemma 6. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \dots, x_9)$ satisfies $x_1, \dots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap \mathbb{N} = \{1, 2\}$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system $\mathcal{A}$.*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement Φ.*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers $x_1, \dots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \dots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 3. *Conditions* `(1)-(3)` *and* `(5)` *hold for $\mathcal{X} = \mathcal{P}_{n^2+1}$. The statement Φ implies that condition* `(4)` *holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [7]. These two facts imply condition `(1)`. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [3]. Thus condition `(2)` holds. Conditions `(3)` and `(5)` hold trivially. The statement Φ implies that β is a threshold number of $\mathcal{P}_{n^2+1}$. Hence, the statement Φ implies that condition `(4)` holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$. □

Proving Landau's conjecture will disprove Statement 3.

Acknowledgement. Agnieszka Kozdęba prepared three diagrams. Apoloniusz Tyszka wrote the article.

References

[1] J. Case and M. Ralston, *Beyond Rogers' non-constructively computable function*, in: *The nature of computation*, Lecture Notes in Comput. Sci., 7921, 45–54, Springer, Heidelberg, 2013, http://link.springer.com/chapter/10.1007/978-3-642-39053-1_6.

[2] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory,* 2nd ed., CRC Press, Boca Raton, FL, 2016.

[3] S. Lloyd, *Ultimate physical limits to computation,* Nature 406 (2000), 1047–1054, http://doi.org/10.1038/35023282.

[4] R. Reitzig, *How can it be decidable whether π has some sequence of digits?*, http://cs.stackexchange.com/questions/367/how-can-it-be-decidable-whether-pi-has-some-sequence-of-digits.

[5] H. Rogers, Jr., *Theory of recursive functions and effective computability*, 2nd ed., MIT Press, Cambridge, MA, 1987.

[6] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002496, Primes of the form $n^2 + 1$,* http://oeis.org/A002496.

[7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A083844, Number of primes of the form $x^2 + 1 < 10^n$,* http://oeis.org/A083844.

[8] Wolfram MathWorld, *Landau's Problems,* http://mathworld.wolfram.com/LandausProblems.html.

Agnieszka Kozdęba

Faculty of Environmental Engineering and Land Surveying
Hugo Kołłątaj University
Balicka 253C, 30-198 Kraków, Poland

Institute of Mathematics
Jagiellonian University
Łojasiewicza 6, 30-348 Kraków, Poland

E-mail address: Agnieszka.Kozdeba@im.uj.edu.pl

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail address: rttyszka@cyf-kr.edu.pl