The physical limits of computation inspire an open problem that concerns decidable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in ZFC as it refers to the current knowledge on $\mathcal{X}$

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Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\{x_i! = x_k : i, k \in \{1, \ldots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\}$. We write some system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1, \ldots, x_9$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. No known system $\mathcal{S} \subseteq B$ with a finite number of solutions in positive integers $x_1, \ldots, x_9$ has a solution $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ satisfying $\max(x_1, \ldots, x_9) > f(9)$. We write some system $\mathcal{A} \subseteq B$ of 8 equations. Let Λ denote the statement: *if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \leqslant f(9)$.* The statement Λ is equivalent to the statement Φ. It heuristically proves the statement Φ. This proof does not yield that $\mathrm{card}(\mathcal{P}_{n^2+1}) = \omega$. Algorithms always terminate. We explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known to us). We explain which elements of a set $\mathcal{X} \subseteq \mathbb{N}$ are classified as known. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning. (1) *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. (2) *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* (3) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$. (4) *There are many elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.* (5) *$\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.* The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (2)-(5). The statement Φ implies condition (1) for $\mathcal{X} = \mathcal{P}_{n^2+1}$. The set $\mathcal{X} = \{k \in \mathbb{N} : (10^{13} < k) \Rightarrow (f(10^{13}), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)-(4) and does not satisfy condition (5) as the set of known elements of $\mathcal{X}$ equals $\{0, \ldots, 10^{13}\}$. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption.

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1. Definitions and the distinction between existing algorithms and known algorithms

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Algorithms always terminate. Semi-algorithms may not terminate. Examples 2–5 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in ZFC) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known to us). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Example 1 explains which elements of a set $\mathcal{X} \subseteq \mathbb{N}$ are classified as known.

Example 1. *Let $\mathcal{T}$ denote the set of twin primes. The known elements of the set* $\{0, \ldots, 10^{13}\} \cup [((((9!)!)!)!)! + 1, \infty) \cap \mathcal{T}$ *form the set* $\{0, \ldots, 10^{13}\}$. *The numbers* $t_1 = 459 \cdot 2^{8529} - 1$ *and* $t_2 = 459 \cdot 2^{8529} + 1$ *belong to $\mathcal{T}$ (Harvey Dubner,* [9, p. 108]*). We classify t_1 and t_2 as known elements of $\mathcal{T}$ because we know the two algebraic expressions (i.e. terms) that define t_1 and t_2 in the field of real numbers. We assume that every known algebraic expression can be physically written. Physics implies that they form a finite set. Let t denote the largest twin prime that is smaller than* $((((9!)!)!)!)!$. *We classify t as an unknown element of $\mathcal{T}$ because no known algebraic expression (i.e. term) defines t in the field of real numbers.*

Definition 1. *Conditions* `(1)`-`(5)` *concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
`(1)` *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
`(2)` *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*
`(3)` *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.
`(4)` *There are many elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.*
`(5)` *$\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.*

Every set $\mathcal{X} \subseteq \mathbb{N}$ studied in this article has at most finitely many known elements, which does not depend on physics. Condition `(3)` implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies conditions `(1)`-`(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning.

Definition 2. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at `https://wolframalpha.com`. □

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [6]–[8]. Let $[\cdot]$ denote the integer part function.

Example 2. *The set* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *satisfies condition* `(3)`.

Example 3. *The set* $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$ *does not satisfy condition* `(3)` *because we know an algorithm with no input that computes* $[\frac{\beta}{\pi}]$. *The set of known elements of $\mathcal{X}$ is empty. Hence, condition* `(5)` *fails for $\mathcal{X}$.*

Example 4. *(*[1], [4], [5, p. 9]*). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & if\ the\ decimal\ expansion\ of\ \pi\ contains\ n\ consecutive\ zeros \\ 0, & otherwise \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ *or there exists* $k \in \mathbb{N}$ *such that*

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

No known algorithm computes the function h.

Example 5. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & if\ the\ continuum\ hypothesis\ holds \\ \emptyset, & otherwise \end{cases}$$

is decidable. This $\mathcal{X}$ satisfies conditions `(1)` *and* `(3)` *and does not satisfy conditions* `(2)`*,* `(4)`*, and* `(5)`*. These facts will hold forever.*

Definition 3. *Let Φ denote the following unproven statement:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ. Theorem 6 heuristically justifies the statement Φ. This proof does not yield that $\mathrm{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. *Condition* `(1)` *remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\mathtt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in condition `(1)`, but the algorithm $\mathtt{Alg}(\mathcal{P}_{n^2+1})$ is unknown for us because its definition is ineffective. □

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Open Problem 1 and Statements 2 and 3.

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies conditions* `(1)`-`(5)`*?*

Open Problem 1 asks: *Are there a set $\mathcal{X} \subseteq \mathbb{N}$ and a constructively defined integer n such that $(\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]) \wedge$ ($\mathcal{X}$ is decidable by a constructively defined algorithm) $\wedge$ (there are many elements of $\mathcal{X}$) $\wedge$ (the infiniteness of $\mathcal{X}$ is conjectured and cannot be decided by any known method) $\wedge$ ($\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements)?*

Statement 2. *The set $\mathcal{X} = \{k \in \mathbb{N} : (10^{13} < k) \Rightarrow (f(10^{13}), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions* `(1)`-`(4)`*. Condition* `(5)` *fails for $\mathcal{X}$.*

Proof. Condition `(4)` holds as $\mathcal{X} \supseteq \{0, \ldots, 10^{13}\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(10^{13}) > f(7) = \beta$, see [3]. Thus condition `(3)` holds. Condition `(2)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^{13} < k) \wedge (f(10^{13}), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer 10^{13} is a threshold number of $\mathcal{X}$. Thus $\mathcal{X}$ satisfies condition `(1)`. Condition `(5)` fails for $\mathcal{X}$ as the set of known elements of $\mathcal{X}$ equals $\{0, \ldots, 10^{13}\}$. □

Proving Landau's conjecture will disprove Statement 2.

Theorem 1. *No set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy conditions* `(1)-(4)` *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies condition `(1)`. Since conditions `(1)-(3)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots \tag{T}$$

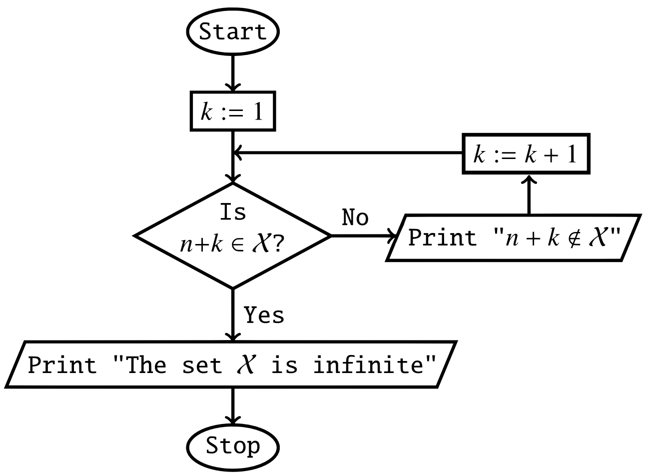

Fig. 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence `(T)` and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in condition `(4)`. □

The physical limits of computation ([3]) disprove the assumption of Theorem 1.

3. Number-theoretic statements Ψ_n

Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

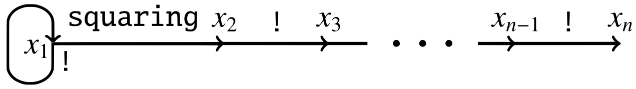

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 2. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $(f(1), \ldots, f(n))$.

Let B_n denote the following system of equations:

$$\left\{x_i! = x_k : \ i, k \in \{1, \ldots, n\}\right\} \cup \left\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\right\}$$

For every positive integer n, no known system $\mathcal{S} \subseteq B_n$ with a finite number of solutions in positive integers $x_1, \ldots, x_n$ has a solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ satisfying $\max(x_1, \ldots, x_n) > f(n)$. For a positive integer n, let Ψ_n denote the following statement: *if a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. *(Wilson's theorem,* [2, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)! + 1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

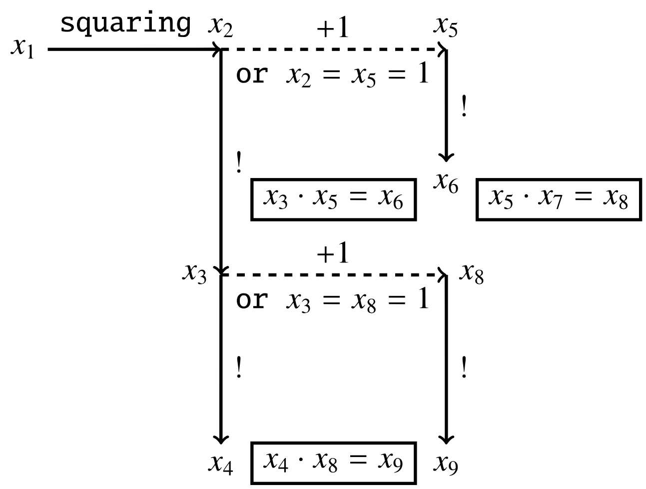

Fig. 3 Construction of the system $\mathcal{A}$

Lemma 5. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{array}{rcl}
x_2 & = & x_1^2 \\
x_3 & = & (x_1^2)! \\
x_4 & = & ((x_1^2)!)! \\
x_5 & = & x_1^2 + 1 \\
x_6 & = & (x_1^2 + 1)! \\
x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 & = & (x_1^2)! + 1 \\
x_9 & = & ((x_1^2)! + 1)!
\end{array}$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Lemma 6. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap \mathbb{N} = \{1, 2\}$. □

Conjecture 1. *The statement* Ψ_9 *is true when is restricted to the system* $\mathcal{A}$.

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer* $x_1 \geqslant 2$ *such that* x_1^2+1 *is prime and greater than* $f(7)$*, then the set* $\mathcal{P}_{n^2+1}$ *is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2+1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)!+1)! \geqslant (f(8)+1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement* Φ.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement* Φ *implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers $x_1, \dots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2+1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1,2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \dots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 3. *Conditions* `(2)-(5)` *hold for* $\mathcal{X} = \mathcal{P}_{n^2+1}$. *The statement* Φ *implies condition* `(1)` *for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form n^2+1 in the interval $[2, 10^{28})$, see [7]. These two facts imply condition `(4)`. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [3]. Thus condition `(3)` holds. Conditions `(2)` and `(5)` hold trivially. The statement Φ implies that β is a threshold number of $\mathcal{P}_{n^2+1}$. Hence, the statement Φ implies condition `(1)` for $\mathcal{X} = \mathcal{P}_{n^2+1}$. □

Proving Landau's conjecture will disprove Statement 3.

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