# On sets $\mathcal{W} \subseteq \mathbb{N}$ whose infinity follows from the existence in $\mathcal{W}$ of an element which is greater than a threshold number computed for $\mathcal{W}$

Apoloniusz Tyszka

**Abstract**

We define computable functions $f, g\colon \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$. For a positive integer $n$, let $\Theta_n$ denote the following statement: if a system $\mathcal{S} \subseteq \Big\{x_i! = x_k : i,k \in \{1,\ldots,n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,n\}\Big\}$ has only finitely many solutions in integers $x_1,\ldots,x_n$ greater than 1, then each such solution $(x_1,\ldots,x_n)$ satisfies $\min(x_1,\ldots,x_n) \leqslant f(n)$. The statement $\Theta_9$ proves that if there exists an integer $x > f(9)$ such that $x^2+1$ (alternatively, $x!+1$) is prime, then there are infinitely many primes of the form $n^2+1$ (respectively, $n!+1$). The statement $\Theta_{16}$ proves that if there exists a twin prime greater than $f(16)+3$, then there are infinitely many twin primes. We formulate a statement which proves that if $2^{2^n}+1$ is composite for some integer $n > g(13)$, then $2^{2^n}+1$ is composite for infinitely many positive integers $n$.

**Key words and phrases:** Brocard's problem, Brocard-Ramanujan equation, composite Fermat numbers, composite numbers of the form $2^{2^n}+1$, prime numbers of the form $n^2+1$, prime numbers of the form $n!+1$, Richert's lemma, twin prime conjecture.

**2010 Mathematics Subject Classification:** 03B30, 11U09.

## 1 Introduction

The following observation concerns the theme described in the title of the article.

**Observation 1.** *If $n \in \mathbb{N}$ and $\mathcal{W} \subseteq \{0,\ldots,n\}$, then we take any integer $m \geqslant n$ as a threshold number for $\mathcal{W}$. If $\mathcal{W} \subseteq \mathbb{N}$ and $W$ is infinite, then we take any non-negative integer $m$ as a threshold number for $\mathcal{W}$.*

We define the set $\mathcal{U} \subseteq \mathbb{N}$ by declaring that a non-negative integer $n$ belongs to $\mathcal{U}$ if and only if $\sin\left(10^{10^{10^{10}}}\right) > 0$. This inequality is practically undecidable, see [6].

**Corollary 1.** *The set $\mathcal{U}$ equals $\emptyset$ or $\mathbb{N}$. The statement "$\mathcal{U} = \emptyset$" remains unproven and the statement "$\mathcal{U} = \mathbb{N}$" remains unproven. Every non-negative integer $m$ is a threshold number for $\mathcal{U}$. For every non-negative integer $k$, the sentence "$k \in \mathcal{U}$" is only theoretically decidable.*

The first-order language of graph theory contains two relation symbols of arity 2: $\sim$ and $=$, respectively for adjacency and equality of vertices. The term first-order imposes the condition that the variables represent vertices and hence the quantifiers apply to vertices only. For a first-order sentence $\Lambda$ about graphs, let Spectrum($\Lambda$) denote the set of all positive integers $n$ such that there is a graph on $n$ vertices satisfying $\Lambda$. By a graph on $n$ vertices we understand a set of $n$ elements with a binary relation which is symmetric and irreflexive.

**Theorem 1.** *([13, p. 171]). If a sentence $\Lambda$ in the language of graph theory has the form $\exists x_1 \ldots x_k \, \forall y_1 \ldots y_l \, \Upsilon(x_1, \ldots, x_k, y_1, \ldots, y_l)$, where $\Upsilon(x_1, \ldots, x_k, y_1, \ldots, y_l)$ is quantifier-free, then either* Spectrum($\Lambda$) $\subseteq [1, (2^k \cdot 4^l) - 1]$ *or* Spectrum($\Lambda$) $\supseteq [k + l, \infty) \cap \mathbb{N}$.

**Corollary 2.** *The number $\left(2^k \cdot 4^l\right) - 1$ is a threshold number for* Spectrum($\Lambda$).

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [11, p. 234].

**Corollary 3.** *If an algorithm $\mathtt{Al}_1$ for every recursive set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Al}_1(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Al}_1(\mathcal{M}) + 1, \infty) \neq \emptyset$.*

**Corollary 4.** *If an algorithm $\mathtt{Al}_2$ for every recursively enumerable set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Al}_2(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\mathtt{Al}_2(\mathcal{M}) + 1, \infty) \neq \emptyset$.*

Let $K = \{j \in \mathbb{N} : \ 2^{\aleph_j} = \aleph_{j+1}\}$.

**Theorem 2.** *If ZFC is consistent, then for every non-negative integer n the sentence*

"$n$ `is a threshold number for` $K$"

*is not provable in ZFC*

*Proof.* There exists a model $\mathcal{E}$ of ZFC such that

$$\forall i \in \{0, \ldots, n+1\}\ \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+1}$$

and

$$\forall i \in \{n+2, n+3, n+4, \ldots\}\ \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+2}$$

see [4] and [7, p. 232]. In the model $\mathcal{E}$, $K = \{0, \ldots, n+1\}$ and $n$ is not a threshold number for $K$. □

**Theorem 3.** *If ZFC is consistent, then for every non-negative integer n the sentence*

"$n$ is not a threshold number for $K$"

*is not provable in ZFC.*

*Proof.* The Generalized Continuum Hypothesis (GCH) is consistent with ZFC, see [7, p. 188] and [7, p. 190]. GCH implies that $K = \mathbb{N}$. Consequently, GCH implies that every non-negative integer $n$ is a threshold number for $K$. □

**Theorem 4.** *([2, p. 35]). There exists a polynomial $D(x_1, \ldots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences*

"The equation $D(x_1, \ldots, x_m) = 0$ is solvable in non-negative integers"

*and*

"The equation $D(x_1, \ldots, x_m) = 0$ is not solvable in non-negative integers"

*are not provable in ZFC.*

Let $\Delta$ denote the set of all non-negative integers $k$ such that the equation $D(x_1, \ldots, x_m) = 0$ has no solutions in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, the set $\Delta$ is computable. Theorem 4 implies the following corollary.

**Corollary 5.** *If ZFC is arithmetically consistent, then for every non-negative integer n the sentences*

"$n$ is a threshold number for $\Delta$"

*and*

"$n$ is not a threshold number for $\Delta$"

*are not provable in ZFC.*

Let $g(1) = 1$, and let $g(n+1) = 2^{2^{g(n)}}$ for every positive integer $n$.

**Hypothesis 1.** *([18]). If a system*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\} \cup \{x_i + 1 = x_k : \ i, k \in \{1, \ldots, n\}\}$$

*has only finitely many solutions in non-negative integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant g(2n)$.*

**Theorem 5.** *([18]). Hypothesis 1 implies that for every $W(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ we can compute a threshold number $b \in \mathbb{N} \setminus \{0\}$ such that any non-negative integers $a_1, \ldots, a_n$ which satisfy*

$$(W(a_1, \ldots, a_n) = 0) \wedge (\max(a_1, \ldots, a_n) > b)$$

*guarantee that the equation $W(x_1, \ldots, x_n) = 0$ has infinitely many solutions in non-negative integers.*

## 2 Basic lemmas

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{V}_1$ denote the system of equations $\{x_1! = x_1\}$, and let $\mathcal{V}_2$ denote the system of equations $\{x_1! = x_1, \ x_1 \cdot x_1 = x_2\}$. For an integer $n \geqslant 3$, let $\mathcal{V}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\} \ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{V}_n$.

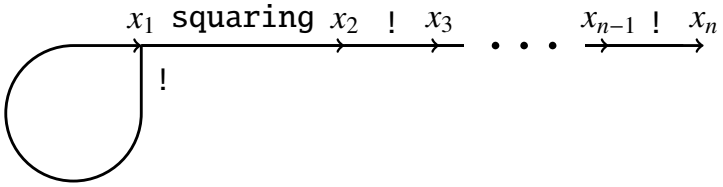


**Fig. 1** Construction of the system $\mathcal{V}_n$

**Lemma 1.** *For every positive integer n, the system $\mathcal{V}_n$ has exactly one solution in integers greater than* 1*, namely $\big(f(1), \ldots, f(n)\big)$.*

Let

$$H_n = \left\{x_i! = x_k : \ i,k \in \{1,\dots,n\}\right\} \cup \left\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\dots,n\}\right\}$$

For a positive integer $n$, let $\Theta_n$ denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers $x_1,\dots,x_n$ greater than 1, then each such solution $(x_1,\dots,x_n)$ satisfies $\min(x_1,\dots,x_n) \leqslant f(n)$. The assumption $\min(x_1,\dots,x_n) \leqslant f(n)$ is weaker than the assumption $\max(x_1,\dots,x_n) \leqslant f(n)$ suggested by Lemma 1.

**Lemma 2.** *For every positive integer n, the system $H_n$ has a finite number of subsystems.*

**Theorem 6.** *Every statement $\Theta_n$ is true with an unknown integer bound that depends on n.*

*Proof.* It follows from Lemma 2. □

**Lemma 3.** *For every integers x and y greater than* 1, $x! \cdot y = y!$ *if and only if* $x + 1 = y$.

**Lemma 4.** *If* $x \geqslant 4$, *then* $\dfrac{(x-1)! + 1}{x} > 1$.

**Lemma 5.** *(Wilson's theorem, [5, p. 89]). For every integer* $x \geqslant 2$, *x is prime if and only if x divides* $(x-1)! + 1$.

## 3 Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [12]. It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [19, p. 297].

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

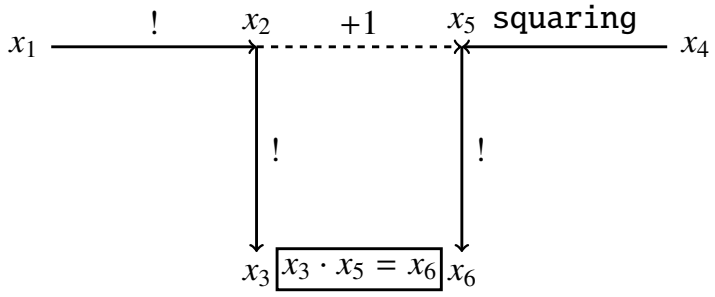


**Fig. 2** Construction of the system $\mathcal{A}$

**Lemma 6.** *For every integers $x_1$ and $x_4$ greater than 1, the system $\mathcal{A}$ is solvable in integers $x_2, x_3, x_5, x_6$ greater than 1 if and only if $x_1! + 1 = x_4^2$. In this case, the integers $x_2, x_3, x_5, x_6$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

*and $x_1 = \min(x_1, \ldots, x_6)$.*

*Proof.* It follows from Lemma 3. □

**Theorem 7.** *The statement $\Theta_6$ proves the following implication: if the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then each such solution $(x_1, x_4)$ satisfies $x_1 \leqslant f(6)$.*

*Proof.* Let positive integers $x_1$ and $x_4$ satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 6, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0, 1\})^4$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Lemma 6 guarantees that $x_1 = \min(x_1, \ldots, x_6)$. By the antecedent and Lemma 6, the system $\mathcal{A}$ has only finitely many solutions in integers $x_1, \ldots, x_6$ greater than 1. Therefore, the statement $\Theta_6$ implies that $x_1 = \min(x_1, \ldots, x_6) \leqslant f(6)$. □

**Hypothesis 2.** *The implication in Theorem 7 is true.*

**Corollary 6.** *Assuming Hypothesis 2, a single query to an oracle for the halting problem decides the problem of the infinitude of the solutions of the equation $x! + 1 = y^2$.*

## 4 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [10, pp. 37–38]. Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

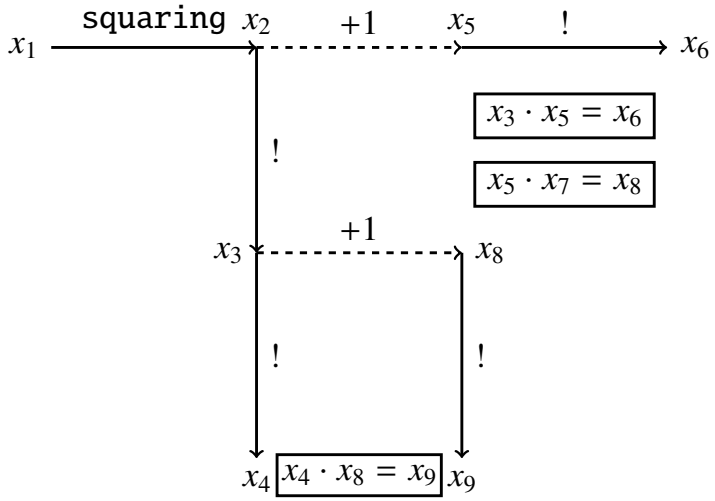


**Fig. 3** Construction of the system $\mathcal{B}$

**Lemma 7.** *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined*

*by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*and* $\min(x_1, \ldots, x_9) = x_1$.

*Proof.* By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. □

**Theorem 8.** *The statement* $\Theta_9$ *proves the following implication: if there exists an integer* $x_1 > f(9)$ *such that* $x_1^2 + 1$ *is prime, then there are infinitely many primes of the form* $n^2 + 1$.

*Proof.* Assume that an integer $x_1$ is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Lemma 7 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement $\Theta_9$ and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. □

**Hypothesis 3.** *The implication in Theorem 8 is true.*

**Corollary 7.** *Assuming Hypothesis 3, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form* $n^2 + 1$.

Let $\mathcal{P}$ denote the set of prime numbers. For a non-negative integer $n$, let $\Omega(n)$ denote the following statement: $\exists m \in \mathbb{N} \cap (n, \infty)\, m^2 + 1 \in \mathcal{P}$. By Theorem 8, assuming the statement $\Theta_9$, we can infer the statement $\forall n \in \mathbb{N}\, \Omega(n)$ from any statement $\Omega(n)$ with $n \geqslant f(9)$. A similar situation holds for inference by the so called *"super-induction method"*, see [20]–[23]. In section 8, we present a theorem whose computer-assisted proof is based on the super-induction method.

## 5 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [1, p. 443] and [16]. Let $\mathcal{G}$ denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system $\mathcal{G}$.

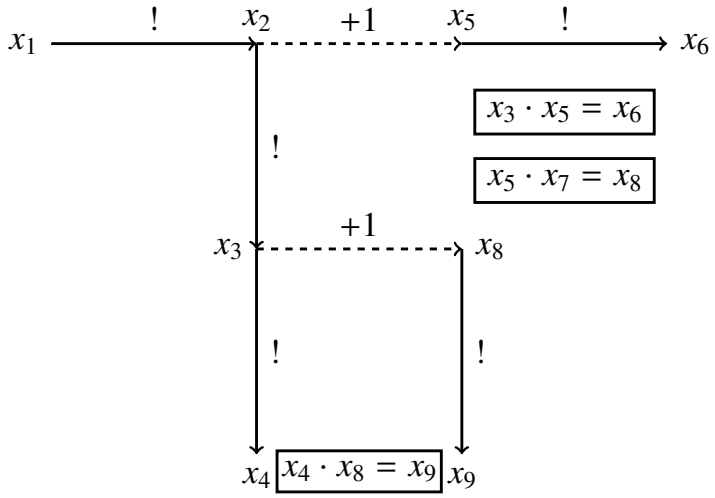


**Fig. 4** Construction of the system $\mathcal{G}$

**Lemma 8.** *For every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than* 1 *if and only if $x_1! + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by*

*the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_4 &= ((x_1!)!)! \\ x_5 &= x_1^! + 1 \\ x_6 &= (x_1! + 1)! \\ x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\ x_8 &= (x_1!)! + 1 \\ x_9 &= ((x_1!)! + 1)! \end{aligned}$$

*and* $\min(x_1, \ldots, x_9) = x_1$.

*Proof.* By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 5. □

**Theorem 9.** *The statement* $\Theta_9$ *proves the following implication: if there exists an integer* $x_1 > f(9)$ *such that* $x_1! + 1$ *is prime, then there are infinitely many primes of the form* $n! + 1$.

*Proof.* Assume that an integer $x_1$ is greater than $f(9)$ and $x_1! + 1$ is prime. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{G}$. Lemma 8 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{G} \subseteq H_9$, the statement $\Theta_9$ and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{G}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 8, there are infinitely many primes of the form $n! + 1$. □

**Hypothesis 4.** *The implication in Theorem 9 is true.*

**Corollary 8.** *Assuming Hypothesis 4, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form* $n! + 1$.

## 6 The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [10, p. 39].

Let $C$ denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{cases}$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system $C$.

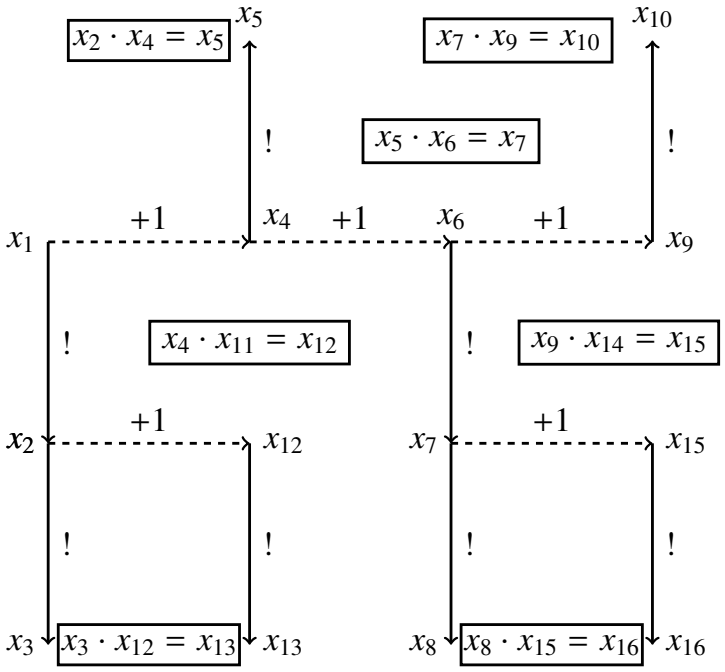


**Fig. 5** Construction of the system $C$

**Lemma 9.** *If $x_4 = 2$, then the system $C$ has no solutions in integers $x_1, \ldots, x_{16}$ greater than* 1.

*Proof.* The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. □

**Lemma 10.** *If $x_4 = 3$, then the system $C$ has no solutions in integers $x_1, \ldots, x_{16}$ greater than* 1.

*Proof.* The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. □

**Lemma 11.** *For every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system $C$ is solvable in integers $x_1$, $x_2$, $x_3$, $x_5$, $x_6$, $x_7$, $x_8$, $x_{10}$, $x_{11}$, $x_{12}$, $x_{13}$, $x_{14}$, $x_{15}$, $x_{16}$ greater than* 1 *if and only if $x_4$ and $x_9$ are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1$, $x_2$, $x_3$, $x_5$, $x_6$, $x_7$, $x_8$, $x_{10}$, $x_{11}$, $x_{12}$, $x_{13}$, $x_{14}$, $x_{15}$, $x_{16}$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

*and* $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3$.

*Proof.* By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system $C$ is solvable in integers $x_1$, $x_2$, $x_3$, $x_5$, $x_6$, $x_7$, $x_8$, $x_{10}$, $x_{11}$, $x_{12}$, $x_{13}$, $x_{14}$, $x_{15}$, $x_{16}$ greater than 1 if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 11 follows from Lemma 5. □

**Theorem 10.** *The statement $\Theta_{16}$ proves the following implication: if there exists a twin prime greater than $f(16)+3$, then there are infinitely many twin primes.*

*Proof.* Assume that the antecedent holds. Then, there exist prime numbers $x_4$ and $x_9$ such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0,1,2,3\}$. By Lemma 11, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0,1\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system $C$. Lemma 11 guarantees that $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement $\Theta_{16}$ and the inequality $\min(x_1, \ldots, x_{16}) > f(16)$ imply that the system $C$ has infinitely many solutions in integers $x_1, \ldots, x_{16}$ greater than 1. According to Lemmas 9–11, there are infinitely many twin primes. □

**Hypothesis 5.** *The implication in Theorem 10 is true.*

**Corollary 9.** *(cf. [3]). Assuming Hypothesis 5, a single query to an oracle for the halting problem decides the twin prime problem.*

## 7 Are there infinitely many composite Fermat numbers?

Primes of the form $2^{2^n}+1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n}+1$ is prime, see [9, p. 1]. Fermat correctly remarked that $2^{2^0}+1 = 3$, $2^{2^1}+1 = 5$, $2^{2^2}+1 = 17$, $2^{2^3}+1 = 257$, and $2^{2^4}+1 = 65537$ are all prime, see [9, p. 1].

**Open Problem.** ([9, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n}+1$?*

Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [8, p. 23].

**Theorem 11.** *([17]). An unproven inequality stated in [17] implies that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$.*

**Lemma 12.** *([9, p. 38]). For every positive integer n, if a prime number p divides $2^{2^n}+1$, then there exists a positive integer k such that $p = k \cdot 2^{n+1}+1$.*

**Corollary 10.** *Since $k \cdot 2^{n+1}+1 \geqslant 2^{n+1}+1 \geqslant n+3$, for every positive integers x, y, and n, the equality $(x+1)(y+1) = 2^{2^n}+1$ implies that $\min(n, x, x+1, y, y+1) = n$.*

Let

$$G_n = \left\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\ldots,n\}\right\} \cup \left\{2^{2^{x_i}} = x_k : \ i,k \in \{1,\ldots,n\}\right\}$$

**Lemma 13.** *The following subsystem of $G_n$*

$$\begin{cases} x_1 \cdot x_1 &= x_1 \\ \forall i \in \{1, \ldots, n-1\}\ 2^{2^{x_i}} &= x_{i+1} \end{cases}$$

*has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(g(1), \ldots, g(n))$.*

For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leqslant g(n)$. The assumption $\min(x_1, \ldots, x_n) \leqslant g(n)$ is weaker than the assumption $\max(x_1, \ldots, x_n) \leqslant g(n)$ suggested by Lemma 13.

**Lemma 14.** *For every positive integer n, the system $G_n$ has a finite number of subsystems.*

**Theorem 12.** *Every statement $\Psi_n$ is true with an unknown integer bound that depends on n.*

*Proof.* It follows from Lemma 14. □

**Lemma 15.** *For every non-negative integers b and c, $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

**Theorem 13.** *The statement $\Psi_{13}$ proves the following implication: if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n.*

*Proof.* Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{1}$$

in positive integers. By Lemma 15, we can transform equation (1) into an equivalent system $\mathcal{F}$ which has 13 variables ($x$, $y$, $z$, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 6.

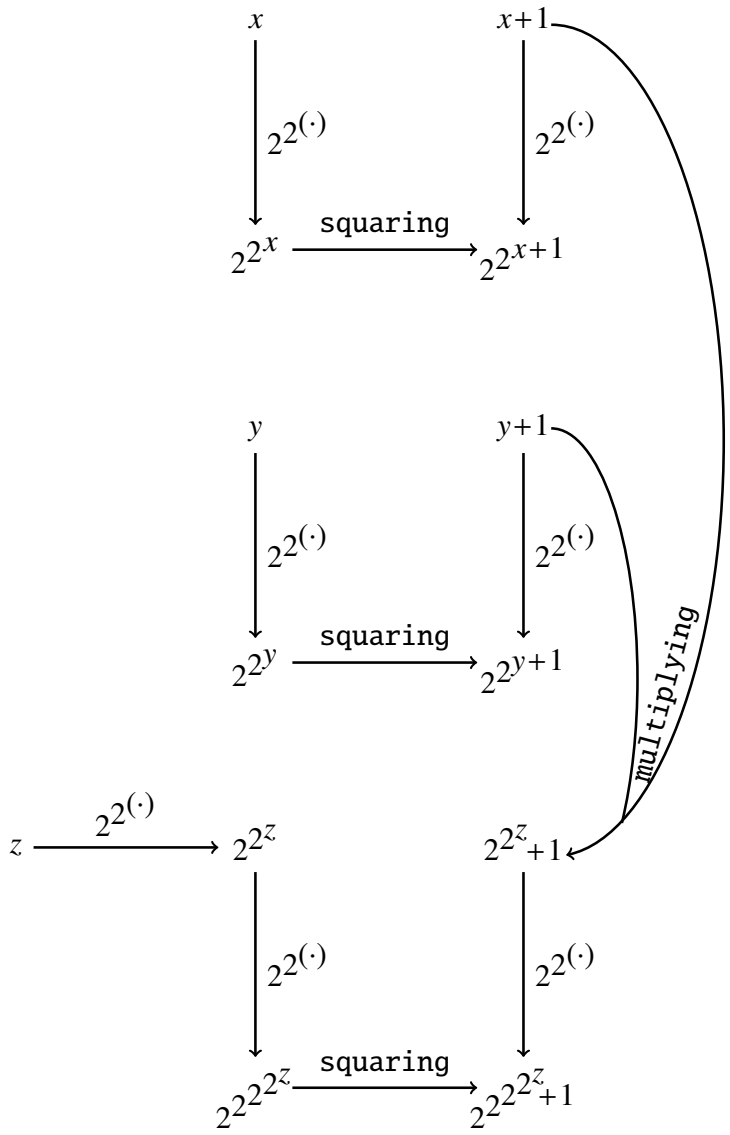


**Fig. 6** Construction of the system $\mathcal{F}$

Assume that $2^{2^n}+1$ is composite for some integer $n > g(13)$. By this and Corollary 10, equation (1) has a solution $(x, y, z) \in (\mathbb{N} \setminus \{0\})^3$ such that $z = n$ and $z = \min(z, x, x+1, y, y+1)$. Hence, the system $\mathcal{F}$ has a solution in positive integers such that $z = n$ and $n$ is the smallest number in the solution sequence. Since $n > g(13)$, the statement $\Psi_{13}$ implies that the system $\mathcal{F}$ has infinitely many solutions in positive integers. Therefore, there are infinitely many positive integers $n$ such that $2^{2^n}+1$ is composite. □

**Hypothesis 6.** *The implication in Theorem 13 is true.*

**Corollary 11.** *Assuming Hypothesis 6, a single query to an oracle for the halting problem decides whether or not the set of composite Fermat numbers is infinite.*

## 8 An application of Richert's lemma

**Lemma 16.** *([14], [15, p. 152]). Let $\{m_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that for some positive integer $k$ the inequality $m_{i+1} \leqslant 2m_i$ holds for all $i > k$. Suppose there exists a non-negative integer $b$ such that the numbers $b+1,\ b+2,\ b+3,\ \ldots,\ b+m_{k+1}$ are all expressible as sums of one or more distinct elements of the set $\{m_1, \ldots, m_k\}$. Then every integer greater than $b$ is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.*

For a positive integer $i$, let $m_i$ denote the integer part of $\dfrac{(i+19)^{i+19}}{(i+19)! \cdot 2^{i+19}}$. Let $\mathcal{T}$ denote the set of all positive integers $x$ which are expressible as a sum of one or more distinct elements of the set $\{m_i : i \in \mathbb{N} \setminus \{0\}\}$. Let $B$ denote the set of all positive integers $x$ which are expressible as a sum of one or more distinct elements of the set $\{m_i : i \in \{1, \ldots, 15\}\}$.

**Theorem 14.** 2761 *is the largest integer which does not belong to $\mathcal{T}$.*

*Proof.* The following *MuPAD* code

```
TEXTWIDTH:=80:
C:={floor((i+19)^(i+19)/((i+19)!*2^(i+19))) $i=1..15+1};
A:={C[i] $i=1..15};
B:={A[1]}:
for i from 2 to 15 do
B:=B union {A[i]} union {B[j]+A[i] $j=1..nops(B)}:
end_for:
{2761} minus B;
{2761+i $i=1..C[15+1]} minus B;
```

first displays the sets $\{m_i : i \in \{1, \ldots, 15+1\}\}$ and $\{m_i : i \in \{1, \ldots, 15\}\}$. Next, it computes the sets $\{2761\} \setminus B$ and $\{2761+1, \ldots, 2761+m_{15+1}\} \setminus B$. The code gives the following output

```
{41, 54, 72, 96, 128, 170, 227, 303, 404, 540, 722, 966, 1293, 1730, 2317, 3105}

{41, 54, 72, 96, 128, 170, 227, 303, 404, 540, 722, 966, 1293, 1730, 2317}

                                    {2761}
```

```
{}
```

Since the set {2761} equals {2761} \ $B$, we conclude that $2761 \notin B$. By this and the inequality $m_{15+1} = 3105 > 2761$, we conclude that $2761 \notin \mathcal{T}$. Since the empty set {} equals {$2761 + 1$, $\ldots$, $2761 + m_{15+1}$} \ $B$, we conclude that each of the integers

$$2761 + 1,\ 2761 + 2,\ 2761 + 3,\ \ldots,\ 2761 + m_{15+1}$$

belongs to $B$. Thus, if we apply Lemma 16 with $k = 15$ and $b = 2761$, we obtain that every integer greater than 2761 belongs to $\mathcal{T}$. □

*MuPAD* is a general-purpose computer algebra system. The commercial version of *MuPAD* is no longer available as a stand-alone product, but only as the *Symbolic Math Toolbox* of *MATLAB*. Fortunately, the presented code can be executed by *MuPAD Light*, which was offered for free for research and education until autumn 2005.

## References


[1] C. K. Caldwell and Y. Gallot, *On the primality of* $n! \pm 1$ *and* $2 \times 3 \times 5 \times \cdots \times p \pm 1$, Math. Comp. 71 (2002), no. 237, 441–448, https://doi.org/10.1090/S0025-5718-01-01315-1.

[2] N. C. A. da Costa and F. A. Doria, *On the foundations of science (LIVRO): essays, first series,* E-papers Serviços Editoriais Ltda, Rio de Janeiro, 2013.

[3] F. G. Dorais, *Can the twin prime problem be solved with a single use of a halting oracle?* July 23, 2011, https://mathoverflow.net/questions/71050.

[4] W. B. Easton, *Powers of regular cardinals,* Ann. Math. Logic 1 (1970), 139–178.

[5] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory,* 2nd ed., CRC Press, Boca Raton, FL, 2016.

[6] J. van der Hoeven, *Undecidability versus undecidability,* Bull. Symbolic Logic 5 (1999), no. 1, 75, https://dx.doi.org/10.2307/421141.

[7] T. Jech, *Set theory,* Springer, Berlin, 2003.

[8] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers,* American Mathematical Society, Providence, RI, 2012.

[9] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry,* Springer, New York, 2001.

[10] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT,* Springer, London, 2012.

[11] P. Odifreddi, *Classical recursion theory: the theory of functions and sets of natural numbers,* North-Holland, Amsterdam, 1989.

[12] M. Overholt, *The Diophantine equation* $n! + 1 = m^2$*,* Bull. London Math. Soc. 25 (1993), no. 2, 104, https://doi.org/10.1112/blms/25.2.104.

[13] O. Pikhurko and O. Verbitsky, *Logical complexity of graphs: a survey;* in: *Model theoretic methods in finite combinatorics,* Contemp. Math. 558, 129–179, Amer. Math. Soc., Providence, RI, 2011, https://dx.doi.org/10.1090/conm/558.

[14] H.-E. Richert, *Über Zerlegungen in paarweise verschiedene Zahlen,* Norsk Mat. Tidsskr. 31 (1949), 120–122.

[15] W. Sierpiński, *Elementary theory of numbers,* 2nd ed. (ed. A. Schinzel), PWN – Polish Scientific Publishers and North-Holland, Warsaw-Amsterdam, 1987.

[16] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002981, Numbers n such that* $n! + 1$ *is prime,* https://oeis.org/A002981.

[17] A. Tyszka, *Is there a computable upper bound for the height of a solution of a Diophantine equation with a unique solution in positive integers?,* Open Comput. Sci. 7 (2017), no. 1, 17–23, https://doi.org/10.1515/comp-2017-0003.

[18] A. Tyszka, *A hypothetical upper bound on the heights of the solutions of a Diophantine equation with a finite number of solutions,* Open Comput. Sci. 8 (2018), no. 1, 109–114, https://dx.doi.org/10.1515/comp-2018-0012.

[19] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics,* 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.

[20] A. A. Zenkin, *Superinduction: a new method for proving general mathematical statements with a computer,* Dokl. Math. 55 (1997), no. 3, 410–413.

[21] A. A. Zenkin, *Superinduction method: logical acupuncture of mathematical infinity* (in Russian), in: *Infinity in mathematics: philosophical and historical aspects* (ed. A. G. Barabashev), Janus-K, Moscow, 1997, 152–168, 173–176.

[22] A. A. Zenkin, *The generalized Waring problem: estimation of the function $G(m,r)$ in terms of the function $g(m-1,r)$ by the superinduction method,* Dokl. Math. 56 (1997), no. 1, 597–600.

[23] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity,* paper presented at the Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, `https://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm`.

Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: `rttyszka@cyf-kr.edu.pl`