The physical limits of computation inspire an open problem that concerns decidable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in mathematics understood as an a priori science because it refers to the current knowledge on $\mathcal{X}$

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Abstract. Algorithms always terminate. We explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in *ZFC*) and known algorithms (i.e. algorithms whose definition is constructive and currently known). Assuming that the infiniteness of a set $\mathcal{X} \subseteq \mathbb{N}$ is false or unproven, we define which elements of $\mathcal{X}$ are classified as known. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning. (1) *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. (2) *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* (3) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$. (4) *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.* (5) *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.* Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form n^2+1. Conditions (2)-(5) hold for $\mathcal{X} = \mathcal{P}_{n^2+1}$. We discuss a conjecture which implies the conjunction of Conditions (1)-(5) for $\mathcal{X} = \mathcal{P}_{n^2+1}$. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. We present a table that shows satisfiable conjunctions consisting of Conditions (1)-(5) and their negations.

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Key words and phrases: conjecturally infinite set $\mathcal{X} \subseteq \mathbb{N}$, constructively defined integer n satisfies $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$, current knowledge on a set $\mathcal{X} \subseteq \mathbb{N}$, distinction between existing algorithms and known algorithms, known elements of a set $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, physical limits of computation, primes of the form n^2+1, $\mathcal{X}$ is decidable by a constructively defined algorithm.

1. Definitions and the distinction between existing algorithms and known algorithms

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. *We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs:* 1 *(one),* + *(addition),* − *(subtraction),* · *(multiplication),* / *(division),* ^ *(exponentiation),* ! *(factorial),* (*(left parenthesis),*) *(right parenthesis).*

Let t denote the largest twin prime that is smaller than $((((((((9!)!)!)!)!)!)!)!)!$. The number t is an unknown element of the set of twin primes.

Lemma 1. *(Wilson's theorem,* [2, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite, see [10]–[12]. Let $[\cdot]$ denote the integer part function. By Lemma 1, for every positive integer n,

$$1+1+\left(\left(n^2\right)!-\left(n^2+1\right)\cdot\left[\frac{\left(n^2\right)!}{n^2+1}\right]\right)\cdot\frac{n^2-1}{n^2}=\begin{cases} n^2+1, & if\ n^2+1\ is\ prime \\ 2, & otherwise \end{cases}$$

Similar identities are unknown for algebraic expressions considered in Definition 1, so Definition 1 seems to be correct.

Definition 2. *Conditions* `(1)-(5)` *concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
`(1)` *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
`(2)` *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*
`(3)` *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.
`(4)` *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.*
`(5)` *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.*

Condition `(3)` implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions `(1)-(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning.

Definition 3. *Let* $\beta = (((24!)!)!)!$.

Lemma 2. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at *https://wolframalpha.com*. □

Example 1. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies Condition* `(3)`.

Example 2. *The set* $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}]\ is\ odd \\ \emptyset, & otherwise \end{cases}$ *does not satisfy Condition* `(3)` *because we know an algorithm with no input that computes $[\frac{\beta}{\pi}]$. The set of known elements of $\mathcal{X}$ is empty. Hence, Condition* `(5)` *fails for $\mathcal{X}$.*

Example 3. *(*[1], [7], [9, p. 9]*). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & if\ the\ decimal\ expansion\ of\ \pi\ contains\ n\ consecutive\ zeros \\ 0, & otherwise \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

No known algorithm computes the function h.

Example 4. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & if\ the\ continuum\ hypothesis\ holds \\ \emptyset, & otherwise \end{cases}$$

is decidable. This $\mathcal{X}$ satisfies Conditions `(1)` *and* `(3)` *and does not satisfy Conditions* `(2)`*,* `(4)`*, and* `(5)`*. These facts will hold forever.*

Definition 4. *Let Φ denote the following unproven statement:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ. Theorem 6 heuristically justifies the statement Φ. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$.

Statement 1. *Condition* `(1)` *remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm `Alg`$(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition `(1)`, but the algorithm `Alg`$(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. □

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in mathematics understood as an a priori science because it refers to the current mathematical knowledge. The same is true for Open Problems 1–3 and Statements 2–4.

Definition 5. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$.

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies Conditions `(1)-(4)`*. Condition* `(5)` *fails for $\mathcal{X}$.*

Proof. Condition `(4)` holds as $\mathcal{X} \supseteq \{0, \ldots, 10^6\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 2, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(10^6) > f(7) = \beta$, see [5]. Thus Condition `(3)` holds. Condition `(2)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer 10^6 is a threshold number of $\mathcal{X}$. Thus $\mathcal{X}$ satisfies Condition `(1)`. Condition `(5)` fails for $\mathcal{X}$ as the set of known elements of $\mathcal{X}$ equals $\{0, \ldots, 10^6\}$. □

Let $j = (((((((((9!)!)!)!)!)!)!)!)!$. For $n \in \mathbb{N} \setminus \{0\}$, let S_n denote $\sum\limits_{k=1}^{n} \frac{1}{k^{1+(1/j)}}$.

Lemma 3. *A known algorithm takes as inputs $m \in \mathbb{N}$ and $n \in \mathbb{N} \setminus \{0\}$ and returns the logical value of the inequality $m \leqslant S_n$. It implies that a known algorithm takes as input $n \in \mathbb{N} \setminus \{0\}$ and returns $[S_n]$.*

Proof. For every positive integer n, S_n is a positive and algebraic number. Now the claims follow from the decidability of real closed fields. □

Let ζ denote Riemann's ζ function.

Lemma 4. $\sum\limits_{k=1}^{\infty} \frac{1}{k^{1+(1/j)}} = \zeta(1+(1/j)) \approx j + 0.6$.

Lemma 5. *The function $\mathbb{N} \ni n \longrightarrow n - \left[\sqrt{n}\right]^2 \in \mathbb{N}$ takes every non-negative integer value infinitely often.*

Statement 3. *Conditions* `(1)-(4)` *hold for*

$$\mathcal{X} = \left\{n \in \mathbb{N} \setminus \{0\} : \left(\left[\sum_{k=1}^{n} \frac{1}{k^{1+(1/j)}}\right] + n - \left[\sqrt{n}\right]^2\right)^2 + 1 \textit{ is prime}\right\}$$

Proof. Lemma 3 implies Condition `(2)`. By Lemmas 3 and 4, a known algorithm with no input returns the smallest positive integer n for which $\sum\limits_{k=1}^{n} \frac{1}{k^{1+(1/j)}} \geqslant j$. This conclusion and Lemmas 4 and 5 imply that $n-1$ is a threshold number of $\mathcal{X}$. Thus Condition `(1)` holds. Since the statement $\mathcal{P}_{n^2+1} \cap \left[j^2+1, \infty\right) \neq \emptyset$ remains unproven, Condition `(3)` holds. The following *MuPAD* code displays the first 10000 elements of $\mathcal{X}$.

```
protocol("the_first_10000_elements_of_X.txt"):
T:=[]:
n:=1:
k:=0:
repeat
a:=floor(EULER+psi(n+1))+n-(floor(sqrt(n)))^2:
if numlib::proveprime(a^2+1)=TRUE then
T:=append(T,n):
k:=k+1:
end_if:
n:=n+1:
until k=10000
end_repeat:
T;
protocol():
```

The inequality $\mathrm{card}(\mathcal{X}) \geqslant 10000$ and the implication

$$\mathcal{P}_{n^2+1} \cap \left[j^2+1, \infty\right) \neq \emptyset \Longrightarrow \mathrm{card}(\mathcal{X}) = \omega$$

show that Condition `(4)` holds. □

Proving Landau's conjecture will disprove Statements 2 and 3.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions* `(1)-(5)`*?*

It may be that a modification of Statement 3 will solve Open Problem 1.

Theorem 1. *No set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy Conditions* (1)-(4) *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition (1). Since Conditions (1)-(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots$$

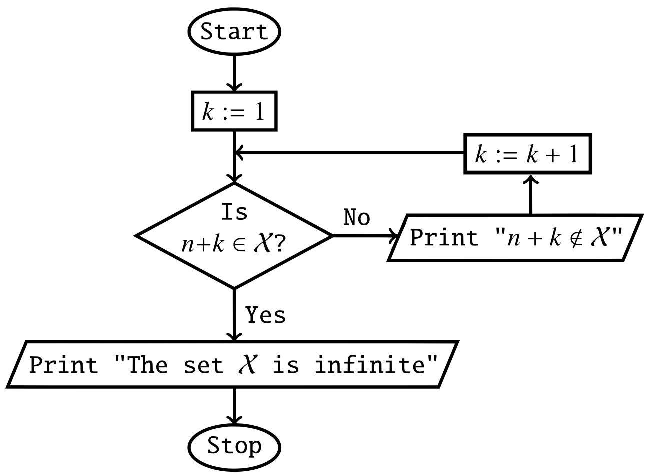

Fig. 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition (4). □

The physical limits of computation ([5]) disprove the assumption of Theorem 1.

3. Number-theoretic statements Ψ_n

Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

Lemma 6. *For every positive integer n, the system* $\mathcal{U}_n$ *has exactly two solutions in positive integers* $x_1, \ldots, x_n$*, namely* $(1, \ldots, 1)$ *and* $(f(1), \ldots, f(n))$*.*

Let B_n denote the following system of equations:

$$\Big\{x_i! = x_k :\ i,k \in \{1,\ldots,n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\ldots,n\}\Big\}$$

For every positive integer n, no known system $\mathcal{S} \subseteq B_n$ with a finite number of solutions in positive integers $x_1,\ldots,x_n$ has a solution $(x_1,\ldots,x_n) \in (\mathbb{N} \setminus \{0\})^n$ satisfying $\max(x_1,\ldots,x_n) > f(n)$. For every positive integer n and for every known system $\mathcal{S} \subseteq B_n$, if the finiteness/infiniteness of the set

$$\{(x_1,\ldots,x_n) \in (\mathbb{N} \setminus \{0\})^n :\ (x_1,\ldots,x_n) \textit{ solves } \mathcal{S}\}$$

is unknown, then the statement

$$\exists x_1,\ldots,x_n \in \mathbb{N} \setminus \{0\}\ ((x_1,\ldots,x_n) \textit{ solves } \mathcal{S}) \wedge (\max(x_1,\ldots,x_n) > f(n))$$

remains unproven.

For a positive integer n, let Ψ_n denote the following statement: *if a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 6 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$, then for every integer $i \in \{1,\ldots,n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1,\ldots,x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution of Open Problem 1

Lemma 7. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 7 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

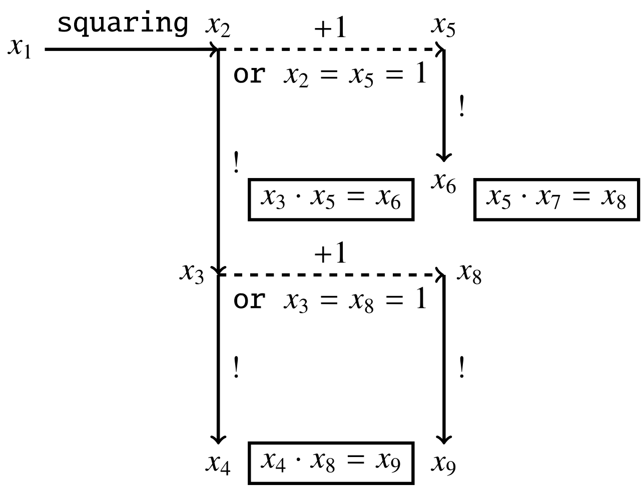

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= \left(x_1^2\right)! \\ x_4 &= \left(\left(x_1^2\right)!\right)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= \left(x_1^2 + 1\right)! \end{aligned} \qquad \begin{aligned} x_7 &= \frac{\left(x_1^2\right)! + 1}{x_1^2 + 1} \\ x_8 &= \left(x_1^2\right)! + 1 \\ x_9 &= \left(\left(x_1^2\right)! + 1\right)! \end{aligned}$$

Proof. By Lemma 7, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 1. □

Lemma 9. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. It is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap (\mathbb{N} \setminus \{0\}) = \{1, 2\}$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system $\mathcal{A}$.*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 8 and 9, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement* Φ.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement* Φ *implies Conjecture 1.*

Proof. By Lemmas 8 and 9, if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 8 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 4. *Conditions* `(2)-(5)` *hold for* $\mathcal{X} = \mathcal{P}_{n^2+1}$. *The statement* Φ *implies that Condition* `(1)` *holds for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [11]. These two facts imply Condition `(4)`. By Lemma 2, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [5]. Thus Condition `(3)` holds. Conditions `(2)` and `(5)` hold trivially. The statement Φ implies that Condition `(1)` holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$ with $n = \beta = (((24!)!)!)!$. □

Proving Landau's conjecture will disprove Statement 4.

Conjecture 2. (*Conditions* `(1)-(5)` *hold for* $\mathcal{X} = \mathcal{P}_{n^2+1}$) $\wedge\ \Phi$.

Conjecture 2 implies that every known proof of the statement Φ does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$.

5. A new heuristic argument for the infiniteness of $\mathcal{P}_{n^2+1}$

The system $\mathcal{A}$ contains four factorials and four multiplications. Let $\mathcal{F}$ denote the family of all systems $\mathcal{S} \subseteq B_9$ which contain at most four factorials and at most four multiplications.

Among known systems $\mathcal{S} \in \mathcal{F}$, the following system C

$$\begin{cases} x_1! = x_2 \\ x_2 \cdot x_9 = x_1 \\ x_2 \cdot x_2 = x_3 \\ x_3 \cdot x_3 = x_4 \\ x_4 \cdot x_4 = x_5 \\ x_5! = x_6 \\ x_6! = x_7 \\ x_7! = x_8 \end{cases}$$

attains the greatest solution in positive integers $x_1, \ldots, x_9$ and has at most finitely many solutions in $(\mathbb{N} \setminus \{0\})^9$. Only the tuples $(1, \ldots, 1)$ and $(2, 2, 4, 16, 256, 256!, (256!)!, ((256!)!)!, 1)$ solve C and belong to $(\mathbb{N} \setminus \{0\})^9$.

For every known system $\mathcal{S} \in \mathcal{F}$, if the finiteness of the set

$$\{(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \ldots, x_9) \textit{ solves } \mathcal{S}\}$$

is unproven and conjectured, then the statement

$$\exists x_1, \ldots, x_9 \in \mathbb{N} \setminus \{0\} \; ((x_1, \ldots, x_9) \textit{ solves } \mathcal{S}) \wedge (\max(x_1, \ldots, x_9) > ((256!)!)!)$$

remains unproven.

Let Γ denote the statement: *if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \leqslant ((256!)!)!$.* The number $46^{512} + 1$ is prime ([6]) and greater than 256!, see also [8, p. 239] for the primality of $150^{2048} + 1$. Hence, the statement Γ is equivalent to the infiniteness of $\mathcal{P}_{n^2+1}$. It heuristically justifies the infiniteness of $\mathcal{P}_{n^2+1}$ in a sophisticated way.

6. Satisfiable conjunctions which consist of Conditions 1–5 and their negations

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)$$

The set $\mathcal{X} = \{0, \ldots, f(7)\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

The set $\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\}, \; otherwise \end{cases}$ satisfies the conjunction

$$(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [4, p. 158] and [8, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [4, p. 159] and [8, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [3, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \textit{ is the sixth prime number of the form } 2^{2^k} + 1\}, \; otherwise \end{cases}$$

satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \textit{ is composite}\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

The following table shows satisfiable conjunctions consisting of Conditions (1)-(5) and their negations.

	(Condition 2) $\wedge$ (Condition 3) $\wedge$ (Condition 4)	(Condition 2) $\wedge$ $\neg$(Condition 3) $\wedge$ (Condition 4)
(Condition 1) $\wedge$ (Condition 5)	Open Problem 1 (conjecturally solved with $\mathcal{X} = \mathcal{P}_{n^2+1}$)	Open Problem 2
(Condition 1) $\wedge$ $\neg$(Condition 5)	$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\}, \ otherwise \end{cases}$
$\neg$(Condition 1) $\wedge$ (Condition 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg$(Condition 1) $\wedge$ $\neg$(Condition 5)	$\mathcal{X} = \{0, \ldots, f(7)\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n\ is \\ the\ sixth\ prime\ number\ of \\ the\ form\ 2^{2^k} + 1\}, \ otherwise \end{cases}$

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