

# The distinction between existing algorithms and constructively defined algorithms inspires theorems and open problems that concern decidable sets $X \subseteq \mathbb{N}$ and cannot be formalized in mathematics understood as an a priori science as they refer to the current knowledge on $X$

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ABSTRACT. Algorithms always terminate. We explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in *ZFC*) and known algorithms (i.e. algorithms whose definition is constructive and currently known). Assuming that the infiniteness of a set  $X \subseteq \mathbb{N}$  is false or unproven, we define which elements of  $X$  are classified as known. No known set  $X \subseteq \mathbb{N}$  satisfies Conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning. (1) A known algorithm with no input returns an integer  $n$  satisfying  $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$ . (2) A known algorithm for every  $k \in \mathbb{N}$  decides whether or not  $k \in X$ . (3) No known algorithm with no input returns the logical value of the statement  $\text{card}(X) = \omega$ . (4) There are many elements of  $X$  and it is conjectured, though so far unproven, that  $X$  is infinite. (5)  $X$  is naturally defined. The infiniteness of  $X$  is false or unproven.  $X$  has the simplest definition among known sets  $Y \subseteq \mathbb{N}$  with the same set of known elements. We define a set  $X \subseteq \mathbb{N}$  which satisfies Conditions (1)–(5) except the requirement that  $X$  is naturally defined. Let  $\mathcal{P}_{n^2+1}$  denote the set of primes of the form  $n^2 + 1$ . Conditions (2)–(5) hold for  $X = \mathcal{P}_{n^2+1}$ . We discuss a conjecture which implies the conjunction of Conditions (1)–(5) for  $X = \mathcal{P}_{n^2+1}$ . No set  $X \subseteq \mathbb{N}$  will satisfy Conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. We present a table that shows satisfiable conjunctions consisting of Conditions (1)–(5) and their negations.

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**Key words and phrases:** conjecturally infinite set  $X \subseteq \mathbb{N}$ , constructively defined integer  $n$  satisfies  $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$ , current knowledge on a set  $X \subseteq \mathbb{N}$ , distinction between existing algorithms and constructively defined algorithms which are currently known, known elements of a set  $X \subseteq \mathbb{N}$  whose infiniteness is false or unproven, physical limits of computation, primes of the form  $n^2 + 1$ ,  $X$  is decidable by a constructively defined algorithm.

## 1. DEFINITIONS AND THE DISTINCTION BETWEEN EXISTING ALGORITHMS AND CONSTRUCTIVELY DEFINED ALGORITHMS WHICH ARE CURRENTLY KNOWN

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known). A definition of an integer  $n$  is called *constructive*, if it provides a known algorithm with no input that returns  $n$ . Definition 1 applies to sets  $X \subseteq \mathbb{N}$  whose infiniteness is false or unproven.

**Definition 1.** We say that a non-negative integer  $k$  is a known element of  $\mathcal{X}$ , if  $k \in \mathcal{X}$  and we know an algebraic expression that defines  $k$  and consists of the following signs: 1 (one), + (addition), - (subtraction),  $\cdot$  (multiplication),  $^{\wedge}$  (exponentiation with exponent in  $\mathbb{N}$ ), ! (factorial of a non-negative integer), ( (left parenthesis), ) (right parenthesis).

Let  $t$  denote the largest twin prime that is smaller than (((((((9!)!)!)!)!)!)!). The number  $t$  is an unknown element of the set of twin primes.

**Lemma 1.** (Wilson's theorem, [4, p. 89]). For every integer  $x \geq 2$ ,  $x$  is prime if and only if  $x$  divides  $(x - 1)! + 1$ .

Edmund Landau's conjecture states that the set  $\mathcal{P}_{n^2+1}$  of primes of the form  $n^2 + 1$  is infinite, see [14]–[16]. Let  $[\cdot]$  denote the integer part function.

**Definition 2.** Conditions (1)–(5) concern sets  $\mathcal{X} \subseteq \mathbb{N}$ .

- (1) A known algorithm with no input returns an integer  $n$  satisfying  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .
- (2) A known algorithm for every  $k \in \mathbb{N}$  decides whether or not  $k \in \mathcal{X}$ .
- (3) No known algorithm with no input returns the logical value of the statement  $\text{card}(\mathcal{X}) = \omega$ .
- (4) There are many elements of  $\mathcal{X}$  and it is conjectured, though so far unproven, that  $\mathcal{X}$  is infinite.
- (5)  $\mathcal{X}$  is naturally defined. The infiniteness of  $\mathcal{X}$  is false or unproven.  $\mathcal{X}$  has the simplest definition among known sets  $\mathcal{Y} \subseteq \mathbb{N}$  with the same set of known elements.

Condition (3) implies that no known proof shows the finiteness/infiniteness of  $\mathcal{X}$ . No known set  $\mathcal{X} \subseteq \mathbb{N}$  satisfies Conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Let  $\beta = (((24!)!)!)!$ .

**Lemma 2.**  $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$ .

*Proof.* We ask Wolfram Alpha at <https://wolframalpha.com>. □

**Example 1.** The set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies Condition (3).

**Example 2.** The set  $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases}$  does not satisfy Condition (3) because we know an algorithm with no input that computes  $[\frac{\beta}{\pi}]$ . The set of known elements of  $\mathcal{X}$  is empty. Hence, Condition (5) fails for  $\mathcal{X}$ .

**Example 3.** ([1], [11], [13, p. 9]). The function

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because  $h = \mathbb{N} \times \{1\}$  or there exists  $k \in \mathbb{N}$  such that

$$h = (\{0, \dots, k\} \times \{1\}) \cup (\{k+1, k+2, k+3, \dots\} \times \{0\})$$

No known algorithm computes the function  $h$ .

**Example 4.** The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds} \\ \emptyset, & \text{otherwise} \end{cases}$$

is decidable. This  $\mathcal{X}$  satisfies Conditions (1) and (3) and does not satisfy Conditions (2), (4), and (5). These facts will hold forever.

Let  $\Phi$  denote the following unproven statement:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement  $\Phi$ . Theorem 6 heuristically justifies the statement  $\Phi$ . This justification does not yield the finiteness/infiniteness of  $\mathcal{P}_{n^2+1}$ .

**Statement 1.** *Condition (1) remains unproven for  $\mathcal{X} = \mathcal{P}_{n^2+1}$ .*

*Proof.* For every set  $\mathcal{X} \subseteq \mathbb{N}$ , there exists an algorithm  $\text{Alg}(\mathcal{X})$  with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This  $n$  satisfies the implication in Condition (1), but the algorithm  $\text{Alg}(\mathcal{P}_{n^2+1})$  is unknown because its definition is ineffective.  $\square$

Proving the statement  $\Phi$  will disprove Statement 1. Statement 1 cannot be formalized in mathematics understood as an a priori science because it refers to the current mathematical knowledge. The same is true for Open Problems 1–5 and Statements 2–5.

**Definition 3.** *We say that an integer  $n$  is a threshold number of a set  $\mathcal{X} \subseteq \mathbb{N}$ , if  $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ .*

If a set  $\mathcal{X} \subseteq \mathbb{N}$  is empty or infinite, then any integer  $n$  is a threshold number of  $\mathcal{X}$ . If a set  $\mathcal{X} \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $\mathcal{X}$  form the set  $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$ .

## 2. THE PHYSICAL LIMITS OF COMPUTATION INSPIRE OPEN PROBLEM 1

Let  $f(1) = 2$ ,  $f(2) = 4$ , and let  $f(n + 1) = f(n)!$  for every integer  $n \geq 2$ .

**Statement 2.** *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

*satisfies Conditions (1)–(4). Condition (5) fails for  $\mathcal{X}$ .*

*Proof.* Condition (4) holds as  $\mathcal{X} \supseteq \{0, \dots, 10^6\}$  and the set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. By Lemma 2, due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $f(10^6) > f(7) = \beta$ , see [7]. Thus Condition (3) holds. Condition (2) holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer  $10^6$  is a threshold number of  $\mathcal{X}$ . Thus  $\mathcal{X}$  satisfies Condition (1). Condition (5) fails for  $\mathcal{X}$  as the set of known elements of  $\mathcal{X}$  equals  $\{0, \dots, 10^6\}$ .  $\square$

For a non-negative integer  $n$ , let  $\theta(n)$  denote the largest integer divisor of  $10^{10^{10}}$  smaller than  $n$ . For a non-negative integer  $n$ , let  $\theta_1(n)$  denote the largest integer divisor of  $10^{10}$  smaller than  $n$ .

**Lemma 3.** *For every integer  $j > 10^{10^{10}}$ ,  $\theta(j) = 10^{10^{10}}$ .*

**Lemma 4.** *For every integer  $j \in (6553600, 7812500]$ ,  $\theta(j) = 6553600$ .*

*Proof.* 6553600 equals  $2^{18} \cdot 5^2$  and divides  $10^{10^{10}}$ .  $7812500 < 2^{24}$ .  $7812500 < 5^{10}$ .

We need to prove that every integer  $j \in (6553600, 7812500)$  does not divide  $10^{10^{10}}$ . It holds as the set

$$\{2^u \cdot 5^v : (u \in \{0, \dots, 23\}) \wedge (v \in \{0, \dots, 9\})\}$$

contains 6553600 and 7812500 as consecutive elements.  $\square$

**Lemma 5.** *The number  $6553600^2 + 1$  is prime.*

*Proof.* The following PARI/GP ([8]) command

```
isprime(6553600^2+1, {flag=2})
```

returns 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([17, p. 226]). It rigorously shows that the number  $6553600^2 + 1$  is prime.  $\square$

In the next lemmas, the execution of the command `isprime(n, {flag=2})` proves the primality of  $n$ .

**Lemma 6.** *The number  $10142101504^2 + 1$  is prime.  $10142101504 > 10^{10}$ .*

**Lemma 7.** *The function*

$$\mathbb{N} \ni n \xrightarrow{\kappa} \underbrace{\text{the\_exponent\_of\_2\_in\_the\_prime\_factorization\_of\_}n+1}_{\in \mathbb{N}} \in \mathbb{N}$$

*takes every non-negative integer value infinitely often.*

Before Open Problem 1,  $\mathcal{X}$  denotes the set  $\{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \text{ is prime}\}$ .

**Lemma 8.** *The set  $\mathcal{X}$  satisfies  $\text{card}(\mathcal{X}) \geq 629450$ .*

*Proof.* By Lemmas 4 and 5, for every even integer  $j \in (6553600, 7812500]$ , the number  $(\theta(j) + \kappa(j))^2 + 1 = (6553600 + 0)^2 + 1$  is prime. Hence,

$$\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500] \subseteq \mathcal{X}$$

Consequently,

$$\text{card}(\mathcal{X}) \geq \text{card}(\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500]) = \frac{7812500 - 6553600}{2} = 629450$$

$\square$

**Lemma 9.**  $10242 \in \mathcal{X}$ .  $10242 \notin \mathcal{X}_1 = \{n \in \mathbb{N} : (\theta_1(n) + \kappa(n))^2 + 1 \text{ is prime}\}$ .

*Proof.* The number  $10240 = 2^{11} \cdot 5$  divides  $10^{10^{10}}$ . Hence,  $\theta(10242) = 10240$ . The number  $(\theta(10242) + \kappa(10242))^2 + 1 = (10240 + 0)^2 + 1$  is prime. The set

$$\{2^u \cdot 5^v : (u \in \{0, \dots, 10\}) \wedge (v \in \{0, \dots, 10\})\}$$

contains 10000 and 12500 as consecutive elements. Hence,  $\theta_1(10242) = 10000$ . The number  $(\theta_1(10242) + \kappa(10242))^2 + 1 = (10000 + 0)^2 + 1 = 17 \cdot 5882353$  is composite.  $\square$

**Statement 3.** *The set  $\mathcal{X}$  satisfies Conditions (1)–(5) except the requirement that  $\mathcal{X}$  is naturally defined.*

*Proof.* Condition (2) holds trivially. Let  $\delta$  denote  $10^{10^{10}}$ . By Lemmas 3 and 7, Condition (1) holds for  $n = \delta$ . Since the statement  $\mathcal{P}_{n^2+1} \cap (\delta^2 + 1, \infty) \neq \emptyset$  remains unproven, Condition (3) holds. Lemma 8 and the implication

$$\mathcal{P}_{n^2+1} \cap (\delta^2 + 1, \infty) \neq \emptyset \implies \text{card}(\mathcal{X}) = \omega$$

show that Condition (4) holds. By Lemma 6, the set  $\mathcal{X}_1$  is infinite. Since Definition 1 applies to sets  $\mathcal{X} \subseteq \mathbb{N}$  whose infiniteness is false or unproven, Condition 5 holds except the requirement that  $\mathcal{X}$  is naturally defined.  $\square$

The set  $\mathcal{X}$  satisfies Condition (5) except the requirement that  $\mathcal{X}$  is naturally defined. It is true because  $\mathcal{X}_1$  is infinite and Definition 1 applies only to sets  $\mathcal{X} \subseteq \mathbb{N}$  whose infiniteness is false or unproven. Ignoring this restriction,  $\mathcal{X}$  still satisfies the same identical condition due to Lemma 9.

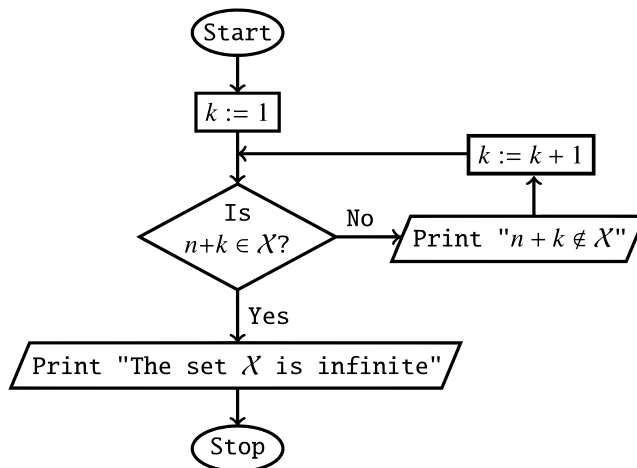
**Open Problem 1.** *Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  which satisfies Conditions (1)–(5)?*

The development of mathematics can change negative answers to Open Problems 1–5.

**Theorem 1.** *No set  $\mathcal{X} \subseteq \mathbb{N}$  will satisfy Conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.*

*Proof.* The proof goes by contradiction. We fix an integer  $n$  that satisfies Condition (1). Since Conditions (1)–(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T)  $n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots$



**Fig. 1** Semi-algorithm that terminates if and only if  $\mathcal{X}$  is infinite

The sentences from the sequence (T) and our assumption imply that for every integer  $m > n$  computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that  $(n, m] \cap \mathcal{X} = \emptyset$ . Thus, at some future day, numerical evidence will support the conjecture that the set  $\mathcal{X}$  is finite, contrary to the conjecture in Condition (4).  $\square$

The physical limits of computation ([7]) disprove the assumption of Theorem 1.

### 3. NUMBER-THEORETIC STATEMENTS $\Psi_n$

Let  $\mathcal{U}_1$  denote the system of equations which consists of the equation  $x_1! = x_1$ . For an integer  $n \geq 2$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

**Lemma 10.** *For every positive integer  $n$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers  $x_1, \dots, x_n$ , namely  $(1, \dots, 1)$  and  $(f(1), \dots, f(n))$ .*

Let  $B_n$  denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For every positive integer  $n$ , no known system  $\mathcal{S} \subseteq B_n$  with a finite number of solutions in positive integers  $x_1, \dots, x_n$  has a solution  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$  satisfying  $\max(x_1, \dots, x_n) > f(n)$ . For every positive integer  $n$  and for every known system  $\mathcal{S} \subseteq B_n$ , if the finiteness/infiniteness of the set

$$\{(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n : (x_1, \dots, x_n) \text{ solves } \mathcal{S}\}$$

is unknown, then the statement

$$\exists x_1, \dots, x_n \in \mathbb{N} \setminus \{0\} ((x_1, \dots, x_n) \text{ solves } \mathcal{S}) \wedge (\max(x_1, \dots, x_n) > f(n))$$

remains unproven.

For a positive integer  $n$ , let  $\Psi_n$  denote the following statement: *if a system  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq f(n)$ .* The statement  $\Psi_n$  says that for subsystems of  $B_n$  with a finite number of solutions, the largest known solution is indeed the largest possible. The statements  $\Psi_1$  and  $\Psi_2$  hold trivially. There is no reason to assume the validity of the statement  $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$ .

**Theorem 2.** *For every statement  $\Psi_n$ , the bound  $f(n)$  cannot be decreased.*

*Proof.* It follows from Lemma 10 because  $\mathcal{U}_n \subseteq B_n$ . □

**Theorem 3.** *For every integer  $n \geq 2$ , the statement  $\Psi_{n+1}$  implies the statement  $\Psi_n$ .*

*Proof.* If a system  $\mathcal{S} \subseteq B_n$  has at most finitely many solutions in positive integers  $x_1, \dots, x_n$ , then for every integer  $i \in \{1, \dots, n\}$  the system  $\mathcal{S} \cup \{x_i! = x_{n+1}\}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_{n+1}$ . The statement  $\Psi_{n+1}$  implies that  $x_i! = x_{n+1} \leq f(n+1) = f(n)!$ . Hence,  $x_i \leq f(n)$ . □

**Theorem 4.** *Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .*

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

4. A CONJECTURAL SOLUTION OF OPEN PROBLEM 1

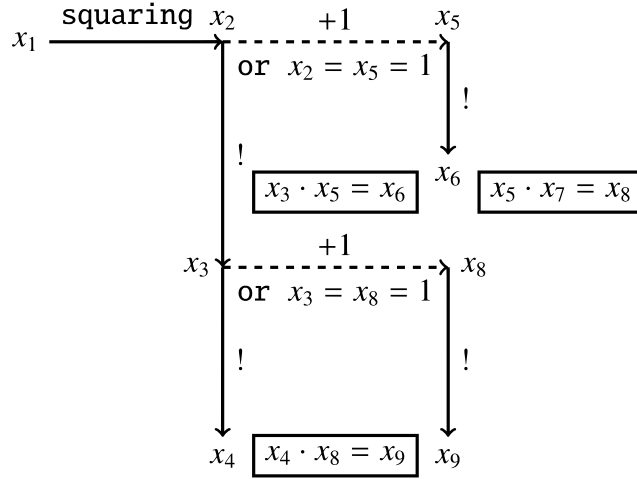
**Lemma 11.** For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Let  $\mathcal{A}$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 11 and the diagram in Figure 2 explain the construction of the system  $\mathcal{A}$ .



**Fig. 2** Construction of the system  $\mathcal{A}$

**Lemma 12.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 11, for every integer  $x_1 \geq 2$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 12 follows from Lemma 1. □

**Lemma 13.** *There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ , which solve the system  $\mathcal{A}$  and satisfy  $x_1 = 1$ . It is true as every such tuple  $(x_1, \dots, x_9)$  satisfies  $x_1, \dots, x_9 \in \{1, 2\}$ .*

*Proof.* The equality  $x_1 = 1$  implies that  $x_2 = x_1 \cdot x_1 = 1$ . Hence,  $x_3 = x_2! = 1$ . Therefore,  $x_4 = x_3! = 1$ . The equalities  $x_5! = x_6$  and  $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$  imply that  $x_5, x_6 \in \{1, 2\}$ . The equalities  $x_8! = x_9$  and  $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$  imply that  $x_8, x_9 \in \{1, 2\}$ . The equality  $x_5 \cdot x_7 = x_8$  implies that  $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap (\mathbb{N} \setminus \{0\}) = \{1, 2\}$ .  $\square$

**Conjecture 1.** *The statement  $\Psi_9$  is true when is restricted to the system  $\mathcal{A}$ .*

**Theorem 5.** *Conjecture 1 proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $f(7)$ , then the set  $\mathcal{P}_{n^2+1}$  is infinite.*

*Proof.* Suppose that the antecedent holds. By Lemma 12, there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{A}$ . Since  $x_1^2 + 1 > f(7)$ , we obtain that  $x_1^2 \geq f(7)$ . Hence,  $(x_1^2)! \geq f(7)! = f(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality  $x_9 > f(9)$  imply that the system  $\mathcal{A}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas 12 and 13, the set  $\mathcal{P}_{n^2+1}$  is infinite.  $\square$

**Theorem 6.** *Conjecture 1 implies the statement  $\Phi$ .*

*Proof.* It follows from Theorem 5 and the equality  $f(7) = (((24!)!)!)!$ .  $\square$

**Theorem 7.** *The statement  $\Phi$  implies Conjecture 1.*

*Proof.* By Lemmas 12 and 13, if positive integers  $x_1, \dots, x_9$  solve the system  $\mathcal{A}$ , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or  $x_1, \dots, x_9 \in \{1, 2\}$ . In the first case, Lemma 12 and the statement  $\Phi$  imply that the inequality  $x_5 \leq (((24!)!)!)! = f(7)$  holds when the system  $\mathcal{A}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_9$ . Hence,  $x_2 = x_5 - 1 < f(7)$  and  $x_3 = x_2! < f(7)! = f(8)$ . Continuing this reasoning in the same manner, we can show that every  $x_i$  does not exceed  $f(9)$ .  $\square$

**Statement 4.** *Conditions (2)-(5) hold for  $\mathcal{X} = \mathcal{P}_{n^2+1}$ . The statement  $\Phi$  implies that Condition (1) holds for  $\mathcal{X} = \mathcal{P}_{n^2+1}$ .*

*Proof.* The set  $\mathcal{P}_{n^2+1}$  is conjecturally infinite. There are 2199894223892 primes of the form  $n^2 + 1$  in the interval  $[2, 10^{28})$ , see [15]. These two facts imply Condition (4). By Lemma 2, due to known physics we are not able to confirm by a direct computation that some element of  $\mathcal{P}_{n^2+1}$  is greater than  $f(7) = (((24!)!)!)! = \beta$ , see [7]. Thus Condition (3) holds. Conditions (2) and (5) hold trivially. The statement  $\Phi$  implies that Condition (1) holds for  $\mathcal{X} = \mathcal{P}_{n^2+1}$  with  $n = \beta = (((24!)!)!)!$ .  $\square$

Proving Landau's conjecture will disprove Statement 4.

**Conjecture 2.** *(Conditions (1)-(5) hold for  $\mathcal{X} = \mathcal{P}_{n^2+1}) \wedge \Phi$ .*

Conjecture 2 implies that every known proof of the statement  $\Phi$  does not yield the finiteness/infiniteness of  $\mathcal{P}_{n^2+1}$ .



5. A NEW HEURISTIC ARGUMENT FOR THE INFINITENESS OF  $\mathcal{P}_{n^2+1}$

The system  $\mathcal{A}$  contains four factorials and four multiplications. Let  $\mathcal{F}$  denote the family of all systems  $\mathcal{S} \subseteq B_9$  which contain at most four factorials and at most four multiplications.

Among known systems  $\mathcal{S} \in \mathcal{F}$ , the following system  $\mathcal{C}$

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2 \cdot x_9 = x_1 \\ x_2 \cdot x_2 = x_3 \\ x_3 \cdot x_3 = x_4 \\ x_4 \cdot x_4 = x_5 \\ x_5! = x_6 \\ x_6! = x_7 \\ x_7! = x_8 \end{array} \right.$$

attains the greatest solution in positive integers  $x_1, \dots, x_9$  and has at most finitely many solutions in  $(\mathbb{N} \setminus \{0\})^9$ . Only the tuples  $(1, \dots, 1)$  and  $(2, 2, 4, 16, 256, 256!, (256!)!, ((256!)!)!, 1)$  solve  $\mathcal{C}$  and belong to  $(\mathbb{N} \setminus \{0\})^9$ .

For every known system  $\mathcal{S} \in \mathcal{F}$ , if the finiteness of the set

$$\{(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \dots, x_9) \text{ solves } \mathcal{S}\}$$

is unproven and conjectured, then the statement

$$\exists x_1, \dots, x_9 \in \mathbb{N} \setminus \{0\} ((x_1, \dots, x_9) \text{ solves } \mathcal{S}) \wedge (\max(x_1, \dots, x_9) > ((256!)!)!)$$

remains unproven.

Let  $\Gamma$  denote the statement: *if the system  $\mathcal{A}$  has at most finitely many solutions in positive integers  $x_1, \dots, x_9$ , then each such solution  $(x_1, \dots, x_9)$  satisfies  $x_1, \dots, x_9 \leq ((256!)!)!$ . The number  $46^{512} + 1$  is prime ([9]) and greater than  $256!$ , see also [12, p. 239] for the primality of  $150^{2048} + 1$ . Hence, the statement  $\Gamma$  is equivalent to the infiniteness of  $\mathcal{P}_{n^2+1}$ . It heuristically justifies the infiniteness of  $\mathcal{P}_{n^2+1}$  in a sophisticated way.*

6. SATISFIABLE CONJUNCTIONS WHICH CONSIST OF CONDITIONS 1-5 AND THEIR NEGATIONS

The set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})$$

The set  $\mathcal{X} = \{0, \dots, f(7)\} \cup \mathcal{P}_{n^2+1}$  satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

The set  $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\}, & \text{otherwise} \end{cases}$  satisfies the conjunction

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

**Open Problem 2.** *Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies the conjunction*

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})?$$

The numbers  $2^{2^k} + 1$  are prime for  $k \in \{0, 1, 2, 3, 4\}$ . It is open whether or not there are infinitely many primes of the form  $2^{2^k} + 1$ , see [6, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form  $2^{2^k} + 1$ , see [6, p. 159] and [12, p. 74]. Most mathematicians believe that  $2^{2^k} + 1$  is composite for every integer  $k \geq 5$ , see [5, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$$

satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

**Open Problem 3.** *Is there a set  $\mathcal{X} \subseteq \mathbb{N}$  that satisfies the conjunction*

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})?$$

It is possible, although very doubtful, that at some future day, the set  $\mathcal{X} = \mathcal{P}_{n^2+1}$  will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set  $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite}\}$  will solve Open Problem 1. The same is true for Open Problems 2 and 3.

The following table shows satisfiable conjunctions consisting of Conditions (1)–(5) and their negations.

	(Condition 2) $\wedge$ (Condition 3) $\wedge$ (Condition 4)	(Condition 2) $\wedge$ $\neg$ (Condition 3) $\wedge$ (Condition 4)
(Condition 1) $\wedge$ (Condition 5)	Open Problem 1 (conjecturally solved with $\mathcal{X} = \mathcal{P}_{n^2+1}$ )	Open Problem 2
(Condition 1) $\wedge$ $\neg$ (Condition 5)	$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\}, & \text{otherwise} \end{cases}$
$\neg$ (Condition 1) $\wedge$ (Condition 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg$ (Condition 1) $\wedge$ $\neg$ (Condition 5)	$\mathcal{X} = \{0, \dots, f(7)\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$

## 7. FINAL REMARKS

Statements 1–4 and Open Problems 1–3 cannot be formalized in mathematics understood as an a priori science. Previously known statements of this type, such as Statement 5, express the current knowledge on particular elements of  $\mathbb{N}$ , which are known to us according to Definition 1. Previously known open problems of this type, such as Open Problems 4 and 5, ask about constructive existence of special elements of  $\mathbb{N}$ .

**Statement 5.** ([2], [3], [6, p. 209], [10]). *The numbers  $2^{2^{22}} + 1$  and  $2^{2^{24}} + 1$  are composite. The known integer divisors of  $2^{2^{22}} + 1$  form the set  $\{-2^{2^{22}} - 1, -1, 1, 2^{2^{22}} + 1\}$ . The known integer divisors of  $2^{2^{24}} + 1$  form the set  $\{-2^{2^{24}} - 1, -1, 1, 2^{2^{24}} + 1\}$ .*

**Open Problem 4.** *Is there a known prime number greater than  $10^{10^{10}}$ ?*

**Open Problem 5.** *Is there a known threshold number of  $\mathcal{P}_{n^2+1}$ ?*

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