

The distinction between constructively defined algorithms and algorithms whose existence is provable in *ZFC* inspires theorems and open problems that concern decidable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in mathematics understood as an a priori science as they refer to the current knowledge on \mathcal{X}

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Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite. We present a new heuristic argument for the infiniteness of \mathcal{P}_{n^2+1} . Landau's conjecture implies the following unproven statement Φ : $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\{x_i! = x_k : i, k \in \{1, \dots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, 9\}\}$. We write some system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers x_1, \dots, x_9 , namely $(1, \dots, 1)$ and $(f(1), \dots, f(9))$. No known system $S \subseteq B$ with a finite number of solutions in positive integers x_1, \dots, x_9 has a solution $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ satisfying $\max(x_1, \dots, x_9) > f(9)$. For every known system $S \subseteq B$, if the finiteness/infiniteness of the set $\{(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \dots, x_9) \text{ solves } S\}$ is unknown, then the statement $\exists x_1, \dots, x_9 \in \mathbb{N} \setminus \{0\} ((x_1, \dots, x_9) \text{ solves } S) \wedge (\max(x_1, \dots, x_9) > f(9))$ remains unproven. We write some system $\mathcal{A} \subseteq B$ of 8 equations. Let Λ denote the statement: *if the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 , then each such solution (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \leq f(9)$* . The statement Λ is equivalent to the statement Φ . It heuristically proves the statement Φ . This proof does not yield the finiteness/infiniteness of \mathcal{P}_{n^2+1} . Algorithms always terminate. We explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in *ZFC*) and known algorithms (i.e. algorithms whose definition is constructive and currently known). Assuming that the infiniteness of a set $\mathcal{X} \subseteq \mathbb{N}$ is false or unproven, we define which elements of \mathcal{X} are classified as known. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning. (1) *A known algorithm with no input returns an integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.* (2) *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* (3) *No known algorithm with no input returns the logical value of the statement $\text{card}(\mathcal{X}) = \omega$.* (4) *There are many elements of \mathcal{X} and it is conjectured, though so far unproven, that \mathcal{X} is infinite.* (5) *\mathcal{X} is naturally defined. The infiniteness of \mathcal{X} is false or unproven. \mathcal{X} has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.* Conditions (2)–(5) hold for $\mathcal{X} = \mathcal{P}_{n^2+1}$. The statement Φ implies the conjunction of Conditions (1)–(5) for $\mathcal{X} = \mathcal{P}_{n^2+1}$. We define a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions (1)–(5) except the requirement that \mathcal{X} is naturally defined. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption. We present a table that shows satisfiable conjunctions consisting of Conditions (1)–(5) and their negations.

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1. DEFINITIONS AND THE DISTINCTION BETWEEN EXISTING ALGORITHMS AND CONSTRUCTIVELY
DEFINED ALGORITHMS WHICH ARE CURRENTLY KNOWN

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n . Definition 1 applies to sets $X \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. We say that a non-negative integer k is a known element of X , if $k \in X$ and we know an algebraic expression that defines k and consists of the following signs: 1 (one), + (addition), − (subtraction), · (multiplication), ^ (exponentiation with exponent in \mathbb{N}), ! (factorial of a non-negative integer), ((left parenthesis),) (right parenthesis).

Let t denote the largest twin prime that is smaller than (((((((9!)!)!)!)!)!)!). The number t is an unknown element of the set of twin primes.

Definition 2. Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.

- (1) A known algorithm with no input returns an integer n satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.
- (2) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in X$.
- (3) No known algorithm with no input returns the logical value of the statement $\text{card}(X) = \omega$.
- (4) There are many elements of X and it is conjectured, though so far unproven, that X is infinite.
- (5) X is naturally defined. The infiniteness of X is false or unproven. X has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.

Condition (3) implies that no known proof shows the finiteness/infiniteness of X . No known set $X \subseteq \mathbb{N}$ satisfies Conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Edmund Landau’s conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [14]–[16]. Let $[\cdot]$ denote the integer part function. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. □

Example 1. The set $X = \mathcal{P}_{n^2+1}$ satisfies Condition (3).

Example 2. The set $X = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases}$ does not satisfy Condition (3) because we know an algorithm with no input that computes $[\frac{\beta}{\pi}]$. The set of known elements of X is empty. Hence, Condition (5) fails for X .

Example 3. ([1], [11], [13, p. 9]). The function

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \dots, k\} \times \{1\}) \cup (\{k + 1, k + 2, k + 3, \dots\} \times \{0\})$$

No known algorithm computes the function h .

Example 4. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds} \\ \emptyset, & \text{otherwise} \end{cases}$$

is decidable. This \mathcal{X} satisfies Conditions (1) and (3) and does not satisfy Conditions (2), (4), and (5). These facts will hold forever.

Let Φ denote the following unproven statement:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ . Theorem 6 heuristically justifies the statement Φ . This justification does not yield the finiteness/infiniteness of \mathcal{P}_{n^2+1} .

Statement 1. *Condition (1) remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $X \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(X)$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(X) \in \{0, \omega\} \\ \max(X), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition (1), but the algorithm $\text{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. \square

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in mathematics understood as an a priori science because it refers to the current mathematical knowledge. The same is true for Open Problems 1–5 and Statements 2–5.

Definition 3. *We say that an integer n is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.*

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of X . If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $[\max(X), \infty) \cap \mathbb{N}$.

2. THE PHYSICAL LIMITS OF COMPUTATION INSPIRE OPEN PROBLEM 1

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$.

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies Conditions (1)–(4). Condition (5) fails for \mathcal{X} .

Proof. Condition (4) holds as $\mathcal{X} \supseteq \{0, \dots, 10^6\}$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than $f(10^6) > f(7) = \beta$, see [7]. Thus Condition (3) holds. Condition (2) holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer 10^6 is a threshold number of \mathcal{X} . Thus \mathcal{X} satisfies Condition (1). Condition (5) fails for \mathcal{X} as the set of known elements of \mathcal{X} equals $\{0, \dots, 10^6\}$. \square

For a non-negative integer n , let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n . For a non-negative integer n , let $\theta_1(n)$ denote the largest integer divisor of 10^{10} smaller than n .

Lemma 2. For every integer $j > 10^{10^{10}}$, $\theta(j) = 10^{10^{10}}$.

Lemma 3. For every integer $j \in (6553600, 7812500]$, $\theta(j) = 6553600$.

Proof. 6553600 equals $2^{18} \cdot 5^2$ and divides $10^{10^{10}}$. $7812500 < 2^{24}$. $7812500 < 5^{10}$. We need to prove that every integer $j \in (6553600, 7812500)$ does not divide $10^{10^{10}}$. It holds as the set

$$\{2^u \cdot 5^v : (u \in \{0, \dots, 23\}) \wedge (v \in \{0, \dots, 9\})\}$$

contains 6553600 and 7812500 as consecutive elements. \square

Lemma 4. The number $6553600^2 + 1$ is prime.

Proof. The following PARI/GP ([8]) command

```
isprime(6553600^2+1,{flag=2})
```

returns 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([17, p. 226]). It rigorously shows that the number $6553600^2 + 1$ is prime. \square

In the next lemmas, the execution of the command `isprime(n,{flag=2})` proves the primality of n .

Lemma 5. The number $10142101504^2 + 1$ is prime. $10142101504 > 10^{10}$.

Lemma 6. The function

$$\mathbb{N} \ni n \xrightarrow{\kappa} \underbrace{\text{the_exponent_of_2_in_the_prime_factorization_of_}n+1}_{\in \mathbb{N}} \in \mathbb{N}$$

takes every non-negative integer value infinitely often.

Before Open Problem 1, \mathcal{X} denotes the set $\{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \text{ is prime}\}$.

Lemma 7. The set \mathcal{X} satisfies $\text{card}(\mathcal{X}) \geq 629450$.

Proof. By Lemmas 3 and 4, for every even integer $j \in (6553600, 7812500]$, the number $(\theta(j) + \kappa(j))^2 + 1 = (6553600 + 0)^2 + 1$ is prime. Hence,

$$\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500] \subseteq \mathcal{X}$$

Consequently,

$$\text{card}(\mathcal{X}) \geq \text{card}(\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500]) = \frac{7812500 - 6553600}{2} = 629450$$

\square

Lemma 8. $10242 \in \mathcal{X}$. $10242 \notin \mathcal{X}_1 = \{n \in \mathbb{N} : (\theta_1(n) + \kappa(n))^2 + 1 \text{ is prime}\}$.

Proof. The number $10240 = 2^{11} \cdot 5$ divides $10^{10^{10}}$. Hence, $\theta(10242) = 10240$. The number $(\theta(10242) + \kappa(10242))^2 + 1 = (10240 + 0)^2 + 1$ is prime. The set

$$\{2^u \cdot 5^v : (u \in \{0, \dots, 10\}) \wedge (v \in \{0, \dots, 10\})\}$$

contains 10000 and 12500 as consecutive elements. Hence, $\theta_1(10242) = 10000$. The number $(\theta_1(10242) + \kappa(10242))^2 + 1 = (10000 + 0)^2 + 1 = 17 \cdot 5882353$ is composite. \square

Statement 3. The set \mathcal{X} satisfies Conditions (1)–(5) except the requirement that \mathcal{X} is naturally defined.

Proof. Condition (2) holds trivially. Let δ denote $10^{10^{10}}$. By Lemmas 2 and 6, Condition (1) holds for $n = \delta$. Since the statement $\mathcal{P}_{n^2+1} \cap (\delta^2 + 1, \infty) \neq \emptyset$ remains unproven, Condition (3) holds. Lemma 7 and the implication

$$\mathcal{P}_{n^2+1} \cap (\delta^2 + 1, \infty) \neq \emptyset \implies \text{card}(\mathcal{X}) = \omega$$

show that Condition (4) holds. By Lemma 5, the set \mathcal{X}_1 is infinite. Since Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, Condition 5 holds except the requirement that \mathcal{X} is naturally defined. \square

The set \mathcal{X} satisfies Condition (5) except the requirement that \mathcal{X} is naturally defined. It is true because \mathcal{X}_1 is infinite and Definition 1 applies only to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven. Ignoring this restriction, \mathcal{X} still satisfies the same identical condition due to Lemma 8.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions (1)–(5)?*

The answers to Open Problems 1–5 may change in time as they depend on the current mathematical knowledge. These answers are currently negative.

Theorem 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)–(4) forever; if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition (1). Since Conditions (1)–(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots$

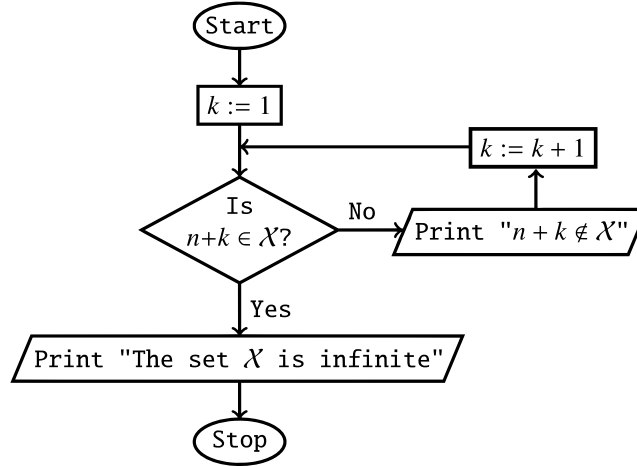


Fig. 1 Semi-algorithm that terminates if and only if \mathcal{X} is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set \mathcal{X} is finite, contrary to the conjecture in Condition (4). \square

The physical limits of computation ([7]) disprove the assumption of Theorem 1.

3. NUMBER-THEORETIC STATEMENTS Ψ_n

Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

Lemma 9. *For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers x_1, \dots, x_n , namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.*

Let B_n denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For every positive integer n , no known system $\mathcal{S} \subseteq B_n$ with a finite number of solutions in positive integers x_1, \dots, x_n has a solution $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ satisfying $\max(x_1, \dots, x_n) > f(n)$. For every positive integer n and for every known system $\mathcal{S} \subseteq B_n$, if the finiteness/infiniteness of the set

$$\{(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n : (x_1, \dots, x_n) \text{ solves } \mathcal{S}\}$$

is unknown, then the statement

$$\exists x_1, \dots, x_n \in \mathbb{N} \setminus \{0\} ((x_1, \dots, x_n) \text{ solves } \mathcal{S}) \wedge (\max(x_1, \dots, x_n) > f(n))$$

remains unproven.

For a positive integer n , let Ψ_n denote the following statement: *if a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

Theorem 2. *For every statement Ψ_n , the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 9 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geq 2$, the statement Ψ_{n+1} implies the statement Ψ_n .*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then for every integer $i \in \{1, \dots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers x_1, \dots, x_{n+1} . The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)!$. Hence, $x_i \leq f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

4. A CONJECTURAL SOLUTION OF OPEN PROBLEM 1

Lemma 10. *For every positive integers x and y , $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 10 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

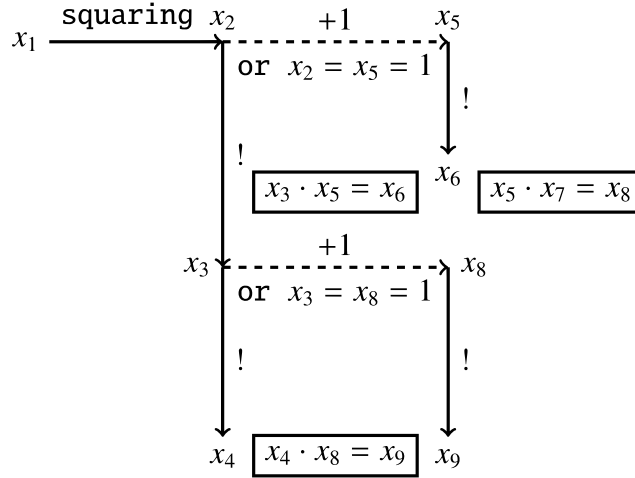


Fig. 2 Construction of the system \mathcal{A}

Lemma 11. (Wilson's theorem, [4, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x-1)! + 1$.

Lemma 12. For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 10, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 12 follows from Lemma 11. \square

Lemma 13. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$. It is true as every such tuple (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap (\mathbb{N} \setminus \{0\}) = \{1, 2\}$. \square

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system \mathcal{A} .*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 12, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 12 and 13, the set \mathcal{P}_{n^2+1} is infinite. \square

Theorem 6. *Conjecture 1 implies the statement Φ .*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. \square

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 12 and 13, if positive integers x_1, \dots, x_9 solve the system \mathcal{A} , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 12 and the statement Φ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 . Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. \square

Statement 4. *Conditions (2)-(5) hold for $\mathcal{X} = \mathcal{P}_{n^2+1}$. The statement Φ implies that Condition (1) holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. The set \mathcal{P}_{n^2+1} is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [15]. These two facts imply Condition (4). By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than $f(7) = (((24!)!)!)! = \beta$, see [7]. Thus Condition (3) holds. Conditions (2) and (5) hold trivially. The statement Φ implies that Condition (1) holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$ with $n = \beta = (((24!)!)!)!$. \square

Proving Landau's conjecture will disprove Statement 4.

Conjecture 2. *(Conditions (1)-(5) hold for $\mathcal{X} = \mathcal{P}_{n^2+1}) \wedge \Phi$.*

Conjecture 2 implies that every known proof of the statement Φ does not yield the finiteness/infiniteness of \mathcal{P}_{n^2+1} .

5. SATISFIABLE CONJUNCTIONS WHICH CONSIST OF CONDITIONS 1-5 AND THEIR NEGATIONS

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})$$

The set $\mathcal{X} = \{0, \dots, f(7)\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

The set $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\}, & \text{otherwise} \end{cases}$ satisfies the conjunction

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [6, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [6, p. 159] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geq 5$, see [5, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$$

satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite}\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

The following table shows satisfiable conjunctions consisting of Conditions (1)-(5) and their negations.

	$(\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4})$	$(\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4})$
$(\text{Condition 1}) \wedge (\text{Condition 5})$	Open Problem 1 (conjecturally solved with $\mathcal{X} = \mathcal{P}_{n^2+1}$)	Open Problem 2
$(\text{Condition 1}) \wedge \neg(\text{Condition 5})$	$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\}, & \text{otherwise} \end{cases}$
$\neg(\text{Condition 1}) \wedge (\text{Condition 5})$	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg(\text{Condition 1}) \wedge \neg(\text{Condition 5})$	$\mathcal{X} = \{0, \dots, f(7)\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \dots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$

6. PREVIOUSLY KNOWN RESULTS OF A SIMILAR TYPE

Statements 1–4 and Open Problems 1–3 cannot be formalized in mathematics understood as an a priori science. Previously known statements of this type, such as Statement 5, express the current knowledge on particular elements of \mathbb{N} , which are known to us according to Definition 1. Previously known open problems of this type, such as Open Problems 4 and 5, ask about constructive existence of special elements of \mathbb{N} .

Statement 5. ([2], [3], [6, p. 209], [10]). *The numbers $2^{2^{22}} + 1$ and $2^{2^{24}} + 1$ are composite. The known integer divisors of $2^{2^{22}} + 1$ form the set $\left\{-2^{2^{22}} - 1, -1, 1, 2^{2^{22}} + 1\right\}$. The known integer divisors of $2^{2^{24}} + 1$ form the set $\left\{-2^{2^{24}} - 1, -1, 1, 2^{2^{24}} + 1\right\}$.*

Open Problem 4. *Is there a known prime number greater than $10^{10^{10^{10}}}$?*

Open Problem 5. *Is there a known threshold number of \mathcal{P}_{n^2+1} ?*

7. A NEW HEURISTIC ARGUMENT FOR THE INFINITENESS OF \mathcal{P}_{n^2+1}

The system \mathcal{A} contains four factorials and four multiplications. Let \mathcal{F} denote the family of all systems $\mathcal{S} \subseteq B_9$ which contain at most four factorials and at most four multiplications.

Among known systems $\mathcal{S} \in \mathcal{F}$, the following system C

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2 \cdot x_9 = x_1 \\ x_2 \cdot x_2 = x_3 \\ x_3 \cdot x_3 = x_4 \\ x_4 \cdot x_4 = x_5 \\ x_5! = x_6 \\ x_6! = x_7 \\ x_7! = x_8 \end{array} \right.$$

attains the greatest solution in positive integers x_1, \dots, x_9 and has at most finitely many solutions in $(\mathbb{N} \setminus \{0\})^9$. Only the tuples $(1, \dots, 1)$ and $(2, 2, 4, 16, 256, 256!, (256!)!, ((256!)!)!, 1)$ solve C and belong to $(\mathbb{N} \setminus \{0\})^9$.

For every known system $\mathcal{S} \in \mathcal{F}$, if the finiteness of the set

$$\{(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \dots, x_9) \text{ solves } \mathcal{S}\}$$

is unproven and conjectured, then the statement

$$\exists x_1, \dots, x_9 \in \mathbb{N} \setminus \{0\} ((x_1, \dots, x_9) \text{ solves } \mathcal{S}) \wedge (\max(x_1, \dots, x_9) > ((256!)!)!)$$

remains unproven.

Let Γ denote the statement: *if the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 , then each such solution (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \leq ((256!)!)!$. The number $46^{512} + 1$ is prime ([9]) and greater than $256!$, see also [12, p. 239] for the primality of $150^{2048} + 1$. Hence, the statement Γ is equivalent to the infiniteness of \mathcal{P}_{n^2+1} . It heuristically justifies the infiniteness of \mathcal{P}_{n^2+1} in a sophisticated way.*

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REFERENCES

- [1] J. Case and M. Ralston, *Beyond Rogers' non-constructively computable function*, in: *The nature of computation*, Lecture Notes in Comput. Sci., 7921, 45–54, Springer, Heidelberg, 2013, http://link.springer.com/chapter/10.1007/978-3-642-39053-1_6.
- [2] R. Crandall, J. Doenias, C. Norrie, J. Young, *The twenty-second Fermat number is composite*, Math. Comp. 64 (1995), 863–868.
- [3] R. Crandall, E. Mayer, J. Papadopoulos, *The twenty-fourth Fermat number is composite*, Math. Comp. 72 (2003), 1555–1572.
- [4] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory*, 2nd ed., CRC Press, Boca Raton, FL, 2016.
- [5] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers*, American Mathematical Society, Providence, RI, 2012.
- [6] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry*, Springer, New York, 2001.
- [7] S. Lloyd, *Ultimate physical limits to computation*, Nature 406 (2000), 1047–1054, <http://doi.org/10.1038/35023282>.
- [8] PARI/GP online documentation, http://pari.math.u-bordeaux.fr/dochtm/html/Arithmetic_functions.html.
- [9] X. M. Pi, *Searching for generalized Fermat primes* (Chinese), J. Math. (Wuhan) 18 (1998), no. 3, 276–280.
- [10] *Proth Search Page*, <http://www.prothsearch.com/fermat.html#Complete>.
- [11] R. Reitzig, *How can it be decidable whether π has some sequence of digits?*, <http://cs.stackexchange.com/questions/367/how-can-it-be-decidable-whether-pi-has-some-sequence-of-digits>.
- [12] P. Ribenboim, *The little book of bigger primes*, 2nd ed., Springer-Verlag, New York, 2004.
- [13] H. Rogers, Jr., *Theory of recursive functions and effective computability*, 2nd ed., MIT Press, Cambridge, MA, 1987.
- [14] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A002496, *Primes of the form $n^2 + 1$* , <http://oeis.org/A002496>.
- [15] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A083844, *Number of primes of the form $x^2 + 1 < 10^n$* , <http://oeis.org/A083844>.
- [16] Wolfram MathWorld, *Landau's Problems*, <http://mathworld.wolfram.com/LandausProblems.html>.
- [17] S. Y. Yan, *Number theory for computing*, 2nd ed., Springer, Berlin, 2002.

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