The distinction between constructively defined algorithms and algorithms whose existence is provable in ZFC inspires theorems and open problems that concern decidable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in mathematics understood as an a priori science as they refer to the current knowledge on $\mathcal{X}$

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Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite. We present a new heuristic argument for the infiniteness of $\mathcal{P}_{n^2+1}$. Landau's conjecture implies the following unproven statement Φ: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let B denote the system of equations: $\{x_i! = x_k : i,k \in \{1,\ldots,9\}\} \cup \{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,9\}\}$. We write some system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1,\ldots,x_9$, namely $(1,\ldots,1)$ and $(f(1),\ldots,f(9))$. No known system $\mathcal{S} \subseteq B$ with a finite number of solutions in positive integers $x_1,\ldots,x_9$ has a solution $(x_1,\ldots,x_9) \in (\mathbb{N} \setminus \{0\})^9$ satisfying $\max(x_1,\ldots,x_9) > f(9)$. For every known system $\mathcal{S} \subseteq B$, if the finiteness/infiniteness of the set $\{(x_1,\ldots,x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1,\ldots,x_9)$ *solves* $\mathcal{S}\}$ is unknown, then the statement $\exists x_1,\ldots,x_9 \in \mathbb{N} \setminus \{0\}$ $((x_1,\ldots,x_9)$ *solves* $\mathcal{S}) \wedge (\max(x_1,\ldots,x_9) > f(9))$ remains unproven. We write some system $\mathcal{A} \subseteq B$ of 8 equations. Let Λ denote the statement: *if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1,\ldots,x_9$, then each such solution $(x_1,\ldots,x_9)$ satisfies $x_1,\ldots,x_9 \leqslant f(9)$*. The statement Λ is equivalent to the statement Φ. It heuristically justifies the statement Φ. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$. Algorithms always terminate. We explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known). Assuming that the infiniteness of a set $\mathcal{X} \subseteq \mathbb{N}$ is false or unproven, we define which elements of $\mathcal{X}$ are classified as known. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions `(1)-(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning. `(1)` *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(2)` *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.* `(3)` *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$. `(4)` *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.* `(5)` *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.* Conditions `(2)-(5)` hold for $\mathcal{X} = \mathcal{P}_{n^2+1}$. The statement Φ implies the conjunction of Conditions `(1)-(5)` for $\mathcal{X} = \mathcal{P}_{n^2+1}$. We define a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions `(1)-(5)` except the requirement that $\mathcal{X}$ is naturally defined. We present a table that shows satisfiable conjunctions of the form #(Condition `1`) $\wedge$ (Condition `2`) $\wedge$ #(Condition `3`) $\wedge$ (Condition `4`) $\wedge$ #(Condition `5`), where # denotes the negation $\neg$ or its absence. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions `(1)-(4)` forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption.

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1. Definitions and the distinction between existing algorithms and constructively defined algorithms which are currently known

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in ZFC) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known). A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. *We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs:* 1 *(one),* + *(addition),* $-$ *(subtraction),* $\cdot$ *(multiplication),* ^ *(exponentiation with exponent in $\mathbb{N}$),* ! *(factorial of a non-negative integer),* (*(left parenthesis),*) *(right parenthesis).*

Let t denote the largest twin prime that is smaller than $((((((((9!)!)!)!)!)!)!)!)!$. The number t is an unknown element of the set of twin primes.

Definition 2. *Conditions* `(1)-(5)` *concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
`(1)` *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
`(2)` *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*
`(3)` *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.
`(4)` *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.*
`(5)` *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.*

Condition `(3)` implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions `(1)-(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning.

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite, see [14]–[16]. Let $[\cdot]$ denote the integer part function. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at *http://wolframalpha.com*. □

Example 1. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies Condition* `(3)`.

Example 2. *The set* $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{\beta}{\pi}] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$ *does not satisfy Condition* `(3)` *because we know an algorithm with no input that computes $[\frac{\beta}{\pi}]$. The set of known elements of $\mathcal{X}$ is empty. Hence, Condition* `(5)` *fails for $\mathcal{X}$.*

Example 3. *(*[1], [11], [13, p. 9]*). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & if\ the\ decimal\ expansion\ of\ \pi\ contains\ n\ consecutive\ zeros \\ 0, & otherwise \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

No known algorithm computes the function h.

Example 4. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & if\ the\ continuum\ hypothesis\ holds \\ \emptyset, & otherwise \end{cases}$$

is decidable. This $\mathcal{X}$ satisfies Conditions `(1)` *and* `(3)` *and does not satisfy Conditions* `(2)`, `(4)`*, and* `(5)`*. These facts will hold forever.*

Let Φ denote the following unproven statement:

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau's conjecture implies the statement Φ. Theorem 6 heuristically justifies the statement Φ. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$.

Statement 1. *Condition* `(1)` *remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\texttt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition `(1)`, but the algorithm $\texttt{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. □

Proving the statement Φ will disprove Statement 1. Statement 1 cannot be formalized in mathematics understood as an a priori science because it refers to the current mathematical knowledge. The same is true for Open Problems 1–5 and Statements 2–5.

Definition 3. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$.

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies Conditions `(1)-(4)`*. Condition* `(5)` *fails for $\mathcal{X}$.*

Proof. Condition `(4)` holds as $\mathcal{X} \supseteq \{0, \dots, 10^6\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(10^6) > f(7) = \beta$, see [7]. Thus Condition `(3)` holds. Condition `(2)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer 10^6 is a threshold number of $\mathcal{X}$. Thus $\mathcal{X}$ satisfies Condition `(1)`. Condition `(5)` fails for $\mathcal{X}$ as the set of known elements of $\mathcal{X}$ equals $\{0, \dots, 10^6\}$. □

For a non-negative integer n, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n. For a non-negative integer n, let $\theta_1(n)$ denote the largest integer divisor of 10^{10} smaller than n.

Lemma 2. *For every integer* $j > 10^{10^{10}}$, $\theta(j) = 10^{10^{10}}$.

Lemma 3. *For every integer* $j \in (6553600, 7812500]$, $\theta(j) = 6553600$.

Proof. 6553600 equals $2^{18} \cdot 5^2$ and divides $10^{10^{10}}$. $7812500 < 2^{24}$. $7812500 < 5^{10}$. We need to prove that every integer $j \in (6553600, 7812500)$ does not divide $10^{10^{10}}$. It holds as the set

$$\left\{2^u \cdot 5^v : (u \in \{0, \ldots, 23\}) \wedge (v \in \{0, \ldots, 9\})\right\}$$

contains 6553600 and 7812500 as consecutive elements. □

Lemma 4. *The number* $6553600^2 + 1$ *is prime.*

Proof. The following `PARI/GP` ([8]) command

```
isprime(6553600^2+1,{flag=2})
```

returns 1. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([17, p. 226]). It rigorously shows that the number $6553600^2 + 1$ is prime. □

In the next lemmas, the execution of the command `isprime(`n`,{flag=2})` proves the primality of n.

Lemma 5. *The number* $10142101504^2 + 1$ *is prime.* $10142101504 > 10^{10}$.

Lemma 6. *The function*

$$\mathbb{N} \ni n \xrightarrow{\kappa} the\_exponent\_of\_2\_in\_the\_prime\_factorization\_of\_\underbrace{n+1} \in \mathbb{N}$$

takes every non-negative integer value infinitely often.

Before Open Problem 1, $\mathcal{X}$ denotes the set $\{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \textit{ is prime}\}$.

Lemma 7. *The set* $\mathcal{X}$ *satisfies* $\mathrm{card}(\mathcal{X}) \geqslant 629450$.

Proof. By Lemmas 3 and 4, for every even integer $j \in (6553600, 7812500]$, the number $(\theta(j) + \kappa(j))^2 + 1 = (6553600 + 0)^2 + 1$ is prime. Hence,

$$\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500] \subseteq \mathcal{X}$$

Consequently,

$$\mathrm{card}(\mathcal{X}) \geqslant \mathrm{card}(\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500]) = \frac{7812500 - 6553600}{2} = 629450$$

□

Lemma 8. $10242 \in \mathcal{X}$. $10242 \notin \mathcal{X}_1 = \{n \in \mathbb{N} : (\theta_1(n) + \kappa(n))^2 + 1 \textit{ is prime}\}$.

Proof. The number $10240 = 2^{11} \cdot 5$ divides $10^{10^{10}}$. Hence, $\theta(10242) = 10240$. The number $(\theta(10242) + \kappa(10242))^2 + 1 = (10240 + 0)^2 + 1$ is prime. The set

$$\left\{2^u \cdot 5^v : (u \in \{0, \ldots, 10\}) \wedge (v \in \{0, \ldots, 10\})\right\}$$

contains 10000 and 12500 as consecutive elements. Hence, $\theta_1(10242) = 10000$. The number $(\theta_1(10242) + \kappa(10242))^2 + 1 = (10000 + 0)^2 + 1 = 17 \cdot 5882353$ is composite. □

Statement 3. *The set* $\mathcal{X}$ *satisfies Conditions* `(1)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.*

Proof. Condition (2) holds trivially. Let δ denote $10^{10^{10}}$. By Lemmas 2 and 6, Condition (1) holds for $n = \delta$. Since the statement $\mathcal{P}_{n^2+1} \cap \left(\delta^2 + 1, \infty\right) \neq \emptyset$ remains unproven, Condition (3) holds. Lemma 7 and the implication

$$\mathcal{P}_{n^2+1} \cap \left(\delta^2 + 1, \infty\right) \neq \emptyset \Longrightarrow \mathrm{card}(\mathcal{X}) = \omega$$

show that Condition (4) holds. By Lemma 5, the set $\mathcal{X}_1$ is infinite. Since Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, Condition 5 holds except the requirement that $\mathcal{X}$ is naturally defined. □

The set $\mathcal{X}$ satisfies Condition (5) except the requirement that $\mathcal{X}$ is naturally defined. It is true because $\mathcal{X}_1$ is infinite and Definition 1 applies only to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven. Ignoring this restriction, $\mathcal{X}$ still satisfies the same identical condition due to Lemma 8.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions* (1)-(5)*?*

The answers to Open Problems 1–5 may change in time as they depend on the current mathematical knowledge. These answers are currently negative.

Theorem 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions* (1)-(4) *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition (1). Since Conditions (1)-(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots$$

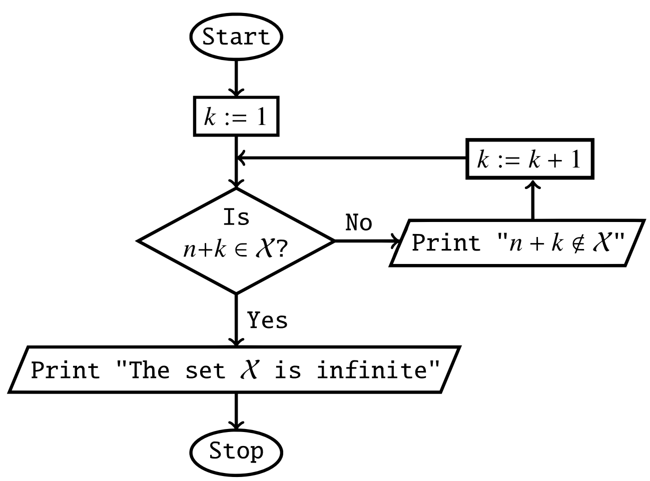

Fig. 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition (4). □

The physical limits of computation ([7]) disprove the assumption of Theorem 1.

3. Number-theoretic statements Ψ_n

Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \qquad\qquad\qquad x_1! &= x_1 \\ \qquad\qquad\quad x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

Lemma 9. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers $x_1, \ldots, x_n$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.*

Let B_n denote the following system of equations:

$$\Big\{x_i! = x_k :\ i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \ldots, n\}\Big\}$$

For every positive integer n, no known system $\mathcal{S} \subseteq B_n$ with a finite number of solutions in positive integers $x_1, \ldots, x_n$ has a solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ satisfying $\max(x_1, \ldots, x_n) > f(n)$. For every positive integer n and for every known system $\mathcal{S} \subseteq B_n$, if the finiteness/infiniteness of the set

$$\{(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n :\ (x_1, \ldots, x_n) \textit{ solves } \mathcal{S}\}$$

is unknown, then the statement

$$\exists x_1, \ldots, x_n \in \mathbb{N} \setminus \{0\}\ ((x_1, \ldots, x_n) \textit{ solves } \mathcal{S}) \wedge (\max(x_1, \ldots, x_n) > f(n))$$

remains unproven.

For a positive integer n, let Ψ_n denote the following statement: *if a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 9 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution of Open Problem 1

Lemma 10. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &=& x_3 \\ x_3! &=& x_4 \\ x_5! &=& x_6 \\ x_8! &=& x_9 \\ x_1 \cdot x_1 &=& x_2 \\ x_3 \cdot x_5 &=& x_6 \\ x_4 \cdot x_8 &=& x_9 \\ x_5 \cdot x_7 &=& x_8 \end{cases}$$

Lemma 10 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

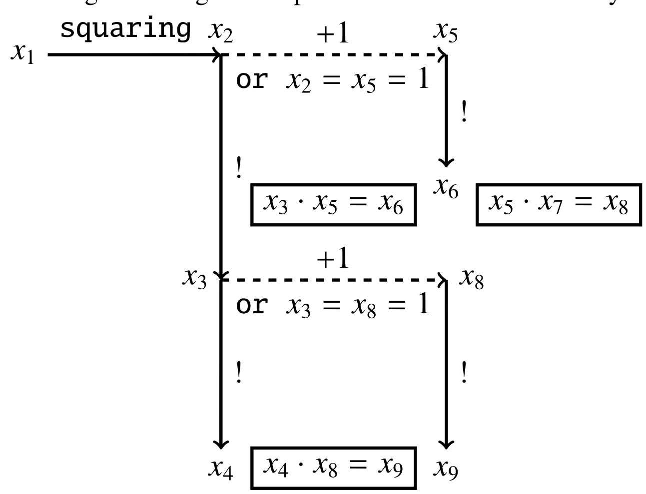

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 11. *(Wilson's theorem,* [4, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Lemma 12. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if x_1^2+1 is prime. In this case, the integers $x_2, \dots, x_9$ are uniquely determined by the following equalities:*

$$\begin{array}{lcl} x_2 &=& x_1^2 \\ x_3 &=& \left(x_1^2\right)! \\ x_4 &=& \left(\left(x_1^2\right)!\right)! \\ x_5 &=& x_1^2+1 \\ x_6 &=& \left(x_1^2+1\right)! \\ x_7 &=& \dfrac{\left(x_1^2\right)!+1}{x_1^2+1} \\ x_8 &=& \left(x_1^2\right)!+1 \\ x_9 &=& \left(\left(x_1^2\right)!+1\right)! \end{array}$$

Proof. By Lemma 10, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if x_1^2+1 divides $(x_1^2)!+1$. Hence, the claim of Lemma 12 follows from Lemma 11. □

Lemma 13. *There are only finitely many tuples* $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$*, which solve the system* $\mathcal{A}$ *and satisfy* $x_1 = 1$*. It is true as every such tuple* $(x_1, \dots, x_9)$ *satisfies* $x_1, \dots, x_9 \in \{1, 2\}$.

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap (\mathbb{N} \setminus \{0\}) = \{1, 2\}$. □

Conjecture 1. *The statement* Ψ_9 *is true when is restricted to the system* $\mathcal{A}$.

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer* $x_1 \geqslant 2$ *such that* $x_1^2 + 1$ *is prime and greater than* $f(7)$*, then the set* $\mathcal{P}_{n^2+1}$ *is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 12, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 12 and 13, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement* Φ.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement* Φ *implies Conjecture 1.*

Proof. By Lemmas 12 and 13, if positive integers $x_1, \dots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 12 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \dots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 4. *Conditions* `(2)`–`(5)` *hold for* $\mathcal{X} = \mathcal{P}_{n^2+1}$*. The statement* Φ *implies that Condition* `(1)` *holds for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [15]. These two facts imply Condition `(4)`. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [7]. Thus Condition `(3)` holds. Conditions `(2)` and `(5)` hold trivially. The statement Φ implies that Condition `(1)` holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$ with $n = \beta = (((24!)!)!)!$. □

Proving Landau's conjecture will disprove Statement 4.

Conjecture 2. (*Conditions* `(1)`–`(5)` *hold for* $\mathcal{X} = \mathcal{P}_{n^2+1}$) $\wedge\ \Phi$.

Conjecture 2 implies that every known proof of the statement Φ does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$.

5. Satisfiable conjunctions which consist of Conditions 1–5 and their negations

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)$$

The set $\mathcal{X} = \{0, \ldots, f(7)\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

The set $\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\}, \ otherwise \end{cases}$ satisfies the conjunction

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [6, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [6, p. 159] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [5, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n\ is\ the\ sixth\ prime\ number\ of\ the\ form\ 2^{2^k} + 1\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1\ is\ composite\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

The following table shows satisfiable conjunctions of the form #(Condition 1) ∧ (Condition 2) ∧ #(Condition 3) ∧ (Condition 4) ∧ #(Condition 5), where # denotes the negation ¬ or its absence.

	(Condition 2) ∧ (Condition 3) ∧ (Condition 4)	(Condition 2) ∧ ¬(Condition 3) ∧ (Condition 4)
(Condition 1) ∧ (Condition 5)	Open Problem 1 (conjecturally solved with $\mathcal{X} = \mathcal{P}_{n^2+1}$)	Open Problem 2
(Condition 1) ∧ ¬(Condition 5)	$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\}, \ otherwise \end{cases}$
¬(Condition 1) ∧ (Condition 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
¬(Condition 1) ∧ ¬(Condition 5)	$\mathcal{X} = \{0, \ldots, f(7)\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n\ is\ the\ sixth\ prime\ number\ of\ the\ form\ 2^{2^k} + 1\}, \ otherwise \end{cases}$

6. Previously known results of a similar type

Statements 1–4 and Open Problems 1–3 cannot be formalized in mathematics understood as an a priori science. Previously known statements of this type, such as Statement 5, express the current knowledge on particular elements of $\mathbb{N}$, which are known to us according to Definition 1. Previously known open problems of this type, such as Open Problems 4 and 5, ask about constructive existence of special elements of $\mathbb{N}$.

Statement 5. *(*[2], [3], [6, p. 209], [10]*). The numbers* $2^{2^{22}}+1$ *and* $2^{2^{24}}+1$ *are composite. The known integer divisors of* $2^{2^{22}}+1$ *form the set* $\left\{-2^{2^{22}}-1,-1,1,2^{2^{22}}+1\right\}$*. The known integer divisors of* $2^{2^{24}}+1$ *form the set* $\left\{-2^{2^{24}}-1,-1,1,2^{2^{24}}+1\right\}$.

Open Problem 4. *Is there a known prime number greater than* $10^{10^{10^{10}}}$*?*

Open Problem 5. *Is there a known threshold number of* $\mathcal{P}_{n^2+1}$*?*

7. A new heuristic argument for the infiniteness of $\mathcal{P}_{n^2+1}$

The system $\mathcal{A}$ contains four factorials and four multiplications. Let $\mathcal{F}$ denote the family of all systems $\mathcal{S} \subseteq B_9$ which contain at most four factorials and at most four multiplications.

Among known systems $\mathcal{S} \in \mathcal{F}$, the following system C

$$\begin{cases} x_1! &= x_2 \\ x_2 \cdot x_9 &= x_1 \\ x_2 \cdot x_2 &= x_3 \\ x_3 \cdot x_3 &= x_4 \\ x_4 \cdot x_4 &= x_5 \\ x_5! &= x_6 \\ x_6! &= x_7 \\ x_7! &= x_8 \end{cases}$$

attains the greatest solution in positive integers $x_1,\ldots,x_9$ and has at most finitely many solutions in $(\mathbb{N}\setminus\{0\})^9$. Only the tuples $(1,\ldots,1)$ and $(2,2,4,16,256,256!,(256!)!,((256!)!)!,1)$ solve C and belong to $(\mathbb{N}\setminus\{0\})^9$.

For every known system $\mathcal{S} \in \mathcal{F}$, if the finiteness of the set

$$\{(x_1,\ldots,x_9) \in (\mathbb{N}\setminus\{0\})^9 : \ (x_1,\ldots,x_9) \textit{ solves } \mathcal{S}\}$$

is unproven and conjectured, then the statement

$$\exists x_1,\ldots,x_9 \in \mathbb{N}\setminus\{0\}\ ((x_1,\ldots,x_9) \textit{ solves } \mathcal{S}) \wedge (\max(x_1,\ldots,x_9) > ((256!)!)!)$$

remains unproven.

Let Γ denote the statement: *if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1,\ldots,x_9$, then each such solution $(x_1,\ldots,x_9)$ satisfies $x_1,\ldots,x_9 \leqslant ((256!)!)!$.* The number $46^{512}+1$ is prime ([9]) and greater than $256!$, see also [12, p. 239] for the primality of $150^{2048}+1$. Hence, the statement Γ is equivalent to the infiniteness of $\mathcal{P}_{n^2+1}$. It heuristically justifies the infiniteness of $\mathcal{P}_{n^2+1}$ in a sophisticated way.

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