STATEMENTS AND OPEN PROBLEMS ON DECIDABLE SETS $X \subseteq \mathbb{N}$ THAT CONTAIN INFORMAL NOTIONS AND REFER TO THE CURRENT KNOWLEDGE ON $X$

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Abstract. Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$: $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, ((24!)!)!]$. We heuristically justify the statement $\Phi$. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$. We present a new heuristic argument for the infiniteness of $\mathcal{P}_{n^2+1}$, which is not based on the statement $\Phi$. The distinction between algorithms whose existence is provable in $\text{ZFC}$ and constructively defined algorithms which are currently known inspires statements and open problems on decidable sets $X \subseteq \mathbb{N}$ that contain informal notions and refer to the current knowledge on $X$.

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Key words and phrases: conjecturally infinite sets $X \subseteq \mathbb{N}$; constructively defined integers $n$ satisfies $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$: known elements of a set $X \subseteq \mathbb{N}$ whose infiniteness is false or unproven; mathematical definitions, statements and open problems with epistemic and informal notions; primes of the form $n^2 + 1$; primes of the form $n! + 1$; $X$ is decidable by a constructively defined algorithm which is currently known.

Sections 1–8 contain purely mathematical results. Nicolas D. Goodman observed that epistemic notions increase the scope of mathematics, see [5]. The article [5] does not discuss the notion of the current mathematical knowledge. This notion and some informal notions occur in Sections 4–8.

1. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{U}_1$ denote the system of equations $\{x_1! = x_1\}$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$
\begin{align*}
{x_1!} &= x_1 \\
{x_1 \cdot x_1} &= x_2 \\
\forall i \in \{2, \ldots, n-1\}, \; x_i! &= x_{i+1}
\end{align*}
$$

Lemma 1. For every positive integer $n$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers $x_1, \ldots, x_n$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.

Let $B_n$ denote the following system of equations:

$${\{x_i! = x_k : \; j, k \in \{1, \ldots, n\}\}} \cup \{x_i \cdot x_j = x_k : \; i, j, k \in \{1, \ldots, n\}\}$$
For every positive integer $n$, no known system $S \subseteq B_n$ with a finite number of solutions in positive integers $x_1, \ldots, x_n$ has a solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ satisfying $\max(x_1, \ldots, x_n) > f(n)$. For every positive integer $n$ and for every known system $S \subseteq B_n$, if the finiteness/infiniteness of the set

$$\{(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n : (x_1, \ldots, x_n) \text{ solves } S\}$$

is unknown, then the statement

$$\exists x_1, \ldots, x_n \in \mathbb{N} \setminus \{0\} ((x_1, \ldots, x_n) \text{ solves } S) \land (\max(x_1, \ldots, x_n) > f(n))$$

remains unproven.

For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n) \in S$ satisfies $x_1, \ldots, x_n \leq f(n)$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible.

The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Psi_n$.

**Theorem 1.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

**Proof.** It follows from Lemma 1 because $U_n \subseteq B_n$. $\square$

**Theorem 2.** For every integer $n \geq 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.

**Proof.** If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement $\Psi_{n+1}$ implies that $x_i = x_{n+1} \leq f(n + 1) = f(n)!$. Hence, $x_i \leq f(n)$. $\square$

**Theorem 3.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $B_n$ has a finite number of subsystems. $\square$

2. A special case of the statement $\Psi_9$ applies to Edmund Landau’s conjecture that the set $P_{n+1}$ of primes of the form $n^2 + 1$ is infinite

**Lemma 2.** For every positive integers $x$ and $y$, $x! \cdot y! = y! \cdot x!$ if and only if $(x + 1 = y) \lor (x = y = 1)$

Let $A$ denote the following system of equations:

$$\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}$$

Lemma 2 and the diagram in Figure 1 explain the construction of the system $A$. 
Lemma 3. (Wilson’s theorem, [4, p. 89]). For every integer \( x \geq 2 \), \( x \) is prime if and only if \( x \) divides \( (x - 1)! + 1 \).

Lemma 4. For every integer \( x_1 \geq 2 \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) is prime. In this case, the integers \( x_2, \ldots, x_9 \) are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2))! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= (x_1^2)! + 1 \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= (x_1^2)! + 1 
\end{align*}
\]

Proof. By Lemma 3, for every integer \( x_1 \geq 2 \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) divides \( (x_1^2)! + 1 \). Hence, the claim of Lemma 4 follows from Lemma 3.

Lemma 5. There are only finitely many tuples \( (x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 \), which solve the system \( \mathcal{A} \) and satisfy \( x_1 = 1 \). It is true as every such tuple \( (x_1, \ldots, x_9) \) satisfies \( x_1, \ldots, x_9 \in \{1, 2\} \).

Proof. The equality \( x_1 = 1 \) implies that \( x_2 = x_1 \cdot x_1 = 1 \). Hence, \( x_3 = x_2! = 1 \). Therefore, \( x_4 = x_3! = 1 \). The equalities \( x_5 = x_6 \) and \( x_5 = 1 \cdot x_6 = x_3 \cdot x_5 = x_6 \) imply that \( x_5, x_6 \in \{1, 2\} \). The equalities \( x_7 = x_8 \) and \( x_7 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9 \) imply that \( x_8, x_9 \in \{1, 2\} \). The equality \( x_5 \cdot x_7 = x_8 \) implies that \( x_7 = \frac{x_8}{x_5} \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \cap (\mathbb{N} \setminus \{0\}) = \{1, 2\} \).

Conjecture 1. The statement \( \Psi_9 \) is true when is restricted to the system \( \mathcal{A} \).
Edmund Landau’s conjecture states that the set $\mathcal{P}_{p^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [14], [16], and [17].

**Theorem 4.** Conjecture[7] proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{p^2+1}$ is infinite.

**Proof.** Suppose that the antecedent holds. By Lemma 4 there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! > (f(8) + 1)! > f(8)! = f(9)$$

Conjecture[1] and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 4 and 5 the set $\mathcal{P}_{p^2+1}$ is infinite. 

\[ \square \]

Landau’s conjecture implies the following unproven statement $\Phi$:

$$\text{card}(\mathcal{P}_{p^2+1}) < \omega \Rightarrow \mathcal{P}_{p^2+1} \subseteq [2,((24!)!)!]$$

Theorem 5 heuristically justifies the statement $\Phi$. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{p^2+1}$.

**Theorem 5.** Conjecture[7] implies the statement $\Phi$.

**Proof.** It follows from Theorem 4 and the equality $f(7) = ((24!)!)!$. 

\[ \square \]

**Theorem 6.** The statement $\Phi$ implies Conjecture 7

**Proof.** By Lemmas 4 and 5 if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 4 and the statement $\Phi$ imply that the inequality $x_5 \leq ((24!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_5 = x_5 - 1 < f(7)$ and $x_5 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every $x_i$ does not exceed $f(9)$.

\[ \square \]

3. A new heuristic argument for the infiniteness of $\mathcal{P}_{p^2+1}$

The system $\mathcal{A}$ contains four factorials and four multiplications. Let $\mathcal{F}$ denote the family of all systems $\mathcal{S} \subseteq \mathcal{B}_9$ which contain at most four factorials and at most four multiplications.

Among known systems $\mathcal{S} \in \mathcal{F}$, the following system $\mathcal{C}$

$$\begin{align*}
x_1! &= x_2 \\
x_2 \cdot x_9 &= x_1 \\
x_2 \cdot x_2 &= x_3 \\
x_3 \cdot x_3 &= x_4 \\
x_4 \cdot x_4 &= x_5 \\
x_5! &= x_6 \\
x_6! &= x_7 \\
x_7! &= x_8 \\
\end{align*}$$

attains the greatest solution in positive integers $x_1, \ldots, x_9$ and has at most finitely many solutions in $(\mathbb{N} \setminus \{0\})^9$. Only the tuples $(1, \ldots, 1)$ and $(2, 2, 4, 16, 256, 256!, (256!)!, (256!)!, \ldots, 1)$ solve $\mathcal{C}$ and belong to $(\mathbb{N} \setminus \{0\})^9$. 
For every known system \( S \in \mathcal{F} \), if the finiteness of the set
\[ \{(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \ldots, x_9) \text{ solves } S\} \]
is unproven and conjectured, then the statement
\[ \exists x_1, \ldots, x_9 \in \mathbb{N} \setminus \{0\} ((x_1, \ldots, x_9) \text{ solves } S) \wedge (\max(x_1, \ldots, x_9) > ((256!)!)) \]
remains unproven.

Let \( \Gamma \) denote the statement: if the system \( A \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_9 \), then each such solution \( (x_1, \ldots, x_9) \) satisfies \( x_1, \ldots, x_9 \leq ((256!)!) \).

The number \( 46512 + 1 \) is prime ([9]) and greater than \( 256! \), see also [12, p. 239] for the primality of \( 1502048 + 1 \). Hence, the statement \( \Gamma \) is equivalent to the infiniteness of \( P_{n^2+1} \). It heuristically justifies the infiniteness of \( P_{n^2+1} \) in a sophisticated way.

4. THE DISTINCTION BETWEEN ALGORITHMS Whose Existence Is provable in ZFC AND CONSTRUCTIVELY Defined Algorithms Which Are Currently Known

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known). A definition of an integer \( n \) is called constructive, if it provides a known algorithm with no input that returns \( n \). Definition 1 applies to sets \( X \subseteq \mathbb{N} \) whose infiniteness is false or unproven.

**Definition 1.** We say that a non-negative integer \( k \) is a known element of \( X \), if \( k \in X \) and we know an algebraic expression that defines \( k \) and consists of the following signs: 1 (one), + (addition), − (subtraction), · (multiplication), ^ (exponentiation with exponent in \( \mathbb{N} \)), ! (factorial of a non-negative integer), ( (left parenthesis), ) (right parenthesis).

The set of known elements of \( X \) is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let \( t \) denote the largest twin prime that is smaller than \((((((9!)!)!)!)!)!)!\). The number \( t \) is an unknown element of the set of twin primes.

**Definition 2.** Conditions (1)–(5) concern sets \( X \subseteq \mathbb{N} \).

(1) A known algorithm with no input returns an integer \( n \) satisfying \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \).
(2) A known algorithm for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \).
(3) No known algorithm with no input returns the logical value of the statement \( \text{card}(X) = \omega \).
(4) There are many elements of \( X \) and it is conjectured, though so far unproven, that \( X \) is infinite.
(5) \( X \) is naturally defined. The infiniteness of \( X \) is false or unproven. \( X \) has the simplest definition among known sets \( Y \subseteq \mathbb{N} \) with the same set of known elements.

Condition (3) implies that no known proof shows the finiteness/infiniteness of \( X \). No known set \( X \subseteq \mathbb{N} \) satisfies Conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.
Let $\lfloor \cdot \rfloor$ denote the integer part function. Let $\beta = (((24!)!)!)!$.

**Lemma 6.** \( \log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))) \approx 1.42298 \).

**Proof.** We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com) \( \Box \)

**Example 1.** The set \( X = \mathcal{P}_{n^2 + 1} \) satisfies Condition (3).

**Example 2.** The set \( X = \begin{cases} \mathbb{N}, & \text{if } \lfloor \frac{\pi}{2} \rfloor \text{ is odd } \\ 0, & \text{otherwise} \end{cases} \) does not satisfy Condition (3) because we know an algorithm with no input that computes \( \lfloor \frac{\pi}{2} \rfloor \). The set of known elements of \( X \) is empty. Hence, Condition (5) fails for \( X \).

**Example 3.** \([1], [11], [13] \text{ p. 9}\). The function \( \mathbb{N} \ni n \rightarrow \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros } \\ 0, & \text{otherwise} \end{cases} \) is computable because \( h = \mathbb{N} \times \{1\} \) or there exists \( k \in \mathbb{N} \) such that \( h = ([0, \ldots, k] \times \{1\}) \cup ([k + 1, k + 2, k + 3, \ldots] \times \{0\}) \). No known algorithm computes the function \( h \).

**Example 4.** The set \( X = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds } \\ 0, & \text{otherwise} \end{cases} \) is decidable. This \( X \) satisfies Conditions (1) and (3) and does not satisfy Conditions (2), (4), and (5). These facts will hold forever.

**Statement 1.** Condition (1) remains unproven for \( X = \mathcal{P}_{n^2 + 1} \).

**Proof.** For every set \( X \subseteq \mathbb{N} \), there exists an algorithm \( \text{Alg}(X) \) with no input that returns

\[ n = \begin{cases} 0, & \text{if } \text{card}(X) \in \{0, \omega\} \\ \max(X), & \text{otherwise} \end{cases} \]

This \( n \) satisfies the implication in Condition (1), but the algorithm \( \text{Alg}(\mathcal{P}_{n^2 + 1}) \) is unknown because its definition is ineffective. \( \Box \)

**Statement 2.** The statement

\[ \exists n \in \mathbb{N} \ (\text{card}(\mathcal{P}_{n^2 + 1}) < \omega \Rightarrow \mathcal{P}_{n^2 + 1} \subseteq [2, n + 3]) \]

remains unproven in ZFC and classical logic without the law of excluded middle.

Statements \([1],[2]\) refer to the current mathematical knowledge. The same is true for Open Problems \([1],[5]\) and Statements \([3],[4]\).

5. **The physical limits of computation inspire Open Problem 1**

**Definition 3.** We say that an integer \( n \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \). If a set \( X \subseteq \mathbb{N} \) is empty or infinite, then any integer \( n \) is a threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is non-empty and finite, then the all threshold numbers of \( X \) form the set \( [\max(X), \infty) \cap \mathbb{N} \).
Statement 3. The set 
\[ X = \{ k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap P_{n+1} \neq \emptyset \} \]
satisfies Conditions (1)–(4). Condition (5) fails for \( X \).

Proof. Condition (4) holds as \( X \supseteq \{0, \ldots, 10^6\} \) and the set \( P_{n+1} \) is conjecturally infinite. By Lemma 6 due to known physics we are not able to confirm by a direct computation that some element of \( P_{n+1} \) is greater than \( f(10^6) > f(7) = \beta \), see [8]. Thus Condition (3) holds. Since the set 
\[ \{ k \in \mathbb{N} : (10^6 < k) \land (f(10^6), f(k)) \cap P_{n+1} \neq \emptyset \} \]
is empty or infinite, the integer \( 10^6 \) is a threshold number of \( X \). Thus \( X \) satisfies Condition (1). Condition (5) fails for \( X \) as the set of known elements of \( X \) equals \( \{0, \ldots, 10^6\} \).

\[ \square \]

Statement 4 provides a stronger example.

Conjecture 2. ([15]). There are infinitely many primes of the form \( k! + 1 \).

The primality of \( k! + 1 \) is only proven when \( k \in \{0, 1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, 320, 340, 399, 427, 872, 1477, 6380, 26951, 110059, 150209, 288465\} \), see [15]. For a non-negative integer \( n \), let \( \rho(n) \) denote \( 299.5 + \frac{500000}{1+n^{0.1}} \cdot (|\sin(n)| - 0.2) \).

Statement 4. The set 
\[ X = \{ n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } \rho(n) \text{ primes of the form } k! + 1 \} \]
satisfies Conditions (1)–(5) except the requirement that \( X \) is naturally defined. Conjecture 2 implies that the set \( \mathbb{N} \setminus X \) is finite.

Proof. For every integer \( n \geq 2 \cdot 10^6 \cdot 50000010 \), 300 is the smallest integer greater than \( \rho(n) \). This proves Condition (1) with \( n = 2 \cdot 10^6 \cdot 50000010 \). The number 300 is greater than the number of known primes of the form \( k! + 1 \), which proves Condition (3). The following code is written in MuPAD Light 2.5.3.

```
m:=0:
for n from 0 to 10^6 do
    if n<1!+1 then r:=0 end_if:
    if n>=1!+1 and n<2!+1 then r:=1 end_if:
    if n>=2!+1 and n<3!+1 then r:=2 end_if:
    if n>=3!+1 then r:=3 end_if:
    b:=299.5+(500000/(1+n^0.1))*(abs(sin(n+0.1))-0.2):
    if r>b then m:=m+1 end_if:
end_for:
print(m):
```

The number \( r \) in the code counts the number of primes of the form \( k! + 1 \) in the interval \([-1, n]\), because \( k! + 1 \) is composite for every \( k \in \{4, 5, 6, 7, 8, 9\} \) and \( 10! + 1 > 10^6 \geq n \). The code computes that \( \text{card}(X \cap [0, 10^6]) = 126385 \), which proves Condition (4).

\[ \square \]
Open Problem 1. Is there a set \( X \subseteq \mathbb{N} \) which satisfies Conditions (1)–(5)?

Open Problem 1 asks about the existence of a year \( t \geq 2022 \) in which the conjunction

\[
\text{(Condition 1)} \land \text{(Condition 2)} \land \text{(Condition 3)} \land \text{(Condition 4)} \land \text{(Condition 5)}
\]

will hold for some \( X \subseteq \mathbb{N} \). For every year \( t \geq 2022 \) and for every \( i \in \{1, 2, 3\} \), a positive solution to Open Problem \( i \) in the year \( t \) may change in the future. Currently, the answers to Open Problems \([1, 5]\) are negative.

Statement 5. No set \( X \subseteq \mathbb{N} \) will satisfy Conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

Proof. The proof goes by contradiction. We fix an integer \( n \) that satisfies Condition (1). Since Conditions (1)–(3) will hold forever, the semi-algorithm in Figure 2 never terminates and sequentially prints the following sentences:

\[
\text{(T)} \quad n + 1 \notin X, \; n + 2 \notin X, \; n + 3 \notin X, \ldots
\]

Fig. 2 Semi-algorithm that terminates if and only if \( X \) is infinite

The sentences from the sequence (T) and our assumption imply that for every integer \( m > n \) computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that \( (n, m) \cap X = \emptyset \). Thus, at some future day, numerical evidence will support the conjecture that the set \( X \) is finite, contrary to the conjecture in Condition (4).

The physical limits of computation ([8]) disprove the assumption of Statement 5.

Statement 6. Conditions (2)–(5) hold for \( X = \mathcal{P}_{n^2+1} \). The statement \( \Phi \) implies Condition (1) for \( X = \mathcal{P}_{n^2+1} \).

Proof. The set \( \mathcal{P}_{n^2+1} \) is conjecturally infinite. There are 2199894223892 primes of the form \( n^2 + 1 \) in the interval \([2, 10^{28}]\), see [16]. These two facts imply Condition (4). By Lemma [9], due to known physics we are not able to confirm by a direct computation that some element of \( \mathcal{P}_{n^2+1} \) is greater than \( f(7) = (((24!)!)!)! = \beta \), see [8]. Thus Condition (3) holds. Conditions (2) and (5) hold trivially. The statement \( \Phi \) implies that Condition (1) holds for \( X = \mathcal{P}_{n^2+1} \) with \( n = (((24!)!)!)! \). \( \square \)
Proving Landau’s conjecture will disprove Statement 6. We do not conjecture that (Conditions (1)–(5) hold for \( X = \mathcal{P}_{n^2+1} \) \& \( \Phi \))

6. Satisfiable conjunctions which consist of Conditions (1)–(5) and their negations

The set \( X = \mathcal{P}_{n^2+1} \) satisfies the conjunction

\[ \neg(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5}) \]

The set \( X = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1} \) satisfies the conjunction

\[ \neg(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5}) \]

The set \( X = \begin{cases} \mathbb{N}, \text{ if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\}, \text{ otherwise} \end{cases} \)

satisfies the conjunction

\[ (\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5}) \]

**Open Problem 2.** Is there a set \( X \subseteq \mathbb{N} \) that satisfies the conjunction

\[ (\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5}) \]?

The numbers \( 2^{2^k} + 1 \) are prime for \( k \in \{0, 1, 2, 3, 4\} \). It is open whether or not there are infinitely many primes of the form \( 2^{2^k} + 1 \), see [7, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form \( 2^{2^k} + 1 \), see [7, p. 159] and [12, p. 74]. Most mathematicians believe that \( 2^{2^k} + 1 \) is composite for every integer \( k \geq 5 \), see [6, p. 23].

The set

\[ X = \begin{cases} \mathbb{N}, \text{ if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\} \cup \{ n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1 \}, \text{ otherwise} \end{cases} \]

satisfies the conjunction

\[ \neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5}) \]

**Open Problem 3.** Is there a set \( X \subseteq \mathbb{N} \) that satisfies the conjunction

\[ \neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5}) \]?

It is possible, although very doubtful, that at some future day, the set \( X = \mathcal{P}_{n^2+1} \) will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set \( X = \{ k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite} \} \) will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

\[ \#(\text{Condition 1}) \land (\text{Condition 2}) \land \#(\text{Condition 3}) \land (\text{Condition 4}) \land \#(\text{Condition 5}) \]

where \# denotes the negation \( \neg \) or the absence of any symbol.
Table 1 Five satisfiable conjunctions

Open Problem 4. Is there a known threshold number of $P_{n^2+1}$?

Open Problem 4 asks about the existence of a year $t \geq 2022$ in which the implication $\text{card}(P_{n^2+1}) < \omega \Rightarrow P_{n^2+1} \subseteq (-\infty, n]$ will hold for some known integer $n$.

Let $T$ denote the set of twin primes.

Open Problem 5. Is there a known threshold number of $T$?

Open Problem 5 asks about the existence of a year $t \geq 2022$ in which the implication $\text{card}(T) < \omega \Rightarrow T \subseteq (-\infty, n]$ will hold for some known integer $n$.

7. PREVIOUSLY KNOWN RESULTS WHICH CORRESPOND TO THE RESULTS OF SECTIONS 4–6

Statements [1–6] and Open Problems [1–5] refer to the current mathematical knowledge. Previously known statements of this type, such as Statement [7], express the current knowledge on particular elements of $\mathbb{N}$, which are known to us according to Definition [1].

Statement 7. ([2, 3, 7, p. 209], [10]). The numbers $2^{22^2} + 1$ and $2^{24^2} + 1$ are composite. The known integer divisors of $2^{22^2} + 1$ form the set \(-2^{22^2} - 1, -1, 1, 2^{22^2} + 1\). The known integer divisors of $2^{24^2} + 1$ form the set \(-2^{24^2} - 1, -1, 1, 2^{24^2} + 1\).

8. SUMMARY

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau’s conjecture states that the set $P_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$: $\text{card}(P_{n^2+1}) < \omega \Rightarrow P_{n^2+1} \subseteq [2, ((241!)^!)!]$. Let $B$ denote the system of equations: $\{x_i! = x_k : j, k \in \{1, \ldots, 9\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\}$. We write some system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1, \ldots, x_9$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. No known system $\mathcal{S} \subseteq B$ with a finite number of solutions in positive integers $x_1, \ldots, x_9$ has a solution $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ satisfying $\max(x_1, \ldots, x_9) > f(9)$. For every known system $\mathcal{S} \subseteq B$, if the finiteness/infinite nature of the set $\{(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \ldots, x_9) \text{ solves } \mathcal{S}\}$ is unknown, then the statement
Theorem 1. For any $k \in \mathbb{N}$, the set $X = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } 299.5 + \frac{500000}{1 + n!} \cdot (\lfloor \sin(n) \rfloor - 0.2) \}$ primes of the form $k! + 1$ satisfies Conditions (1)-(5) except the requirement that $X$ is naturally defined. The conjecture that there are infinitely many primes of the form $k! + 1$ implies that the set $\mathbb{N} \setminus X$ is finite. Table 1 shows satisfiable conjunctions of the form $\#(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})$, where $\#$ denotes the negation $\neg$ or the absence of any symbol. No set $X \subseteq \mathbb{N}$ will satisfy Conditions (1)-(5) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption.

References

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