

On sets $\mathcal{W} \subseteq \mathbb{N}$ whose infinitude follows from the existence in \mathcal{W} of an element which is greater than a threshold number computed for \mathcal{W}

Abstract

We define computable functions $f, g: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$. For a positive integer n , let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has only finitely many solutions in integers x_1, \dots, x_n greater than 1, then each such solution (x_1, \dots, x_n) satisfies $\min(x_1, \dots, x_n) \leq f(n)$. The statement Θ_9 proves that if there exists an integer $x > f(9)$ such that $x^2 + 1$ (alternatively, $x! + 1$) is prime, then there are infinitely many primes of the form $n^2 + 1$ (respectively, $n! + 1$). The statement Θ_{16} proves that if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes. We formulate a statement which proves that if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n .

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation, composite Fermat numbers, composite numbers of the form $2^{2^n} + 1$, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, Richert's lemma, twin prime conjecture.

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1 Introduction

The following observation concerns the theme described in the title of the article.

Observation 1. *If $n \in \mathbb{N}$ and $\mathcal{W} \subseteq \{0, \dots, n\}$, then we take any integer $m \geq n$ as a threshold number for \mathcal{W} . If $\mathcal{W} \subseteq \mathbb{N}$ and \mathcal{W} is infinite, then we take any non-negative integer m as a threshold number for \mathcal{W} .*

We define the set $\mathcal{U} \subseteq \mathbb{N}$ by declaring that a non-negative integer n belongs to \mathcal{U} if and only if $\sin\left(10^{10^{10^{10}}}\right) > 0$. This inequality is practically undecidable, see [7].

Corollary 1. *The set \mathcal{U} equals \emptyset or \mathbb{N} . The statement “ $\mathcal{U} = \emptyset$ ” remains unproven and the statement “ $\mathcal{U} = \mathbb{N}$ ” remains unproven. Every non-negative integer m is a threshold number for \mathcal{U} . For every non-negative integer k , the sentence “ $k \in \mathcal{U}$ ” is only theoretically decidable.*

The first-order language of graph theory contains two relation symbols of arity 2: \sim and $=$, respectively for adjacency and equality of vertices. The term first-order imposes the condition that the variables represent vertices and hence the quantifiers apply to vertices only. For a first-order sentence Λ about graphs, let $\text{Spectrum}(\Lambda)$ denote the set of all positive integers n such that there is a graph on n vertices satisfying Λ . By a graph on n vertices we understand a set of n elements with a binary relation which is symmetric and irreflexive.

Theorem 1. ([15, p. 171]). *If a sentence Λ in the language of graph theory has the form $\exists x_1 \dots x_k \forall y_1 \dots y_l \Upsilon(x_1, \dots, x_k, y_1, \dots, y_l)$, where $\Upsilon(x_1, \dots, x_k, y_1, \dots, y_l)$ is quantifier-free, then either $\text{Spectrum}(\Lambda) \subseteq [1, (2^k \cdot 4^l) - 1]$ or $\text{Spectrum}(\Lambda) \supseteq [k + l, \infty) \cap \mathbb{N}$.*

Corollary 2. *The number $(2^k \cdot 4^l) - 1$ is a threshold number for $\text{Spectrum}(\Lambda)$.*

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [13, p. 234].

Corollary 3. *If an algorithm Al_1 for every recursive set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\text{Al}_1(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\text{Al}_1(\mathcal{M}) + 1, \infty) \neq \emptyset$.*

Corollary 4. *If an algorithm Al_2 for every recursively enumerable set $\mathcal{W} \subseteq \mathbb{N}$ finds a non-negative integer $\text{Al}_2(\mathcal{W})$, then there exists a finite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\mathcal{M} \cap [\text{Al}_2(\mathcal{M}) + 1, \infty) \neq \emptyset$.*

Let $K = \{j \in \mathbb{N} : 2^{\aleph_j} = \aleph_{j+1}\}$.

Theorem 2. *If ZFC is consistent, then for every non-negative integer n the sentence*

" n is a threshold number for K "

is not provable in ZFC.

Proof. There exists a model \mathcal{E} of ZFC such that

$$\forall i \in \{0, \dots, n+1\} \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+1}$$

and

$$\forall i \in \{n+2, n+3, n+4, \dots\} \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+2}$$

see [5] and [8, p. 232]. In the model \mathcal{E} , $K = \{0, \dots, n+1\}$ and n is not a threshold number for K . \square

Theorem 3. *If ZFC is consistent, then for every non-negative integer n the sentence*

" n is not a threshold number for K "

is not provable in ZFC.

Proof. The Generalized Continuum Hypothesis (GCH) is consistent with ZFC, see [8, p. 188] and [8, p. 190]. GCH implies that $K = \mathbb{N}$. Consequently, GCH implies that every non-negative integer n is a threshold number for K . \square

Theorem 4. ([2, p. 35]). *There exists a polynomial $D(x_1, \dots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences*

"The equation $D(x_1, \dots, x_m) = 0$ is solvable in non-negative integers"

and

"The equation $D(x_1, \dots, x_m) = 0$ is not solvable in non-negative integers"

are not provable in ZFC.

Let Δ denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has no solutions in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, the set Δ is computable. Theorem 4 implies the following corollary.

Corollary 5. *If ZFC is arithmetically consistent, then for every non-negative integer n the sentences*

" n is a threshold number for Δ "

and

" n is not a threshold number for Δ "

are not provable in ZFC.

Let $g(1) = 1$, and let $g(n + 1) = 2^{2^{g(n)}}$ for every positive integer n .

Hypothesis 1. ([20]). *If a system*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{x_i + 1 = x_k : i, k \in \{1, \dots, n\}\}$$

has only finitely many solutions in non-negative integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq g(2n)$.

Theorem 5. ([20]). *Hypothesis 1 implies that for every $W(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ we can compute a threshold number $b \in \mathbb{N} \setminus \{0\}$ such that any non-negative integers a_1, \dots, a_n which satisfy*

$$(W(a_1, \dots, a_n) = 0) \wedge (\max(a_1, \dots, a_n) > b)$$

guarantee that the equation $W(x_1, \dots, x_n) = 0$ has infinitely many solutions in non-negative integers.

2 Basic lemmas

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{V}_1 denote the system of equations $\{x_1! = x_1\}$, and let \mathcal{V}_2 denote the system of equations $\{x_1! = x_1, x_1 \cdot x_1 = x_2\}$. For an integer $n \geq 3$, let \mathcal{V}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{V}_n .

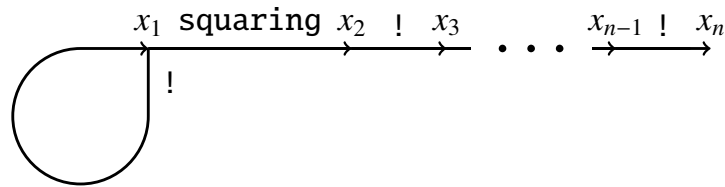


Fig. 1 Construction of the system \mathcal{V}_n

Lemma 1. *For every positive integer n , the system \mathcal{V}_n has exactly one solution in integers greater than 1, namely $(f(1), \dots, f(n))$.*

Let

$$H_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers x_1, \dots, x_n greater than 1, then each such solution (x_1, \dots, x_n) satisfies $\min(x_1, \dots, x_n) \leq f(n)$. The assumption $\min(x_1, \dots, x_n) \leq f(n)$ is weaker than the assumption $\max(x_1, \dots, x_n) \leq f(n)$ suggested by Lemma 1.

Lemma 2. *For every positive integer n , the system H_n has a finite number of subsystems.*

Theorem 6. *Every statement Θ_n is true with an unknown integer bound that depends on n .*

Proof. It follows from Lemma 2. □

Lemma 3. *For every integers x and y greater than 1, $x! \cdot y = y!$ if and only if $x + 1 = y$.*

Lemma 4. *If $x \geq 4$, then $\frac{(x-1)! + 1}{x} > 1$.*

Lemma 5. *(Wilson's theorem, [6, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x-1)! + 1$.*

3 Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [14]. It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [21, p. 297].

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{array} \right.$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

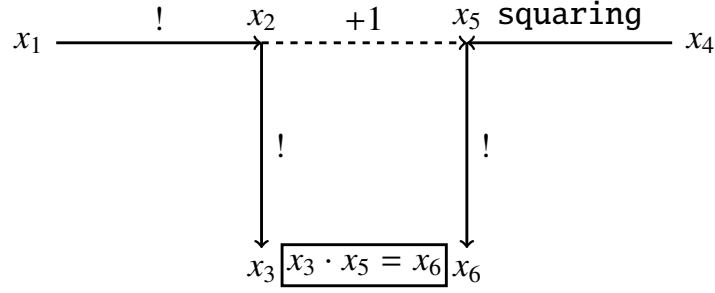


Fig. 2 Construction of the system \mathcal{A}

Lemma 6. For every integers x_1 and x_4 greater than 1, the system \mathcal{A} is solvable in integers x_2, x_3, x_5, x_6 greater than 1 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

and $x_1 = \min(x_1, \dots, x_6)$.

Proof. It follows from Lemma 3. □

Theorem 7. The statement Θ_6 proves the following implication: if the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then each such solution (x_1, x_4) satisfies $x_1 \leq f(6)$.

Proof. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 6, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0, 1\})^4$ such that the tuple (x_1, \dots, x_6) solves the system \mathcal{A} . Lemma 6 guarantees that $x_1 = \min(x_1, \dots, x_6)$. By the antecedent and Lemma 6, the system \mathcal{A} has only finitely many solutions in integers x_1, \dots, x_6 greater than 1. Therefore, the statement Θ_6 implies that $x_1 = \min(x_1, \dots, x_6) \leq f(6)$. □

Hypothesis 2. The implication in Theorem 7 is true.

Corollary 6. Assuming Hypothesis 2, a single query to an oracle for the halting problem decides the problem of the infinitude of the solutions of the equation $x! + 1 = y^2$.

4 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [12, pp. 37–38]. Let \mathcal{B} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system \mathcal{B} .

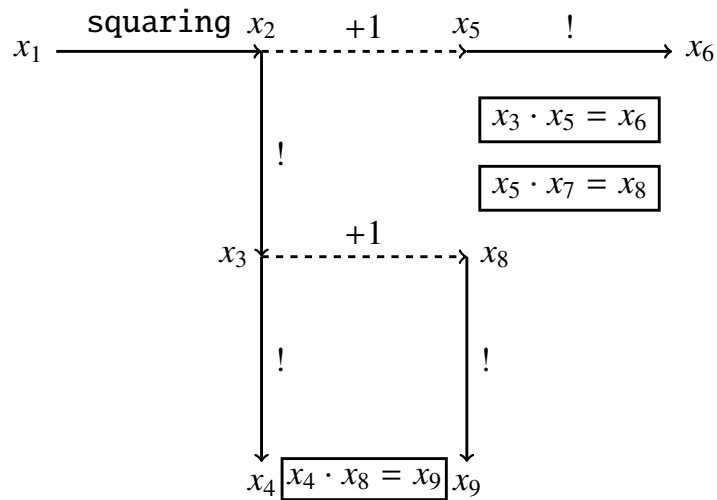


Fig. 3 Construction of the system \mathcal{B}

Lemma 7. For every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in integers x_2, \dots, x_9 greater than 1 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined

by the following equalities:

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

and $\min(x_1, \dots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in integers x_2, \dots, x_9 greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. \square

Theorem 8. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 7, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{B} . Lemma 7 guarantees that $\min(x_1, \dots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \dots, x_9) = x_1 > f(9)$ imply that the system \mathcal{B} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. \square

Hypothesis 3. *The implication in Theorem 8 is true.*

Corollary 7. *Assuming Hypothesis 3, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n^2 + 1$.*

The following question is open: *Is it possible to compute the largest prime number of the form $n^2 + 1$, if the set of these primes is finite?* The unproven statement Θ_9 implies this claim although does not imply that there are infinitely many primes of the form $n^2 + 1$.

Let \mathcal{P} denote the set of prime numbers. For a non-negative integer n , let $\Omega(n)$ denote the following statement: $\exists m \in \mathbb{N} \cap (n, \infty) m^2 + 1 \in \mathcal{P}$. By Theorem 8, assuming the statement Θ_9 , we can infer the statement $\forall n \in \mathbb{N} \Omega(n)$ from any statement $\Omega(n)$ with $n \geq f(9)$. A similar situation holds for inference by the so called "*super-induction method*", see [22]–[25]. In section 8, we present Richert's lemma which is frequently used in proofs by super-induction.

5 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [1, p. 443] and [18]. Let \mathcal{G} denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system \mathcal{G} .

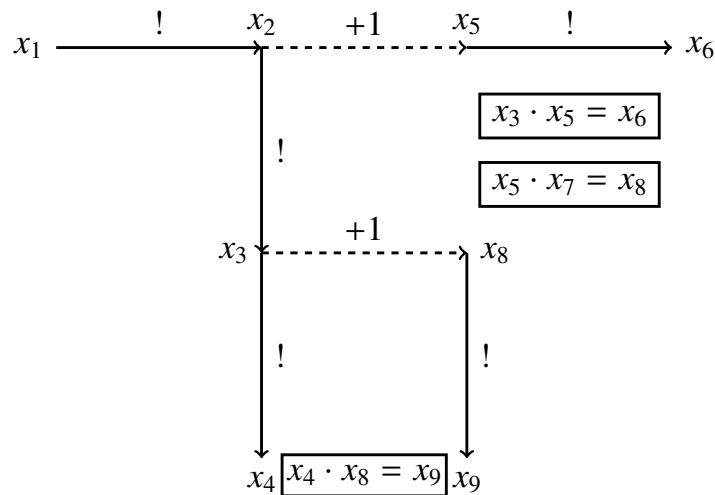


Fig. 4 Construction of the system \mathcal{G}

Lemma 8. *For every integer $x_1 \geq 2$, the system \mathcal{G} is solvable in integers x_2, \dots, x_9 greater than 1 if and only if $x_1! + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by*

the following equalities:

$$\begin{aligned}
x_2 &= x_1! \\
x_3 &= (x_1!)! \\
x_4 &= ((x_1!)!)! \\
x_5 &= x_1! + 1 \\
x_6 &= (x_1! + 1)! \\
x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\
x_8 &= (x_1!)! + 1 \\
x_9 &= ((x_1!)! + 1)!
\end{aligned}$$

and $\min(x_1, \dots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geq 2$, the system \mathcal{G} is solvable in integers x_2, \dots, x_9 greater than 1 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 5. \square

Theorem 9. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1! + 1$ is prime. By Lemma 8, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{G} . Lemma 8 guarantees that $\min(x_1, \dots, x_9) = x_1$. Since $\mathcal{G} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \dots, x_9) = x_1 > f(9)$ imply that the system \mathcal{G} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 8, there are infinitely many primes of the form $n! + 1$. \square

Hypothesis 4. *The implication in Theorem 9 is true.*

Corollary 8. *Assuming Hypothesis 4, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n! + 1$.*

The following question is open: *Is it possible to compute the largest prime number of the form $n! + 1$, if the set of these primes is finite?* The unproven statement Θ_9 implies this claim although does not imply that there are infinitely many primes of the form $n! + 1$.

6 The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [12, p. 39].

Let C denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{array} \right.$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system C .

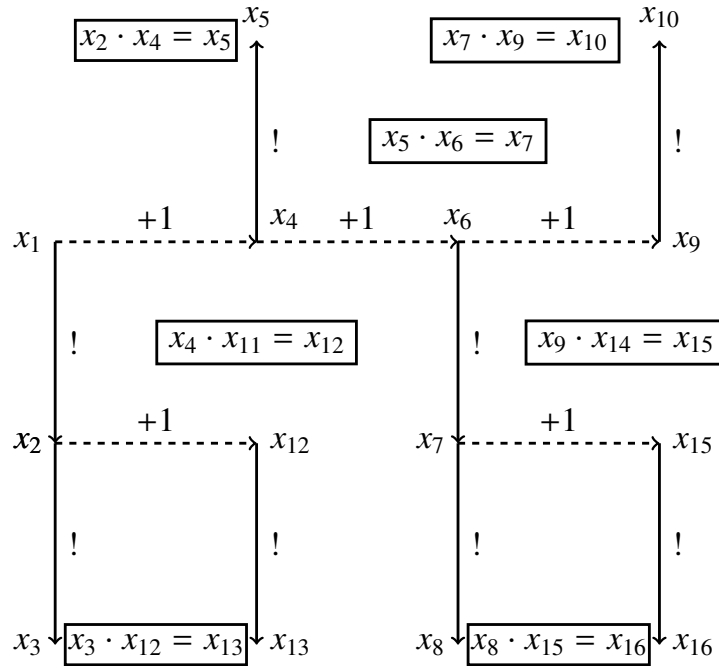


Fig. 5 Construction of the system C

Lemma 9. *If $x_4 = 2$, then the system C has no solutions in integers x_1, \dots, x_{16} greater than 1.*

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. \square

Lemma 10. *If $x_4 = 3$, then the system C has no solutions in integers x_1, \dots, x_{16} greater than 1.*

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. \square

Lemma 11. *For every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$\begin{aligned}
 x_1 &= x_4 - 1 \\
 x_2 &= (x_4 - 1)! \\
 x_3 &= ((x_4 - 1)!)! \\
 x_5 &= x_4! \\
 x_6 &= x_9 - 1 \\
 x_7 &= (x_9 - 1)! \\
 x_8 &= ((x_9 - 1)!)! \\
 x_{10} &= x_9! \\
 x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
 x_{12} &= (x_4 - 1)! + 1 \\
 x_{13} &= ((x_4 - 1)! + 1)! \\
 x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
 x_{15} &= (x_9 - 1)! + 1 \\
 x_{16} &= ((x_9 - 1)! + 1)!
 \end{aligned}$$

and $\min(x_1, \dots, x_{16}) = x_1 = x_9 - 3$.

Proof. By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ greater than 1 if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | ((x_4 - 1)! + 1)) \wedge (x_9 | ((x_9 - 1)! + 1))$$

Hence, the claim of Lemma 11 follows from Lemma 5. \square

Theorem 10. *The statement Θ_{16} proves the following implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.*

Proof. Assume that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 11, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0, 1\})^{14}$ such that the tuple (x_1, \dots, x_{16}) solves the system C . Lemma 11 guarantees that $\min(x_1, \dots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement Θ_{16} and the inequality $\min(x_1, \dots, x_{16}) > f(16)$ imply that the system C has infinitely many solutions in integers x_1, \dots, x_{16} greater than 1. According to Lemmas 9–11, there are infinitely many twin primes. \square

Hypothesis 5. *The implication in Theorem 10 is true.*

Corollary 9. (cf. [3]). *Assuming Hypothesis 5, a single query to an oracle for the halting problem decides the twin prime problem.*

The following question is open: *Is it possible to compute the largest twin prime, if the set of twin primes is finite?* The unproven statement Θ_{16} implies this claim although does not imply that there are infinitely many twin primes.

7 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [11, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [11, p. 1].

Open Problem. ([11, p. 159]). *Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [10, p. 23].

Theorem 11. ([19]). *An unproven inequality stated in [19] implies that $2^{2^n} + 1$ is composite for every integer $n \geq 5$.*

Lemma 12. ([11, p. 38]). *For every positive integer n , if a prime number p divides $2^{2^n} + 1$, then there exists a positive integer k such that $p = k \cdot 2^n + 1 + 1$.*

Corollary 10. *Since $k \cdot 2^n + 1 + 1 \geq 2^n + 1 + 1 \geq n + 3$, for every positive integers x, y , and n , the equality $(x + 1)(y + 1) = 2^{2^n} + 1$ implies that $\min(n, x, x + 1, y, y + 1) = n$.*

Let

$$G_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Lemma 13. *The following subsystem of G_n*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(g(1), \dots, g(n))$.

For a positive integer n , let Ψ_n denote the following statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $\min(x_1, \dots, x_n) \leq g(n)$. The assumption $\min(x_1, \dots, x_n) \leq g(n)$ is weaker than the assumption $\max(x_1, \dots, x_n) \leq g(n)$ suggested by Lemma 13.

Lemma 14. *For every positive integer n , the system G_n has a finite number of subsystems.*

Theorem 12. *Every statement Ψ_n is true with an unknown integer bound that depends on n .*

Proof. It follows from Lemma 14. □

Lemma 15. *For every non-negative integers b and c , $b+1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Theorem 13. *The statement Ψ_{13} proves the following implication: if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n .*

Proof. Let us consider the equation

$$(x+1)(y+1) = 2^{2^z} + 1 \tag{1}$$

in positive integers. By Lemma 15, we can transform equation (1) into an equivalent system \mathcal{F} which has 13 variables (x, y, z , and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 6.

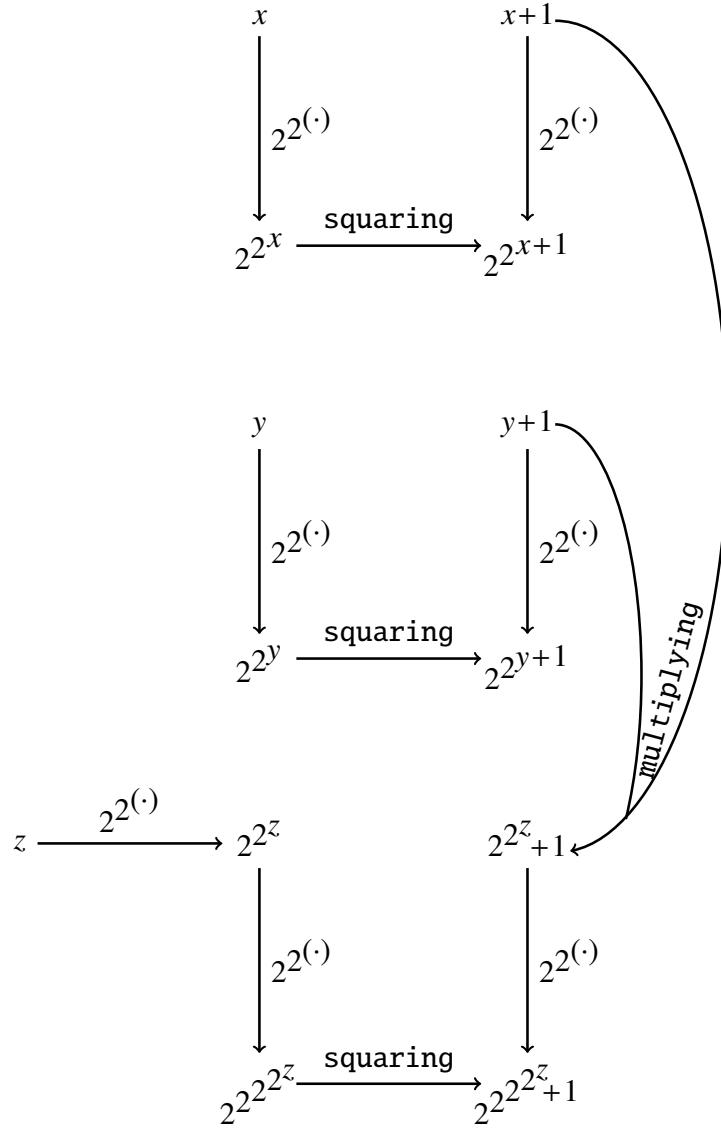


Fig. 6 Construction of the system \mathcal{F}

Assume that $2^{2^n} + 1$ is composite for some integer $n > g(13)$. By this and Corollary 10, equation (1) has a solution $(x, y, z) \in (\mathbb{N} \setminus \{0\})^3$ such that $z = n$ and $z = \min(z, x, x + 1, y, y + 1)$. Hence, the system \mathcal{F} has a solution in positive integers such that $z = n$ and n is the smallest number in the solution sequence. Since $n > g(13)$, the statement Ψ_{13} implies that the system \mathcal{F} has infinitely many solutions in positive integers. Therefore, there are infinitely many positive integers n such that $2^{2^n} + 1$ is composite. \square

Hypothesis 6. *The implication in Theorem 13 is true.*

Corollary 11. *Assuming Hypothesis 6, a single query to an oracle for the halting problem decides whether or not the set of composite Fermat numbers is infinite.*

The following question is open: *Is it possible to compute the largest composite Fermat number, if the set of these numbers is finite?* The unproven statement Ψ_{13} implies this claim although does not imply that there are infinitely many composite Fermat numbers.

8 Subsets of $\mathbb{N} \setminus \{0\}$ which are cofinite by Richert's lemma and the halting of a computer program

The following lemma is known as Richert's lemma.

Lemma 16. *([4], [16], [17, p. 152]). Let $\{m_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that for some positive integer k the inequality $m_{i+1} \leq 2m_i$ holds for all $i > k$. Suppose there exists a non-negative integer b such that the numbers $b + 1, b + 2, b + 3, \dots, b + m_{k+1}$ are all expressible as sums of one or more distinct elements of the set $\{m_1, \dots, m_k\}$. Then every integer greater than b is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$.*

Corollary 12. *If the sequence $\{m_i\}_{i=1}^{\infty}$ is computable and the flowchart algorithm in Figure 7 terminates, then almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$ and the algorithm returns all positive integers which are not expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$.*

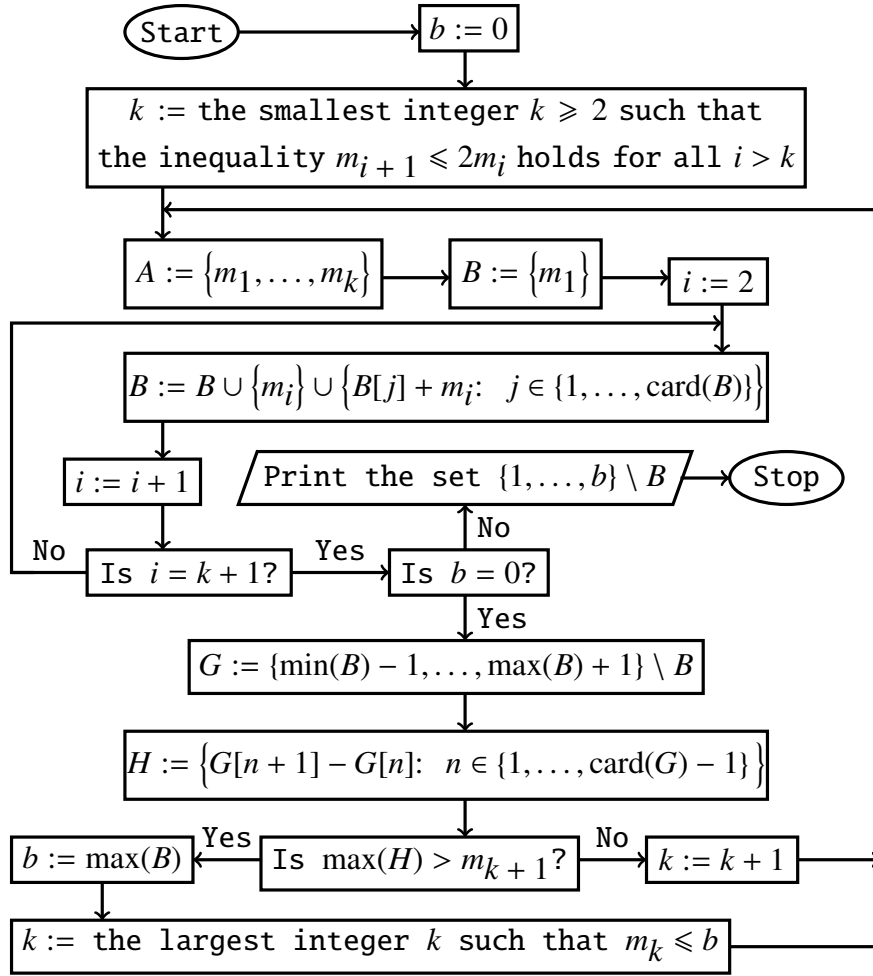


Fig. 7 The algorithm which uses Richert's lemma

The above algorithm works correctly because the inequality $\max(H) > m_{k+1}$ holds true if and only if the set B contains m_{k+1} consecutive integers.

Theorem 14. ([9, Theorem 2.3]). *If there exists $\varepsilon > 0$ such that the inequality $m_{i+1} \leq (2 - \varepsilon) \cdot m_i$ holds for every sufficiently large i , then the flowchart algorithm in Figure 7 terminates if and only if almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \dots\}$.*

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