On sets $\mathcal{W} \subseteq \mathbb{N}$ whose infinitude follows from the existence in $\mathcal{W}$ of an element which is greater than a threshold number computed for $\mathcal{W}$

Abstract

We define computable functions $f, g\colon \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$. For a positive integer n, let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq \Big\{x_i! = x_k : i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\Big\}$ has only finitely many solutions in integers $x_1, \ldots, x_n$ greater than 1, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leqslant f(n)$. The statement Θ_9 proves that if there exists an integer $x > f(9)$ such that $x^2 + 1$ (alternatively, $x! + 1$) is prime, then there are infinitely many primes of the form $n^2 + 1$ (respectively, $n! + 1$). The statement Θ_{16} proves that if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes. We formulate a statement which proves that if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n.

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation, composite Fermat numbers, composite numbers of the form $2^{2^n} + 1$, infinite set, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, Richert's lemma, twin prime conjecture.

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1 Introduction

Euclid indirectly proved that there are infinitely many prime numbers. A stronger theorem states that for every integer $n > 1$ there exists a prime number p such that $n < p < 2n$, see [19, p. 145]. This theorem is a Π_1 statement.

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [14, p. 39]. The following statement

`(A) "For every non-negative integer` n `there exists a twin prime which belongs to the interval` $\left(10^n, 10^{n+1}\right)$`"`

is a Π_1 statement which strengthens the twin prime conjecture, cf. [3, p. 43]. The statement `(A)` is equivalent to the non-halting of a Turing machine. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger Π_1 statements, see [1].

In this article, we study sets $\mathcal{W} \subseteq \mathbb{N}$ whose infinitude follows from the existence in $\mathcal{W}$ of an element which is greater than a threshold number computed for $\mathcal{W}$. If $\mathcal{W}$ is computable, then this property implies that the infinity of $\mathcal{W}$ is equivalent to the halting of a Turing machine. If $n \in \mathbb{N}$ and $\mathcal{W} \subseteq \{0, \ldots, n\}$, then any integer $m \geqslant n$ is a threshold number for $\mathcal{W}$. If $\mathcal{W} \subseteq \mathbb{N}$ and $\mathcal{W}$ is empty or infinite, then any non-negative integer m is a threshold number for $\mathcal{W}$.

We define the set $\mathcal{U} \subseteq \mathbb{N}$ by declaring that a non-negative integer n belongs to $\mathcal{U}$ if and only if $\sin\left(10^{10^{10^{10}}}\right) > 0$. This inequality is practically undecidable, see [9]. The set $\mathcal{U}$ equals $\emptyset$ or $\mathbb{N}$. The statement "$\mathcal{U} = \emptyset$" remains unproven and the statement "$\mathcal{U} = \mathbb{N}$" remains unproven. Every non-negative integer m is a threshold number for $\mathcal{U}$. For every non-negative integer k, the sentence "$k \in \mathcal{U}$" is only theoretically decidable.

The first-order language of graph theory contains two relation symbols of arity 2: $\sim$ and $=$, respectively for adjacency and equality of vertices. The term first-order imposes the condition that the variables represent vertices and hence the quantifiers apply to vertices only. For a first-order sentence Λ about graphs, let $\mathrm{Spectrum}(\Lambda)$ denote the set of all positive integers n such that there is a graph on n vertices satisfying Λ. By a graph on n vertices we understand a set of n elements with a binary relation which is symmetric and irreflexive.

Theorem 1. *([17, p. 171]). If a sentence Λ in the language of graph theory has the form $\exists x_1 \ldots x_k \, \forall y_1 \ldots y_l \, \Upsilon(x_1, \ldots, x_k, y_1, \ldots, y_l)$, where $\Upsilon(x_1, \ldots, x_k, y_1, \ldots, y_l)$ is quantifier-free, then either* $\mathrm{Spectrum}(\Lambda) \subseteq [1, (2^k \cdot 4^l) - 1]$ *or* $\mathrm{Spectrum}(\Lambda) \supseteq [k + l, \infty) \cap \mathbb{N}$.

Corollary 1. *The number $\left(2^k \cdot 4^l\right) - 1$ is a threshold number for* $\mathrm{Spectrum}(\Lambda)$.

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [15, p. 234].

Corollary 2. *If an algorithm* $\mathtt{Al}_1$ *for every recursive set* $\mathcal{W} \subseteq \mathbb{N}$ *finds a non-negative integer* $\mathtt{Al}_1(\mathcal{W})$*, then there exists a finite set* $\mathcal{M} \subseteq \mathbb{N}$ *such that* $\mathcal{M} \cap [\mathtt{Al}_1(\mathcal{M}) + 1, \infty) \neq \emptyset$.

Corollary 3. *If an algorithm* $\mathtt{Al}_2$ *for every recursively enumerable set* $\mathcal{W} \subseteq \mathbb{N}$ *finds a non-negative integer* $\mathtt{Al}_2(\mathcal{W})$*, then there exists a finite set* $\mathcal{M} \subseteq \mathbb{N}$ *such that* $\mathcal{M} \cap [\mathtt{Al}_2(\mathcal{M})+1, \infty) \neq \emptyset$.

Let $K = \{j \in \mathbb{N} : \ 2^{\aleph_j} = \aleph_{j+1}\}$.

Theorem 2. *If ZFC is consistent, then for every non-negative integer n the sentence*

"n `is a threshold number for` K"

is not provable in ZFC.

Proof. There exists a model $\mathcal{E}$ of ZFC such that

$$\forall i \in \{0, \ldots, n+1\} \ \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+1}$$

and

$$\forall i \in \{n+2, n+3, n+4, \ldots\} \ \mathcal{E} \models 2^{\aleph_i} = \aleph_{i+2}$$

see [7] and [10, p. 232]. In the model $\mathcal{E}$, $K = \{0, \ldots, n+1\}$ and n is not a threshold number for K. □

Theorem 3. *If ZFC is consistent, then for every non-negative integer n the sentence*

"n `is not a threshold number for` K"

is not provable in ZFC.

Proof. The Generalized Continuum Hypothesis (GCH) is consistent with ZFC, see [10, p. 188] and [10, p. 190]. GCH implies that $K = \mathbb{N}$. Consequently, GCH implies that every non-negative integer n is a threshold number for K. □

Theorem 4. *([4, p. 35]). There exists a polynomial* $D(x_1, \ldots, x_m)$ *with integer coefficients such that if ZFC is arithmetically consistent, then the sentences*

"`The equation` $D(x_1, \ldots, x_m) = 0$ `is solvable in non-negative integers`"

and

"`The equation` $D(x_1, \ldots, x_m) = 0$ `is not solvable in non-negative integers`"

are not provable in ZFC.

Let Δ denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has no solutions in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, we know a computer program which for every non-negative integer x decides whether or not x belongs to Δ. Theorem 4 implies Theorem 5.

Theorem 5. *If ZFC is arithmetically consistent, then for every non-negative integer n the sentences*

$$\text{"}n \texttt{ is a threshold number for } \Delta\text{"}$$

and

$$\text{"}n \texttt{ is not a threshold number for } \Delta\text{"}$$

are not provable in ZFC.

Let $\sigma \colon \mathbb{N}^{m+1} \to \mathbb{N}$ be a computable bijection. Let $\mathcal{H} \subseteq \mathbb{N}^{m+1}$ be the solution set of the equation $D(x_1, \dots, x_m) + 0 \cdot x_{m+1} = 0$.

Theorem 6. *We know a computer program which for every non-negative integer x decides whether or not x belongs to $\mathcal{H}$. The set $\sigma(\mathcal{H})$ is empty or infinite. In both cases, every non-negative integer n is a threshold number for $\sigma(\mathcal{H})$. If ZFC is arithmetically consistent, then the sentences "$\sigma(\mathcal{H}) = \emptyset$", "$\sigma(\mathcal{H}) \neq \emptyset$", "$\sigma(\mathcal{H})$ `is finite`", and "$\sigma(\mathcal{H})$ `is infinite`" are not provable in ZFC.*

Proof. We leave the proof to the reader. □

Let $g(1) = 1$, and let $g(n+1) = 2^{2^{g(n)}}$ for every positive integer n.

Hypothesis 1. *([22]). If a system*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \dots, n\}\} \cup \{x_i + 1 = x_k : \ i, k \in \{1, \dots, n\}\}$$

has only finitely many solutions in non-negative integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant g(2n)$.

Theorem 7. *([22]). Hypothesis 1 implies that for every $W(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ we can compute a threshold number $b \in \mathbb{N} \setminus \{0\}$ such that any non-negative integers $a_1, \dots, a_n$ which satisfy*

$$(W(a_1, \dots, a_n) = 0) \wedge (\max(a_1, \dots, a_n) > b)$$

guarantee that the equation $W(x_1, \dots, x_n) = 0$ has infinitely many solutions in non-negative integers.

2 Basic lemmas

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{V}_1$ denote the system of equations $\{x_1! = x_1\}$, and let $\mathcal{V}_2$ denote the system of equations $\{x_1! = x_1,\ x_1 \cdot x_1 = x_2\}$. For an integer $n \geqslant 3$, let $\mathcal{V}_n$ denote the following system of equations:

$$\begin{cases} \phantom{\forall i \in \{2,\ldots,n-1\}}\ x_1! &= x_1 \\ \phantom{\forall i \in \{2,\ldots,n-1\}}\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2,\ldots,n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{V}_n$.

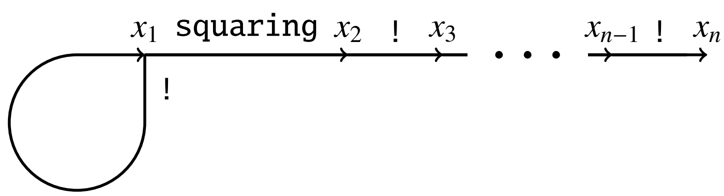

Fig. 1 Construction of the system $\mathcal{V}_n$

Lemma 1. *For every positive integer n, the system $\mathcal{V}_n$ has exactly one solution in integers greater than 1, namely $\big(f(1), \ldots, f(n)\big)$.*

Let

$$H_n = \Big\{x_i! = x_k :\ i,k \in \{1,\ldots,n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\ldots,n\}\Big\}$$

For a positive integer n, let Θ_n denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers $x_1,\ldots,x_n$ greater than 1, then each such solution $(x_1,\ldots,x_n)$ satisfies $\min(x_1,\ldots,x_n) \leqslant f(n)$. The assumption $\min(x_1,\ldots,x_n) \leqslant f(n)$ is weaker than the assumption $\max(x_1,\ldots,x_n) \leqslant f(n)$ suggested by Lemma 1.

Lemma 2. *For every positive integer n, the system H_n has a finite number of subsystems.*

Theorem 8. *Every statement Θ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 2. □

Lemma 3. *For every integers x and y greater than 1, $x! \cdot y = y!$ if and only if $x+1 = y$.*

Lemma 4. *If $x \geqslant 4$, then* $\dfrac{(x-1)!+1}{x} > 1$.

Lemma 5. *(Wilson's theorem, [8, p. 89]). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

3 Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [16]. It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [23, p. 297].

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

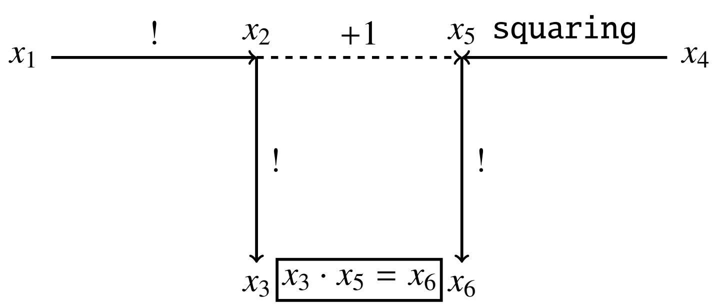

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 6. *For every integers x_1 and x_4 greater than 1, the system $\mathcal{A}$ is solvable in integers x_2, x_3, x_5, x_6 greater than 1 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

and $x_1 = \min(x_1, \ldots, x_6)$.

Proof. It follows from Lemma 3. □

Theorem 9. *The statement Θ_6 proves the following implication: if the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then each such solution (x_1, x_4) satisfies $x_1 \leqslant f(6)$.*

Proof. Let positive integers x_1 and x_4 satisfy $x_1!+1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 6, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0, 1\})^4$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Lemma 6 guarantees that $x_1 = \min(x_1, \ldots, x_6)$. By the antecedent and Lemma 6, the system $\mathcal{A}$ has only finitely many solutions in integers $x_1, \ldots, x_6$ greater than 1. Therefore, the statement Θ_6 implies that $x_1 = \min(x_1, \ldots, x_6) \leqslant f(6)$. $\square$

Hypothesis 2. *The implication in Theorem 9 is true.*

Corollary 4. *Assuming Hypothesis 2, a single query to an oracle for the halting problem decides the problem of the infinitude of the solutions of the equation $x! + 1 = y^2$.*

4 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [14, pp. 37–38]. Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

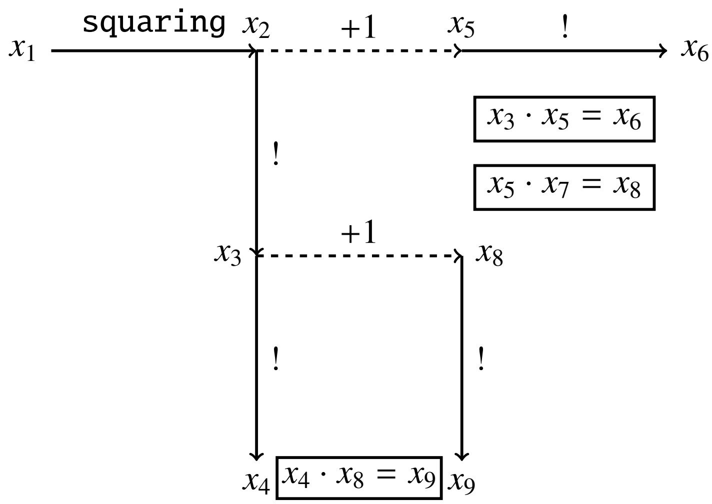

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 7. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \dots, x_9$ greater than* 1 *if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \dots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

and $\min(x_1, \dots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in integers $x_2, \dots, x_9$ greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 5. □

Theorem 10. *The statement Θ_9 proves the following implication: if there exists an integer $x_1 > f(9)$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 7, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{B}$. Lemma 7 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 7, there are infinitely many primes of the form $n^2 + 1$. □

Hypothesis 3. *The implication in Theorem 10 is true.*

Corollary 5. *Assuming Hypothesis 3, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form* $n^2 + 1$.

The following question is open: *Is it possible to effectively determine the largest prime number of the form* $n^2 + 1$*, if the set of these primes is finite?* The unproven statement Θ_9 implies this claim although does not imply that there are infinitely many primes of the form $n^2 + 1$.

Let $J = \{0\} \cup \{i \in \{1\} : \ 2^{\aleph_i} = \aleph_{i+1}\}$.

Theorem 11. *It is impossible to uniquely determine an integer* $j \in \{0, 1\}$ *which is the largest element of* J.

Proof. If ZFC is inconsistent, then for every integer $n \in \mathbb{N}$ the sentence

"n is the largest element of J"

is provable in ZFC. If ZFC is consistent, then by Easton's theorem ([7] and [10, p. 232]) for every integer $n \in \{0, 1\}$ there exists a model of ZFC in which $J = \{0, \ldots, n\}$. □

Let $\mathcal{P}$ denote the set of prime numbers. For a non-negative integer n, let $\Omega(n)$ denote the following statement: $\exists m \in \mathbb{N} \cap (n, \infty) \ m^2 + 1 \in \mathcal{P}$. By Theorem 10, assuming the statement Θ_9, we can infer the statement $\forall n \in \mathbb{N} \ \Omega(n)$ from any statement $\Omega(n)$ with $n \geqslant f(9)$. A similar situation holds for inference by the so called *"super-induction method"*, see [24]–[27]. In section 8, we present Richert's lemma which is frequently used in proofs by super-induction.

5 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [2, p. 443] and [20]. Let $\mathcal{G}$ denote the following system of equations:

$$\left\{\begin{array}{rcl} x_1! &=& x_2 \\ x_2! &=& x_3 \\ x_3! &=& x_4 \\ x_5! &=& x_6 \\ x_8! &=& x_9 \\ x_3 \cdot x_5 &=& x_6 \\ x_4 \cdot x_8 &=& x_9 \\ x_5 \cdot x_7 &=& x_8 \end{array}\right.$$

Lemma 3 and the diagram in Figure 4 explain the construction of the system $\mathcal{G}$.

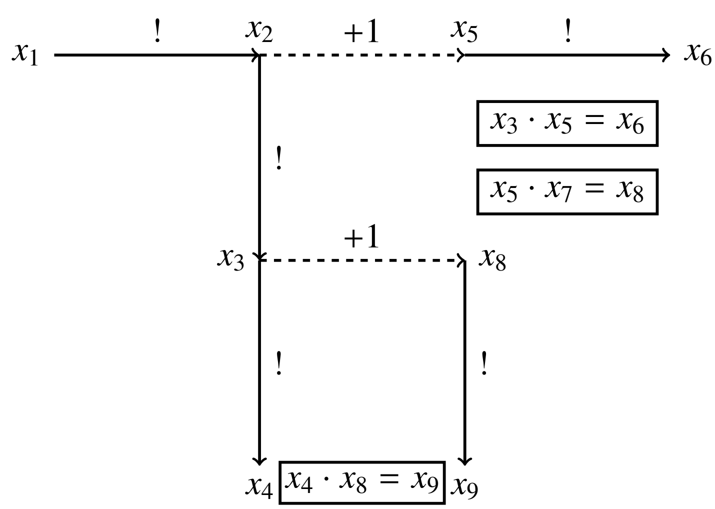

Fig. 4 Construction of the system $\mathcal{G}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by* *the following equalities:*

$$\begin{array}{rcl} x_2 & = & x_1! \\ x_3 & = & (x_1!)! \\ x_4 & = & ((x_1!)!)! \\ x_5 & = & x_1^! + 1 \\ x_6 & = & (x_1! + 1)! \\ x_7 & = & \dfrac{(x_1!)! + 1}{x_1! + 1} \\ x_8 & = & (x_1!)! + 1 \\ x_9 & = & ((x_1!)! + 1)! \end{array}$$

and $\min(x_1, \ldots, x_9) = x_1$.

Proof. By Lemmas 3 and 4, for every integer $x_1 \geqslant 2$, the system $\mathcal{G}$ is solvable in integers $x_2, \ldots, x_9$ greater than 1 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 5. □

Theorem 12. *The statement* Θ_9 *proves the following implication: if there exists an integer* $x_1 > f(9)$ *such that* $x_1! + 1$ *is prime, then there are infinitely many primes of the form* $n! + 1$.

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1! + 1$ is prime. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{G}$. Lemma 8 guarantees that $\min(x_1, \ldots, x_9) = x_1$. Since $\mathcal{G} \subseteq H_9$, the statement Θ_9 and the inequality $\min(x_1, \ldots, x_9) = x_1 > f(9)$ imply that the system $\mathcal{G}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 8, there are infinitely many primes of the form $n! + 1$. □

Hypothesis 4. *The implication in Theorem 12 is true.*

Corollary 6. *Assuming Hypothesis 4, a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form* $n! + 1$.

The following question is open: *Is it possible to effectively determine the largest prime number of the form* $n! + 1$*, if the set of these primes is finite?* The unproven statement Θ_9 implies this claim although does not imply that there are infinitely many primes of the form $n! + 1$.

6 The twin prime conjecture

Let C denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_4! &= x_5 \\ x_6! &= x_7 \\ x_7! &= x_8 \\ x_9! &= x_{10} \\ x_{12}! &= x_{13} \\ x_{15}! &= x_{16} \\ x_2 \cdot x_4 &= x_5 \\ x_5 \cdot x_6 &= x_7 \\ x_7 \cdot x_9 &= x_{10} \\ x_4 \cdot x_{11} &= x_{12} \\ x_3 \cdot x_{12} &= x_{13} \\ x_9 \cdot x_{14} &= x_{15} \\ x_8 \cdot x_{15} &= x_{16} \end{cases}$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system C.

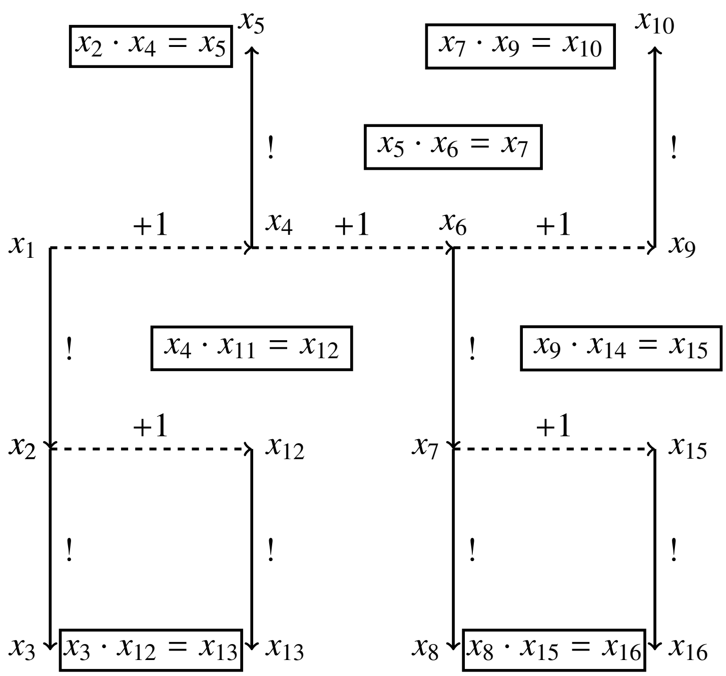

Fig. 5 Construction of the system C

Lemma 9. *If $x_4 = 2$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. □

Lemma 10. *If $x_4 = 3$, then the system C has no solutions in integers $x_1, \ldots, x_{16}$ greater than 1.*

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. □

Lemma 11. *For every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12},*

x_{13}, x_{14}, x_{15}, x_{16} are uniquely determined by the following equalities:

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

and $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3$.

Proof. By Lemmas 3 and 4, for every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} greater than 1 if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 11 follows from Lemma 5. □

Theorem 13. *The statement Θ_{16} proves the following implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.*

Proof. Assume that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 11, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0, 1\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Lemma 11 guarantees that $\min(x_1, \ldots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement Θ_{16} and the inequality $\min(x_1, \ldots, x_{16}) > f(16)$ imply that the system C has infinitely many solutions in integers $x_1, \ldots, x_{16}$ greater than 1. According to Lemmas 9–11, there are infinitely many twin primes. □

Hypothesis 5. *The implication in Theorem 13 is true.*

Corollary 7. *(cf. [5]). Assuming Hypothesis 5, a single query to an oracle for the halting problem decides the twin prime problem.*

The following question is open: *Is it possible to effectively determine the largest twin prime, if the set of twin primes is finite?* The unproven statement Θ_{16} implies this claim although does not imply that there are infinitely many twin primes.

7 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [13, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [13, p. 1].

Open Problem. ([13, p. 159]). *Are there infinitely many composite numbers of the form* $2^{2^n} + 1$*?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [12, p. 23].

Theorem 14. *([21]). An unproven inequality stated in [21] implies that* $2^{2^n} + 1$ *is composite for every integer* $n \geqslant 5$.

Lemma 12. *([13, p. 38]). For every positive integer n, if a prime number p divides* $2^{2^n} + 1$*, then there exists a positive integer k such that* $p = k \cdot 2^{n+1} + 1$.

Corollary 8. *Since* $k \cdot 2^{n+1} + 1 \geqslant 2^{n+1} + 1 \geqslant n + 3$*, for every positive integers x, y, and n, the equality* $(x + 1)(y + 1) = 2^{2^n} + 1$ *implies that* $\min(n, x, x + 1, y, y + 1) = n$.

Let

$$G_n = \left\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \dots, n\}\right\} \cup \left\{2^{2^{x_i}} = x_k : \ i, k \in \{1, \dots, n\}\right\}$$

Lemma 13. *The following subsystem of* G_n

$$\begin{cases} x_1 \cdot x_1 & = & x_1 \\ \forall i \in \{1, \dots, n-1\} \ 2^{2^{x_i}} & = & x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$*, namely* $(g(1), \dots, g(n))$.

For a positive integer n, let Ψ_n denote the following statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leqslant g(n)$. The assumption $\min(x_1, \ldots, x_n) \leqslant g(n)$ is weaker than the assumption $\max(x_1, \ldots, x_n) \leqslant g(n)$ suggested by Lemma 13.

Lemma 14. *For every positive integer n, the system G_n has a finite number of subsystems.*

Theorem 15. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 14. □

Lemma 15. *For every non-negative integers b and c, $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

Theorem 16. *The statement Ψ_{13} proves the following implication: if $2^{2^n} + 1$ is composite for some integer $n > g(13)$, then $2^{2^n} + 1$ is composite for infinitely many positive integers n.*

Proof. Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \qquad (1)$$

in positive integers. By Lemma 15, we can transform equation (1) into an equivalent system $\mathcal{F}$ which has 13 variables (x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 6.

$$
\begin{tikzcd}
 & x \arrow[d, "2^{2^{(\cdot)}}"] & x+1 \arrow[d, "2^{2^{(\cdot)}}"] \arrow[dddd, bend left=60, "\texttt{multiplying}"] \\
 & 2^{2^x} \arrow[r, "\texttt{squaring}"] & 2^{2^{x+1}} \\
 & y \arrow[d, "2^{2^{(\cdot)}}"] & y+1 \arrow[d, "2^{2^{(\cdot)}}"] \arrow[dd, bend left=40] \\
 & 2^{2^y} \arrow[r, "\texttt{squaring}"] & 2^{2^{y+1}} \\
z \arrow[r, "2^{2^{(\cdot)}}"] & 2^{2^z} \arrow[d, "2^{2^{(\cdot)}}"] & 2^{2^z}+1 \arrow[d, "2^{2^{(\cdot)}}"] \\
 & 2^{2^{2^{2^z}}} \arrow[r, "\texttt{squaring}"] & 2^{2^{2^{2^z}+1}}
\end{tikzcd}
$$

Fig. 6 Construction of the system $\mathcal{F}$

Assume that $2^{2^n} + 1$ is composite for some integer $n > g(13)$. By this and Corollary 8, equation (1) has a solution $(x, y, z) \in (\mathbb{N} \setminus \{0\})^3$ such that $z = n$ and $z = \min(z, x, x + 1, y, y + 1)$. Hence, the system $\mathcal{F}$ has a solution in positive integers such that $z = n$ and n is the smallest number in the solution sequence. Since $n > g(13)$, the statement Ψ_{13} implies that the system $\mathcal{F}$ has infinitely many solutions in positive integers. Therefore, there are infinitely many positive integers n such that $2^{2^n} + 1$ is composite. □

Hypothesis 6. *The implication in Theorem 16 is true.*

Corollary 9. *Assuming Hypothesis 6, a single query to an oracle for the halting problem decides whether or not the set of composite Fermat numbers is infinite.*

The following question is open: *Is it possible to effectively determine the largest composite Fermat number, if the set of these numbers is finite?* The unproven statement Ψ_{13} implies this claim although does not imply that there are infinitely many composite Fermat numbers.

8 Subsets of $\mathbb{N} \setminus \{0\}$ which are cofinite by Richert's lemma and the halting of a computer program

The following lemma is known as Richert's lemma.

Lemma 16. *([6], [18], [19, p. 152]). Let $\{m_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that for some positive integer k the inequality $m_{i+1} \leqslant 2m_i$ holds for all $i > k$. Suppose there exists a non-negative integer b such that the numbers $b+1,\ b+2,\ b+3,\ \ldots,\ b+m_{k+1}$ are all expressible as sums of one or more distinct elements of the set $\{m_1, \ldots, m_k\}$. Then every integer greater than b is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.*

Corollary 10. *If the sequence $\{m_i\}_{i=1}^{\infty}$ is computable and the flowchart algorithm in Figure 7 terminates, then almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$ and the algorithm returns all positive integers which are not expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.*

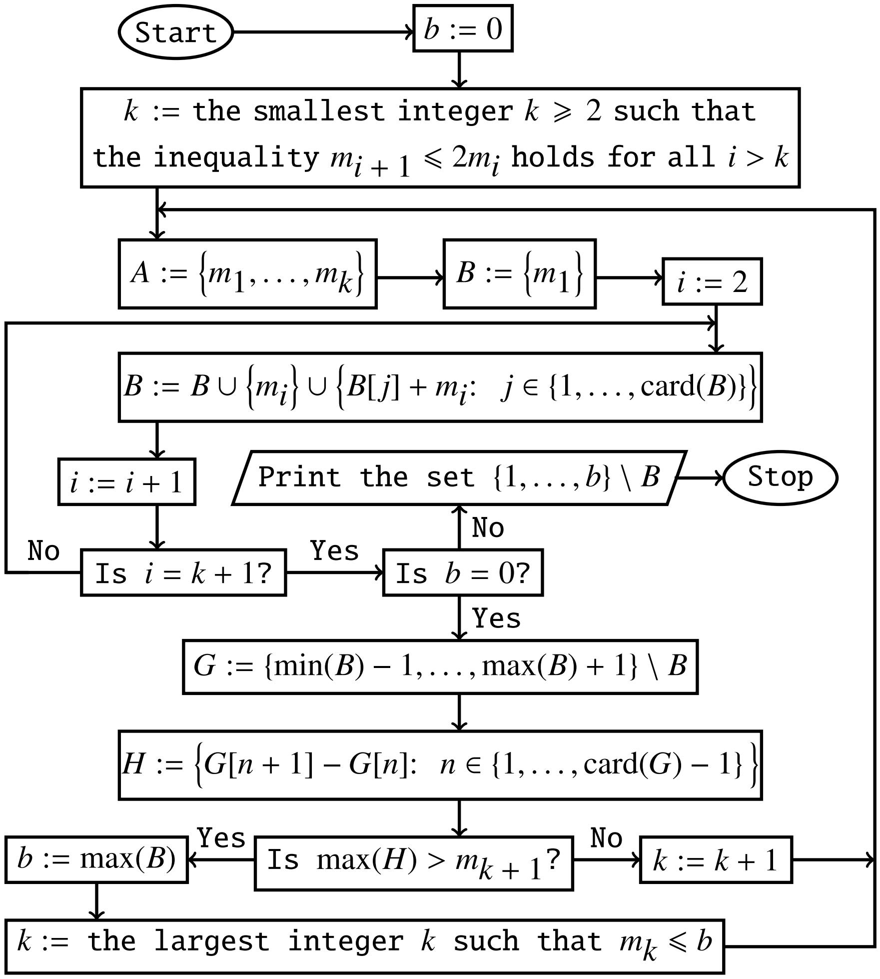

Fig. 7 The algorithm which uses Richert's lemma

The above algorithm works correctly because the inequality $\max(H) > m_{k+1}$ holds true if and only if the set B contains m_{k+1} consecutive integers.

Theorem 17. *([11, Theorem 2.3]). If there exists $\varepsilon > 0$ such that the inequality $m_{i+1} \leqslant (2-\varepsilon) \cdot m_i$ holds for every sufficiently large i, then the flowchart algorithm in Figure 7 terminates if and only if almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.*

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