# On sets $\mathcal{X} \subseteq \mathbb{N}$ for which we know an algorithm that computes a threshold number $t(\mathcal{X}) \in \mathbb{N}$ such that $\mathcal{X}$ is infinite if and only if there exists an element of $\mathcal{X}$ which is greater than $t(\mathcal{X})$

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**Abstract**

We define computable functions $g, h : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$. For an integer $n \geqslant 3$, let $\Psi_n$ denote the following statement: *if a system* $\mathcal{S} \subseteq \{x_i! = x_k : (i,k \in \{1,\ldots,n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,n\}\}$ *has only finitely many solutions in positive integers* $x_1,\ldots,x_n$*, then each such solution* $(x_1,\ldots,x_n)$ *satisfies* $x_1,\ldots,x_n \leqslant g(n)$. For a positive integer $n$, let $\Gamma_n$ denote the following statement: *if a system* $S \subseteq \{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,n\}\} \cup \{2^{2^{x_i}} = x_k : i,k \in \{1,\ldots,n\}\}$ *has only finitely many solutions in positive integers* $x_1,\ldots,x_n$*, then each such solution* $(x_1,\ldots,x_n)$ *satisfies* $x_1,\ldots,x_n \leqslant h(n)$. We prove: **(1)** if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers, then the statement $\Psi_6$ guarantees that each such solution $(x,y)$ belongs to the set $\{(4,5),(5,11),(7,71)\}$, **(2)** the statement $\Psi_9$ proves the following implication: if there exists a positive integer $x$ such that $x^2+1$ is prime and $x^2 + 1 > g(7)$, then there are infinitely many primes of the form $n^2+1$, **(3)** the statement $\Psi_9$ proves the following implication: if there exists an integer $x \geqslant g(6)$ such that $x! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$, **(4)** the statement $\Psi_{16}$ proves the following implication: if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes, **(5)** the statement $\Gamma_{13}$ proves the following implication: if $n \in \mathbb{N} \setminus \{0\}$ and $2^{2^n}+1$ is composite and greater than $h(12)$, then $2^{2^n}+1$ is composite for infinitely many positive integers $n$.

**Key words and phrases:** Brocard's problem, Brocard-Ramanujan equation, composite Fermat numbers, Dickson's conjecture, halting of a Turing machine, infinite subset of $\mathbb{N}$, prime numbers of the form $n^2+1$, prime numbers of the form $n!+1$, Richert's lemma, twin prime conjecture.

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## 1 Introduction

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [15, p. 39]. The following statement

(1) "For every non-negative integer $n$ there exist prime exist numbers $p$ and $q$ such that $p+2=q$ and $p \in [10^n, 10^{n+1}]$"

is a $\Pi_1$ statement which strengthens the twin prime conjecture, see [3, p. 43], cf. [5, pp. 337–338]. Statement (1) is equivalent to the non-halting of a Turing machine. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger $\Pi_1$ statements, see [1].

In this article, we study sets $\mathcal{X} \subseteq \mathbb{N}$ for which we know an algorithm that computes a threshold number $t(\mathcal{X}) \in \mathbb{N}$ such that $\mathcal{X}$ is infinite if and only if there exists an element of $\mathcal{X}$ which is greater than $t(\mathcal{X})$. If $\mathcal{X}$ is computable, then this property implies that the infinity of $\mathcal{X}$ is equivalent to the halting of a Turing machine. If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer $m$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $\{\max(\mathcal{X}), \max(\mathcal{X})+1, \max(\mathcal{X})+2, \ldots\}$.

**Theorem 1.** *([4, p. 35]). There exists a polynomial $D(x_1,\ldots,x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences* "The equation $D(x_1,\ldots,x_m)=0$ is solvable in non-negative integers" *and* "The equation $D(x_1,\ldots,x_m)=0$ is not solvable in non-negative integers *are not provable in ZFC.*

Let $\mathcal{Y}$ denote the set of all non-negative integers $k$ such that the equation $D(x_1, \ldots, x_m) = 0$ has no solutions in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Let $\gamma\colon \mathbb{N}^{m+1} \to \mathbb{N}$ be a computable bijection, and let $\mathcal{E} \subseteq \mathbb{N}^{m+1}$ be the solution set of the equation $D(x_1, \ldots, x_m) + 0 \cdot x_{m+1} = 0$. Theorem 1 implies Theorems 2 and 3.

**Theorem 2.** *If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences "$n$ is a threshold number of $\mathcal{Y}$" and "$n$ is not a threshold number of $\mathcal{Y}$" are not provable in ZFC.*

**Theorem 3.** *We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \gamma(\mathcal{E})$. The set $\gamma(\mathcal{E})$ is empty or infinite. In both cases, every non-negative integer $n$ is a threshold number of $\gamma(\mathcal{E})$. If ZFC is arithmetically consistent, then the sentences "$\gamma(\mathcal{E})$ is empty", "$\gamma(\mathcal{E})$ is not empty", "$\gamma(\mathcal{E})$ is finite", and "$\gamma(\mathcal{E})$ is infinite" are not provable in ZFC.*

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [16, p. 234].

**Corollary 1.** *If an algorithm $\mathtt{Alg}_1$ for every recursive set $\mathcal{R} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_1(\mathcal{R})$, then there exists a finite set $\mathcal{W} \subseteq \mathbb{N}$ such that $\mathcal{W} \cap [\mathtt{Alg}_1(\mathcal{W}) + 1, \infty) \neq \emptyset$. If an algorithm $\mathtt{Alg}_2$ for every recursively enumerable set $\mathcal{R} \subseteq \mathbb{N}$ finds a non-negative integer $\mathtt{Alg}_2(\mathcal{R})$, then there exists a finite set $\mathcal{W} \subseteq \mathbb{N}$ such that $\mathcal{W} \cap [\mathtt{Alg}_2(\mathcal{W}) + 1, \infty) \neq \emptyset$.*

## 2 Basic definitions and lemmas

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $h(1) = 1$, and let $h(n+1) = 2^{2^{h(n)}}$ for every positive integer $n$. Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geqslant 3$. For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{2\}\ x_i! &= x_{i+1} \\ x_1 \cdot x_2 &= x_3 \\ x_2 \cdot x_2 &= x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

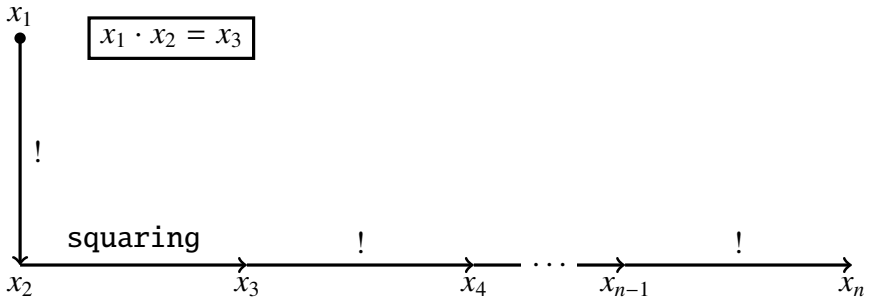


**Fig. 1** Construction of the system $\mathcal{U}_n$

**Lemma 1.** *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $\big(2, 2, g(3), \ldots, g(n)\big)$.*

Let

$$B_n = \Big\{x_i! = x_k :\ (i, k \in \{1, \ldots, n\}) \wedge (i \neq k)\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \ldots, n\}\Big\}$$

For an integer $n \geqslant 3$, let $\Psi_n$ denote the following statement: *if a system $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant g(n)$.* The statement $\Psi_n$ says that for subsystems of $B_n$ the largest known solution is indeed the largest possible.

**Hypothesis 1.** *The statements $\Psi_3, \ldots, \Psi_{16}$ are true.*

**Theorem 4.** *Every statement $\Psi_n$ is true with an unknown integer bound that depends on n.*

*Proof.* For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

**Theorem 5.** *For every statement $\Psi_n$, the bound $g(n)$ cannot be decreased.*

*Proof.* It follows from Lemma 1 because $\mathcal{U}_n \subseteq B_n$. □

**Lemma 2.** *For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 3.** *For every positive integers $x$ and $y$, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** *For every positive integers $x$ and $y$, $x + 1 = y$ if and only if*

$$(1 \neq y) \wedge (x! \cdot y = y!)$$

**Lemma 5.** *For every non-negative integers $b$ and $c$, $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

**Lemma 6.** *(Wilson's theorem, [7, p. 89]). For every integer $x \geqslant 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.*

## 3 Heuristic arguments against the statement $\forall n \in \mathbb{N} \setminus \{0, 1, 2\}\ \Psi_n$

Let

$$G_n = \{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \ldots, n\}\} \cup \{x_i + 1 = x_k :\ i, k \in \{1, \ldots, n\}\}$$

**Hypothesis 2.** *([25, p. 109]. If a system $\mathcal{S} \subseteq G_n$ has only finitely many solutions in non-negative integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant h(2n)$.*

**Hypothesis 3.** *If a system $\mathcal{S} \subseteq G_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(2n)$.*

Observations 1 and 2 heuristically justify Hypothesis 3.

**Observation 1.** *(cf. [25, p. 110, Observation 1]). For every system $\mathcal{S} \subseteq G_n$ which involves all the variables $x_1, \ldots, x_n$, the following new system*

$$\left( \bigcup_{x_i \cdot x_j = x_k \in S} \{x_i \cdot x_j = x_k\} \right) \cup \{x_k! = y_k :\ k \in \{1, \ldots, n\}\} \cup \left( \bigcup_{x_i + 1 = x_k \in S} \{1 \neq x_k,\ y_i \cdot x_k = y_k\} \right)$$

*is equivalent to $\mathcal{S}$. If the system $\mathcal{S}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then the new system has only finitely many solutions in positive integers $x_1, \ldots, x_n, y_1, \ldots, y_n$.*

*Proof.* It follows from Lemma 4. □

**Observation 2.** *The equation $x_1! = x_1$ has exactly two solutions in positive integers, namely $x_1 = 1$ and $x_1 = f(1)$. The system $\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \end{cases}$ has exactly two solutions in positive integers, namely $(1, 1)$ and $(f(1), f(2))$. For every integer $n \geqslant 3$, the following system*

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

*has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.*

For a positive integer $n$, let $\Phi_n$ denote the following statement: *if a system*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_i! = x_k : i, k \in \{1, \ldots, n\}\} \cup \{1 \neq x_k : k \in \{1, \ldots, n\}\}$$

*has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.*

**Theorem 6.** *The statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Phi_n$ implies Hypothesis 3.*

*Proof.* It follows from Lemma 4. □

Let $\mathcal{R}ng$ denote the class of all rings $\boldsymbol{K}$ that extend $\mathbb{Z}$, and let

$$E_n = \{1 = x_k : k \in \{1, \ldots, n\}\} \cup \{x_i + x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

Th. Skolem proved that every Diophantine equation can be algorithmically transformed into an equivalent system of Diophantine equations of degree at most 2, see [21, pp. 2–3] and [12, pp. 3–4]. The following result strengthens Skolem's theorem.

**Lemma 7.** *([23, p. 720]). Let $D(x_1, \ldots, x_p) \in \mathbb{Z}[x_1, \ldots, x_p]$. Assume that $\deg(D, x_i) \geqslant 1$ for each $i \in \{1, \ldots, p\}$. We can compute a positive integer $n > p$ and a system $T \subseteq E_n$ which satisfies the following two conditions:*

`Condition 1.` *If $\boldsymbol{K} \in \mathcal{R}ng \cup \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$, then*

$$\forall \tilde{x}_1, \ldots, \tilde{x}_p \in \boldsymbol{K} \left( D(\tilde{x}_1, \ldots, \tilde{x}_p) = 0 \Longleftrightarrow \exists \tilde{x}_{p+1}, \ldots, \tilde{x}_n \in \boldsymbol{K}\ (\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{x}_{p+1}, \ldots, \tilde{x}_n) \text{ solves } T \right)$$

`Condition 2.` *If $\boldsymbol{K} \in \mathcal{R}ng \cup \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$, then for each $\tilde{x}_1, \ldots, \tilde{x}_p \in \boldsymbol{K}$ with $D(\tilde{x}_1, \ldots, \tilde{x}_p) = 0$, there exists a unique tuple $(\tilde{x}_{p+1}, \ldots, \tilde{x}_n) \in \boldsymbol{K}^{n-p}$ such that the tuple $(\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{x}_{p+1}, \ldots, \tilde{x}_n)$ solves $T$.*

*Conditions 1 and 2 imply that for each $\boldsymbol{K} \in \mathcal{R}ng \cup \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$, the equation $D(x_1, \ldots, x_p) = 0$ and the system $T$ have the same number of solutions in $\boldsymbol{K}$.*

Let $\alpha, \beta$, and $\gamma$ denote variables.

**Lemma 8.** *([19, p. 100]) For each positive integers $x, y, z$, $x + y = z$ if and only if*

$$(zx + 1)(zy + 1) = z^2(xy + 1) + 1$$

**Corollary 2.** *We can express the equation $x + y = z$ as an equivalent system $\mathcal{F}$, where $\mathcal{F}$ involves $x, y, z$ and 9 new variables, and where $\mathcal{F}$ consists of equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$.*

*Proof.* The new 9 variables express the following polynomials:

$$zx, \quad zx + 1, \quad zy, \quad zy + 1, \quad z^2, \quad xy, \quad xy + 1, \quad z^2(xy + 1), \quad z^2(xy + 1) + 1$$

□

**Lemma 9.** *(cf. [25, p. 110, Lemma 4]). Let $D(x_1, \ldots, x_p) \in \mathbb{Z}[x_1, \ldots, x_p]$. Assume that $\deg(D, x_i) \geqslant 1$ for each $i \in \{1, \ldots, p\}$. We can compute a positive integer $n > p$ and a system $T \subseteq G_n$ which satisfies the following two conditions:*

`Condition 3.` *For every positive integers $\tilde{x}_1, \ldots, \tilde{x}_p$,*

$$D(\tilde{x}_1, \ldots, \tilde{x}_p) = 0 \Longleftrightarrow \exists \tilde{x}_{p+1}, \ldots, \tilde{x}_n \in \mathbb{N} \setminus \{0\}\ (\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{x}_{p+1}, \ldots, \tilde{x}_n) \text{ solves } T$$

`Condition 4.` *If positive integers $\tilde{x}_1, \ldots, \tilde{x}_p$ satisfy $D(\tilde{x}_1, \ldots, \tilde{x}_p) = 0$, then there exists a unique tuple $(\tilde{x}_{p+1}, \ldots, \tilde{x}_n) \in (\mathbb{N} \setminus \{0\})^{n-p}$ such that the tuple $(\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{x}_{p+1}, \ldots, \tilde{x}_n)$ solves $T$.*

*Conditions 3 and 4 imply that the equation $D(x_1, \ldots, x_p) = 0$ and the system $T$ have the same number of solutions in positive integers.*

*Proof.* Let the system $T$ be given by Lemma 7. We replace in $T$ each equation of the form $1 = x_k$ by the equation $x_k \cdot x_k = x_k$. Next, we apply Corollary 2 and replace in $T$ each equation of the form $x_i + x_j = x_k$ by an equivalent system of equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$. □

**Theorem 7.** *Hypothesis 3 implies that there is an algorithm which takes as input a Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set.*

*Proof.* It follows from Lemma 9. □

**Open Problem 1.** *Is there an algorithm which takes as input a Diophantine equation, and returns an integer such that this integer is greater than the moduli of integer (non-negative integer, positive integer) solutions, if the solution set is finite?*

Matiyasevich's conjecture on finite-fold Diophantine representations ([14]) implies a negative answer to Open Problem 1, see [13, p. 42].

The statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Phi_n$ implies that there is an algorithm which takes as input a factorial Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is a bit strange because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [11, p. 300].

## 4 Brocard's problem

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

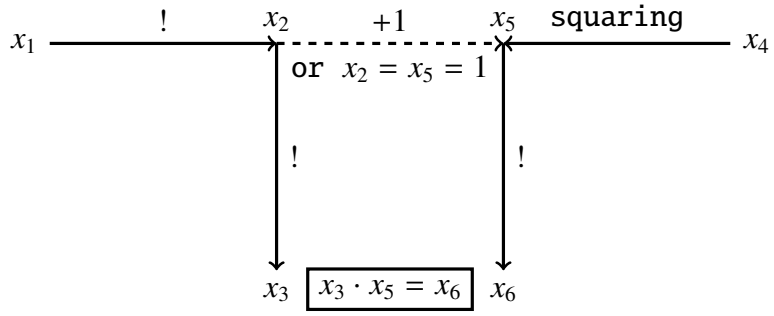


**Fig. 2** Construction of the system $\mathcal{A}$

**Lemma 10.** *For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system $\mathcal{A}$ is solvable in positive integers $x_2$, $x_3$, $x_5$, $x_6$ if and only if $x_1! + 1 = x_4^2$. In this case, the integers $x_2, x_3, x_5, x_6$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

*Proof.* It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [26, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [17].

**Theorem 8.** *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement $\Psi_6$ guarantees that each such solution $(x_1, x_4)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

*Proof.* Suppose that the antecedent holds. Let positive integers $x_1$ and $x_4$ satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 10, the system $\mathcal{A}$ is solvable in positive integers $x_2, x_3, x_5, x_6$. Since $\mathcal{A} \subseteq B_6$, the statement $\Psi_6$ implies that $x_6 = (x_1! + 1)! \leqslant g(6) = g(5)!$. Hence, $x_1! + 1 \leqslant g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

## 5 Are there infinitely many prime numbers of the form $n^2 + 1$?

Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

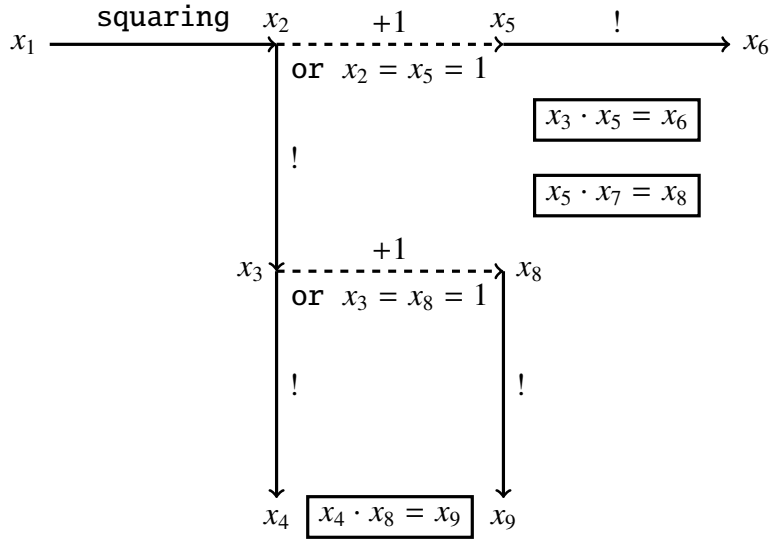


**Fig. 3** Construction of the system $\mathcal{B}$

**Lemma 11.** *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

*Proof.* By Lemma 2, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 11 follows from Lemma 6. □

**Lemma 12.** *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.*

*Proof.* If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [15, pp. 37–38].

**Theorem 9.** *The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

*Proof.* Suppose that the antecedent holds. By Lemma 11, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geqslant g(7)$. Hence, $(x_1^2)! \geqslant g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement $\Psi_9$ and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 11 and 12, there are infinitely many primes of the form $n^2 + 1$. □

## 6 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [2, p. 443] and [22].

**Theorem 10.** *The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geqslant g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

*Proof.* We leave the analogous proof to the reader. □

## 7 The twin prime conjecture and Dickson's conjecture

Let $C$ denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{cases}$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system $C$.

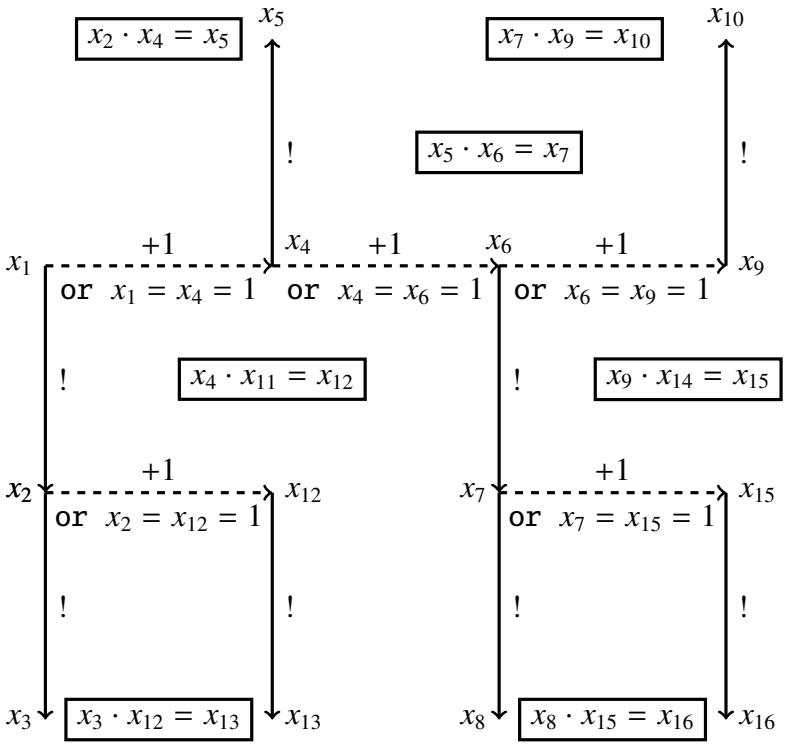


**Fig. 4** Construction of the system $C$

**Lemma 13.** *For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system $C$ is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if $x_4$ and $x_9$ are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

*Proof.* By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system $C$ is solvable in positive integers $x_1$, $x_2$, $x_3$, $x_5$, $x_6$, $x_7$, $x_8$, $x_{10}$, $x_{11}$, $x_{12}$, $x_{13}$, $x_{14}$, $x_{15}$, $x_{16}$ if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 13 follows from Lemma 6. □

**Lemma 14.** *There are only finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system $C$ and satisfy*

$$(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$$

*Proof.* If a tuple $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system $C$ and

$$(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$$

then $x_1, \ldots, x_{16} \leqslant 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. □

**Theorem 11.** *The statement $\Psi_{16}$ proves the following implication: $(*)$ if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.*

*Proof.* Suppose that the antecedent holds. Then, there exist prime numbers $x_4$ and $x_9$ such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma 13, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system $C$. Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geqslant g(14)$. Therefore, $(x_9 - 1)! \geqslant g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement $\Psi_{16}$ and the inequality $x_{16} > g(16)$ imply that the system $C$ has infinitely many solutions in positive integers $x_1, \ldots, x_{16}$. According to Lemmas 13 and 14, there are infinitely many twin primes. □

Let $\mathbb{P}(x)$ denote the predicate "*$x$ is a prime number*". Dickson's conjecture ([15, p. 36], [27, p. 109]) implies that the existential theory of $(\mathbb{N}, =, +, \mathbb{P})$ is decidable, see [27, Theorem 2, p. 109]. For a positive integer $n$, let $\Theta_n$ denote the following statement: *for every system $\mathcal{S} \subseteq \{x_i + 1 = x_j : i, j \in \{1, \ldots, n\}\} \cup \{\mathbb{P}(x_i) : i \in \{1, \ldots, n\}\}$ the solvability of $\mathcal{S}$ in non-negative integers is decidable.*

**Lemma 15.** *If the existential theory of $(\mathbb{N}, =, +, \mathbb{P})$ is decidable, then the statements $\Theta_n$ are true.*

*Proof.* For every non-negative integers $x$ and $y$, $x + 1 = y$ if and only if

$$\exists u, v \in \mathbb{N} \ ((u + u = v) \wedge \mathbb{P}(v) \wedge (x + u = y))$$

□

**Theorem 12.** *The conjunction of the implication $(*)$ and the statement $\Theta_{g(14)+2}$ implies that the twin prime conjecture is decidable.*

*Proof.* By the statement $\Theta_{g(14)+2}$, we can decide the truth of the sentence

$$\exists x_1 \ldots \exists x_{g(14)+2} \left( (\forall i \in \{1, \ldots, g(14) + 1\} \ x_i + 1 = x_{i+1}) \wedge \mathbb{P}(x_{g(14)}) \wedge \mathbb{P}(x_{g(14)+2}) \right) \qquad (2)$$

If sentence (2) is false, then the twin prime conjecture is false. If sentence (2) is true, then there exists a twin prime greater than $g(14)$. In this case, the twin prime conjecture follows from Theorem 11. □

## 8 A hypothesis which implies that any prime number $p > 24$ proves that there are infinitely many prime numbers

For a positive integer $n$, let $\Gamma(n)$ denote $(n - 1)!$. Let $\lambda(5) = \Gamma(5)$, and let $\lambda(n + 1) = \Gamma(\lambda(n))$ for every integer $n \geqslant 5$. For an integer $n \geqslant 5$, let $\mathcal{J}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n - 1\} \setminus \{3\} \ \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{cases}$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system $\mathcal{J}_n$.

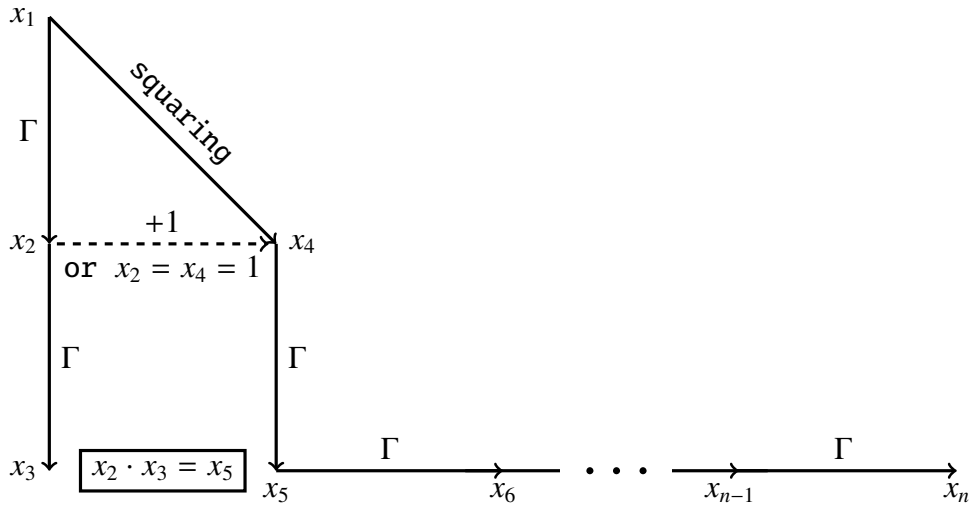


**Fig. 5** Construction of the system $\mathcal{J}_n$

**Observation 3.** *For every integer $n \geqslant 5$, the system $\mathcal{J}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$.

For an integer $n \geqslant 5$, let $\Delta_n$ denote the following statement: *if a system* $\mathcal{S} \subseteq \left\{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\right\} \cup \left\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\right\}$ *has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant \lambda(n)$.*

**Hypothesis 4.** *The statements $\Delta_5, \ldots, \Delta_{14}$ are true.*

Lemmas 3 and 6 imply that the statements $\Delta_n$ have essentially the same consequences as the statements $\Psi_n$.

**Theorem 13.** *The statement $\Delta_6$ implies that any prime number $p > 24$ proves that there are infinitely many prime numbers.*

*Proof.* It follows from Lemmas 3 and 6. We leave the details to the reader. □

## 9 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [10, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [10, p. 1].

**Open Problem 2.** *([10, p. 159]). Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [9, p. 23].

**Theorem 14.** *([24]). An unproven inequality stated in [24] implies that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$.*

Let

$$H_n = \left\{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\right\} \cup \left\{2^{2^{x_i}} = x_k : \ i, k \in \{1, \ldots, n\}\right\}$$

**Lemma 16.** *The following subsystem of $H_n$*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \ldots, n-1\} \ 2^{2^{x_i}} = x_{i+1} \end{cases}$$

*has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \ldots, h(n))$.*

For a positive integer $n$, let $\Gamma_n$ denote the following statement: *if a system $S \subseteq H_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant h(n)$*. The statement $\Gamma_n$ says that for subsystems of $H_n$ the largest known solution is indeed the largest possible.

**Hypothesis 5.** *The statements $\Gamma_1, \ldots, \Gamma_{13}$ are true.*

The truth of the statement $\forall n \in \mathbb{N} \setminus \{0\}\, \Gamma_n$ is doubtful because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [11, p. 300].

**Lemma 17.** *For every positive integer $n$, the system $H_n$ has a finite number of subsystems.*

**Theorem 15.** *Every statement $\Gamma_n$ is true with an unknown integer bound that depends on $n$.*

*Proof.* It follows from Lemma 17. □

**Theorem 16.** *The statement $\Gamma_{13}$ proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers $z$.*

*Proof.* Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{3}$$

in positive integers. By Lemma 5, we can transform equation (3) into an equivalent system $\mathcal{G}$ which has 13 variables ($x$, $y$, $z$, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 6.

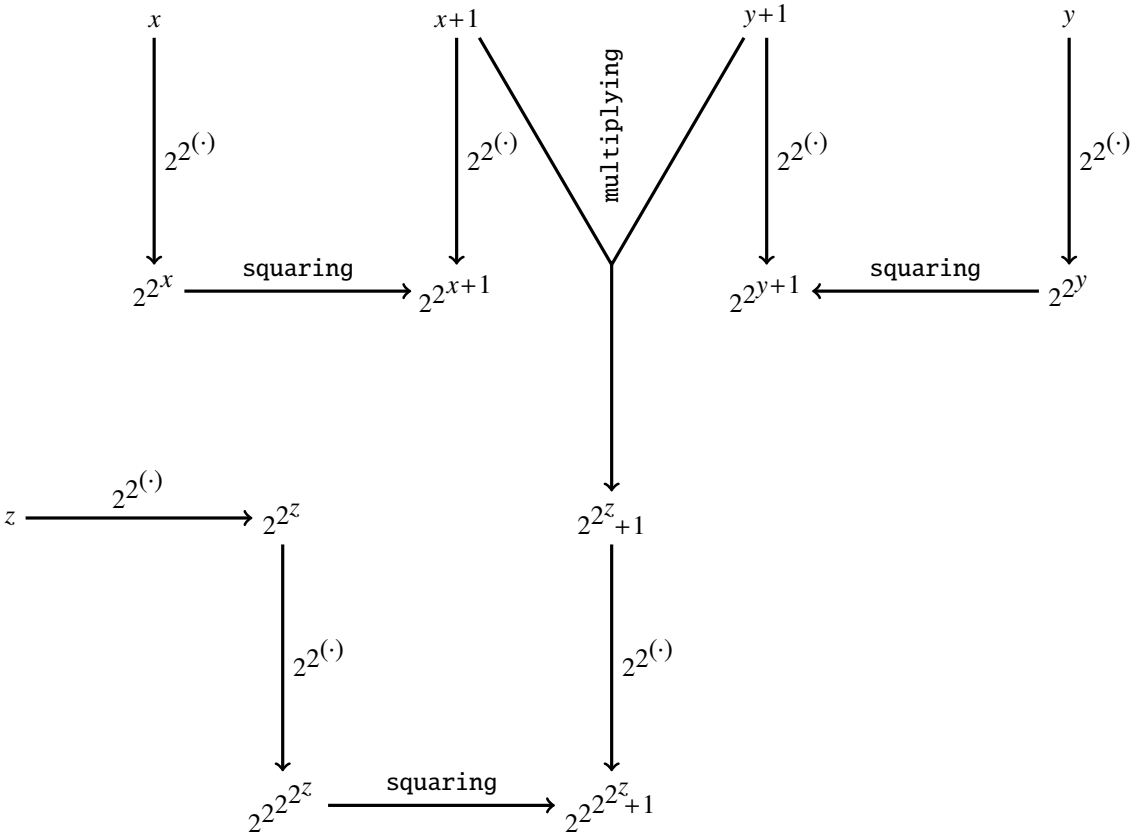


**Fig. 6** Construction of the system $\mathcal{G}$

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z}+1}} > h(13)$. By this, the statement $\Gamma_{13}$ implies that the system $\mathcal{G}$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

## 10 Subsets of $\mathbb{N}$ whose infinitude is unconditionally equivalent to the halting of a Turing machine

The following lemma is known as Richert's lemma.

**Lemma 18.** *([6], [18], [20, p. 152]). Let $\{m_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that for some positive integer $k$ the inequality $m_{i+1} \leqslant 2m_i$ holds for all $i > k$. Suppose there exists a non-negative integer $b$ such that the numbers $b+1,\ b+2,\ b+3,\ \ldots,\ b+m_{k+1}$ are all expressible as sums of one or more distinct elements of the set $\{m_1, \ldots, m_k\}$. Then every integer greater than $b$ is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.*

Let $\mathcal{T}$ denote the set of all positive integers $i$ such that every integer $j \geqslant i$ is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$. Obviously, $\mathcal{T} = \emptyset$ or $\mathcal{T} = [d, \infty) \cap \mathbb{N}$ for some positive integer $d$.

**Corollary 3.** *If the sequence $\{m_i\}_{i=1}^{\infty}$ is computable and the algorithm in Figure 7 terminates, then almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$. In particular, if the sequence $\{m_i\}_{i=1}^{\infty}$ is computable and the algorithm in Figure 7 terminates, then the set $\mathcal{T}$ is infinite.*

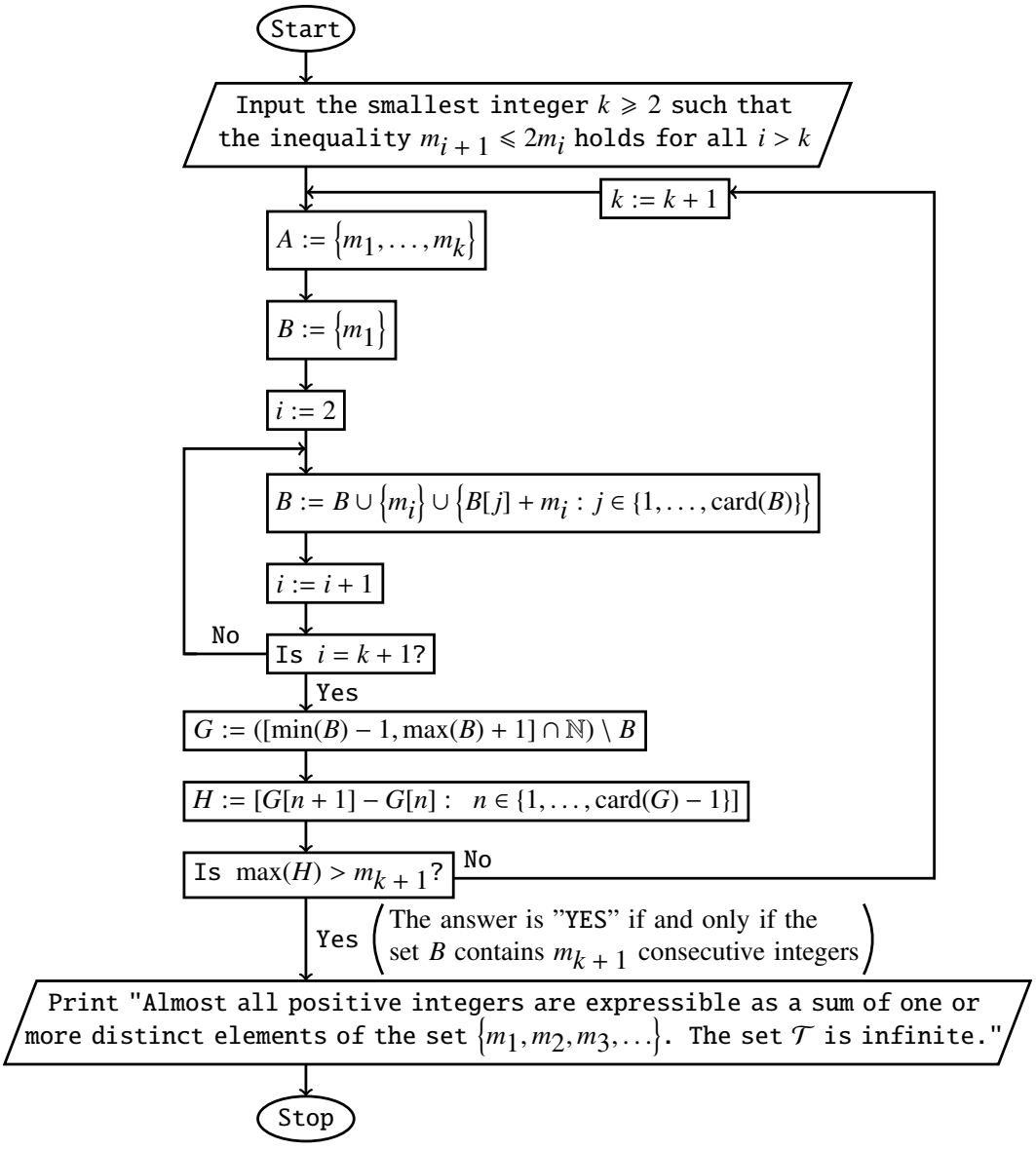


**Fig. 7** The algorithm which uses Richert's lemma

**Theorem 17.** *([8, Theorem 2.3]). If there exists $\varepsilon > 0$ such that the inequality $m_{i+1} \leqslant (2-\varepsilon) \cdot m_i$ holds for every sufficiently large $i$, then the algorithm in Figure 7 terminates if and only if almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.*

**Corollary 4.** *If there exists $\varepsilon > 0$ such that the inequality $m_{i+1} \leqslant (2-\varepsilon) \cdot m_i$ holds for every sufficiently large $i$, then the algorithm in Figure 7 terminates if and only if the set $\mathcal{T}$ is infinite.*

We show how the algorithm in Figure 7 works for a concrete sequence $\{m_i\}_{i=1}^{\infty}$. Let $[\cdot]$ denote the integer part function. For a positive integer $i$, let $t_i = \dfrac{(i+19)^i + 19}{(i+19)! \cdot 2^i + 19}$, and let $m_i = [t_i]$.

**Lemma 19.** *The inequality $m_{i+1} \leqslant 2m_i$ holds for every positive integer $i$.*

*Proof.* For every positive integer $i$,

$$\frac{m_i}{m_{i+1}} = \frac{[t_i]}{[t_{i+1}]} > \frac{t_i - 1}{t_{i+1}} = \frac{t_i}{t_{i+1}} - \frac{1}{t_{i+1}} \geqslant \frac{t_i}{t_{i+1}} - \frac{1}{t_2} =$$

$$2 \cdot \frac{i+20}{i+19} \cdot \left(1 - \frac{1}{i+20}\right)^{i+20} - \frac{21! \cdot 2^{21}}{21^{21}} > 2 \cdot \left(1 - \frac{1}{21}\right)^{21} - \frac{21! \cdot 2^{21}}{21^{21}} = \frac{4087158528442715204485120000}{5842587018385982521381124421}$$

The last fraction was computed by *MuPAD* and is greater than $\frac{1}{2}$. □

**Theorem 18.** *The algorithm in Figure 7 terminates for the sequence $\{m_i\}_{i=1}^{\infty}$.*

*Proof.* By Lemma 19, we take $k = 2$ as the initial value of $k$. The following *MuPAD* code

```
k:=2:
repeat
C:={floor((i+19)^(i+19)/((i+19)!*2^(i+19))) $i=1..k+1}:
A:={floor((i+19)^(i+19)/((i+19)!*2^(i+19))) $i=1..k}:
B:={A[1]}:
for i from 2 to nops(A) do
B:=B union {A[i]} union {B[j]+A[i] $j=1..nops(B)}:
end_for:
G:={y $y=B[1]-1..B[nops(B)]+1} minus B:
H:={G[n+1]-G[n] $n=1..nops(G)-1}:
k:=k+1:
until H[nops(H)]>C[nops(C)] end_repeat:
print(Unquoted, "Almost all positive integers are expressible"):
print(Unquoted, "as a sum of one or more distinct elements of"):
print(Unquoted, "the set {m_1,m_2,m_3,...}. The set T is infinite."):
```

implements the algorithm in Figure 7 because *MuPAD* automatically orders every finite set of integers and the inequality `H[nops(H)]>C[nops(C)]` holds true if and only if the set $B$ contains $m_{k+1}$ consecutive integers. The author checked that the execution of the code terminates. □

*MuPAD* is a general-purpose computer algebra system. *MuPAD* is no longer available as a stand-alone computer program, but only as the *Symbolic Math Toolbox* of *MATLAB*. Fortunately, the presented code can be executed by *MuPAD Light*, which was offered for free for research and education until autumn 2005.

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