Is it possible to compute an integer d such that any twin prime greater than d proves that the set of twin primes is infinite?

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Abstract

For a positive integer, let $\Gamma(n)$ denote $(n-1)!$. Let $f(5) = 24!$, and let $f(n+1) = \Gamma(f(n))$ for every integer $n \geqslant 5$. For an integer $n \geqslant 5$, let $T(n)$ denote the statement: if a system of equations $\mathcal{S} \subseteq \left\{\Gamma(x_i) = x_k :\ i,k \in \{1,\dots,n\}\right\} \cup \left\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\dots,n\}\right\}$ has at most finitely many solutions in positive integers $x_1,\dots,x_n$, then each such solution $(x_1,\dots,x_n)$ satisfies $\min(x_1,\dots,x_n) \leqslant f(n)$. We conjecture that the statements $T(5),\dots,T(14)$ are true. The statement $T(6)$ implies that if $x!+1$ is a square for at most finitely many non-negative integers x then each such x satisfies $x \leqslant f(6)$. The statement $T(9)$ proves the implication: if there exists an integer $x > f(9)$ such that x^2+1 is prime, then there are infinitely many primes of the form n^2+1. The statement $T(14)$ proves the implication: if there exists a twin prime greater than $f(14)+2$, then there are infinitely many twin primes.

Key words and phrases: Brocard's problem, Brocard-Ramanujan equation, prime numbers of the form n^2+1, single query to an oracle for the halting problem, twin prime conjecture.

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1. Introduction and basic lemmas

In this article, we study a conjecture which applies to Brocard's problem, the problem of the infinitude of primes of the form n^2+1, and the twin prime problem. The conjecture allows us to compute an integer b_6 such that if $x!+1$ is a square for at most finitely many non-negative integers x then each such x satisfies $x \leqslant b_6$. The conjecture allows us to compute an integer b_9 such that any prime number of the form n^2+1 which is greater than b_9 proves that the set of prime numbers of the form n^2+1 is infinite. The conjecture allows us to compute an integer b_{14} such that any twin prime greater than $b_{14}+2$ proves that the set of twin primes is infinite.

For a positive integer, let $\Gamma(n)$ denote $(n-1)!$.

Lemma 1. *For every positive integers x and y, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 2. *(Wilson's theorem, [1, p. 89]). For every integer $x \geqslant 2$, x is prime if and only if x divides* $\Gamma(x)+1$.

Lemma 3. *For every integer $x \geqslant 5$, we have* $x \leqslant \sqrt{\Gamma(x)+1}$.

Lemma 4. *For every integer $x \geqslant 5$, we have $x \leqslant \frac{\Gamma(x)+1}{x}$.*

2. A conjecture on the statements $\Psi(n,b)$

For a positive integer n, let G_n denote the following system of equations:

$$\Big\{\Gamma(x_i) = x_k : \ i,k \in \{1,\ldots,n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\ldots,n\}\Big\}$$

For positive integers n and b, let $\Psi(n,b)$ denote the statement: if a system $\mathcal{S} \subseteq G_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$ then each such solution $(x_1,\ldots,x_n)$ satisfies $\min(x_1,\ldots,x_n) \leqslant b$.

Theorem 1. *For every positive integer n, there exists an integer $b \geqslant 4$ such that the statement $\Psi(n,b)$ is true.*

Proof. It follows from the fact that the system G_n has a finite number of subsystems. □

Let $f(5) = 24!$, and let $f(n+1) = \Gamma(f(n))$ for every integer $n \geqslant 5$. For an integer $n \geqslant 5$, let $\mathcal{U}_n \subseteq G_n$ be the system of equations illustrated in Figure 1. Lemma 1 explains the construction of the system $\mathcal{U}_n$.

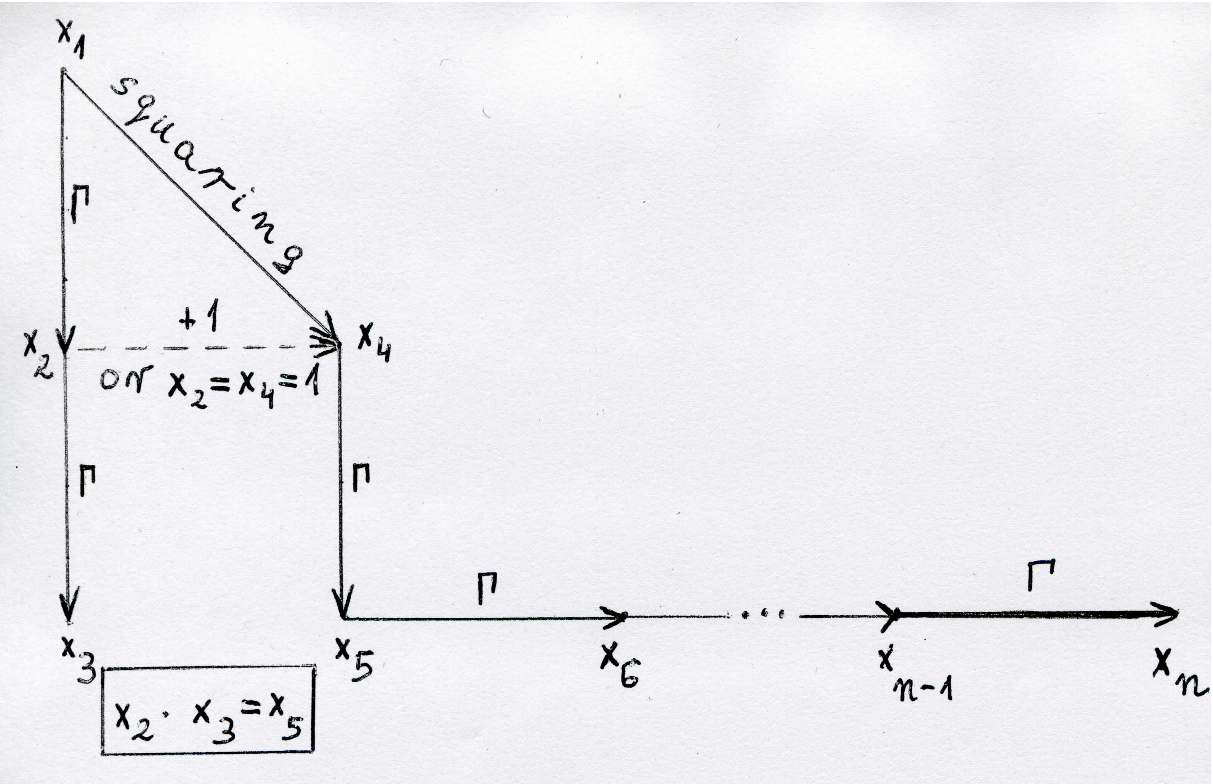

Fig. 1 Construction of the system $\mathcal{U}_n$

For every integer $n \geqslant 5$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1,\ldots,1)$ and $(5, 24, 23!, 25, f(5), \ldots, f(n))$.

Conjecture. *For every integer $n \in \{5,\ldots,14\}$, the statement $\Psi(n, f(n))$ is true.*

3. Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $\Gamma(x) + 1 = y^2$, see [3]. It is conjectured that $\Gamma(x) + 1$ is a square only for $x \in \{5, 6, 8\}$, see [4, p. 297].

Let $\mathcal{A} \subseteq G_6$ be the system of equations illustrated in Figure 2. Lemma 1 explains the construction of the system $\mathcal{A}$.

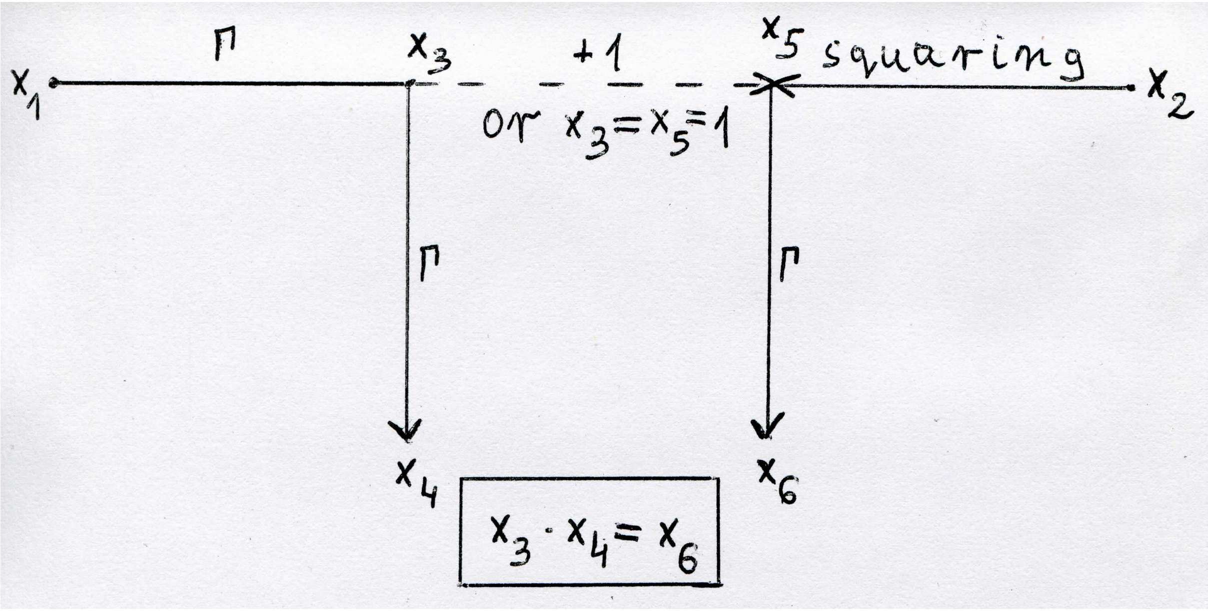

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 5. *The system $\mathcal{A}$ has only finitely many solutions $(x_1, \dots, x_6) \in (\mathbb{N} \setminus \{0\})^6$ with $x_1 \in \{1, 2\}$. For every integer $x_1 \geqslant 3$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \dots, x_6$ if and only if $\Gamma(x_1) + 1$ is a square. In this case, $x_1 \geqslant 5$, the numbers $x_2, \dots, x_6$ are uniquely determined by x_1, and $x_1 = \min(x_1, \dots, x_6)$ (which follows from $x_1 \geqslant 5$ and Lemma 3).*

Proof. It follows from Lemma 1. □

Theorem 2. *For every positive integer b, if $\Gamma(x_1) + 1$ is a square for at most finitely many positive integers x_1, then the statement $\Psi(6, b)$ implies that each such x_1 satisfies $x_1 \leqslant b$.*

Proof. Let us assume that for a positive integer x_1 there exists a positive integer x_2 such that $\Gamma(x_1) + 1 = x_2^2$. Then, $x_1 \geqslant 5$. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_6) \in (\mathbb{N} \setminus \{0\})^5$ such that the tuple $(x_1, \dots, x_6)$ solves the system $\mathcal{A}$. Lemma 5 guarantees that $x_1 = \min(x_1, \dots, x_6)$. By the antecedent and Lemma 5, the system $\mathcal{A}$ has only finitely many solutions in positive integers $x_1, \dots, x_6$. Therefore, the statement $\Psi(6, b)$ implies that $x_1 = \min(x_1, \dots, x_6) \leqslant b$. □

4. Are there infinitely many prime numbers of the form $n^2 + 1$?

Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [2, pp. 37–38].

Let $\mathcal{B} \subseteq G_9$ be the system of equations illustrated in Figure 3. Lemma 1 explains the construction of the system $\mathcal{B}$.

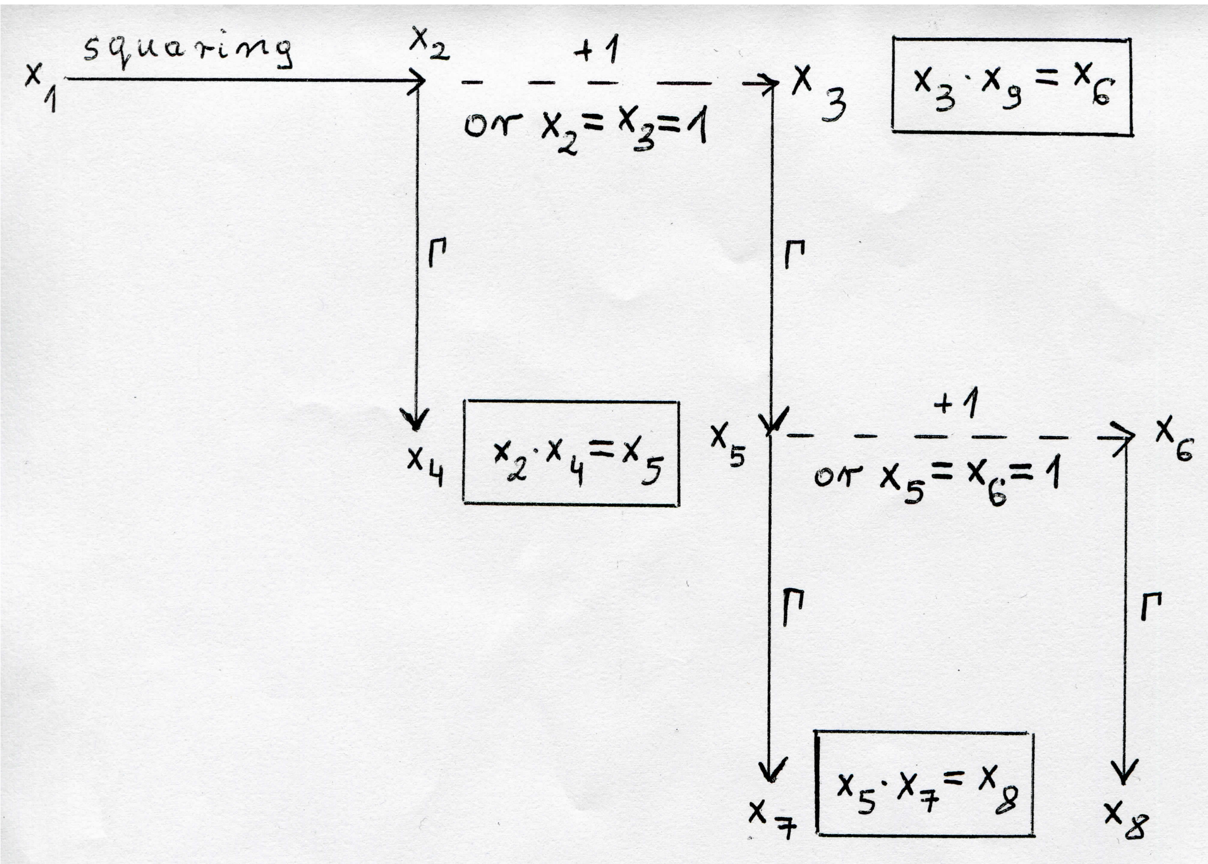

Fig. 3 Construction of the system $\mathcal{B}$

Lemma 6. *The system $\mathcal{B}$ has only finitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ with $x_1 = 1$. For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the numbers $x_2, \ldots, x_9$ are uniquely determined by x_1, and $x_1 = \min(x_1, \ldots, x_9)$.*

Proof. By Lemma 1, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $\Gamma(x_1^2 + 1) + 1$. By Lemma 2, the last is true if and only if $x_1^2 + 1$ is prime. The inequality $x_1 \geqslant 2$ and Lemma 4 imply that $x_1 = \min(x_1, \ldots, x_9)$. □

Theorem 3. *For every integer $b \geqslant 4$, the statement $\Psi(9, b)$ proves the implication: if there exists an integer $x_1 > b$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Let us assume that an integer x_1 is greater than b and $x_1^2 + 1$ is prime. Since $b \geqslant 4$, we obtain that $x_1 \geqslant 5$. By Lemma 6, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, \ldots, x_9)$ solves the system $\mathcal{B}$. Lemma 6 guarantees that $x_1 = \min(x_1, \ldots, x_9)$. Since $\mathcal{B} \subseteq G_9$, we obtain that the statement $\Psi(9, b)$ and the inequality $b < x_1 = \min(x_1, \ldots, x_9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemma 6, there are infinitely many primes of the form $n^2 + 1$. □

5. The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [2, p. 39].

Let $\mathcal{C} \subseteq G_{14}$ be the system of equations illustrated in Figure 4. Lemma 1 explains the construction of the system $\mathcal{C}$.

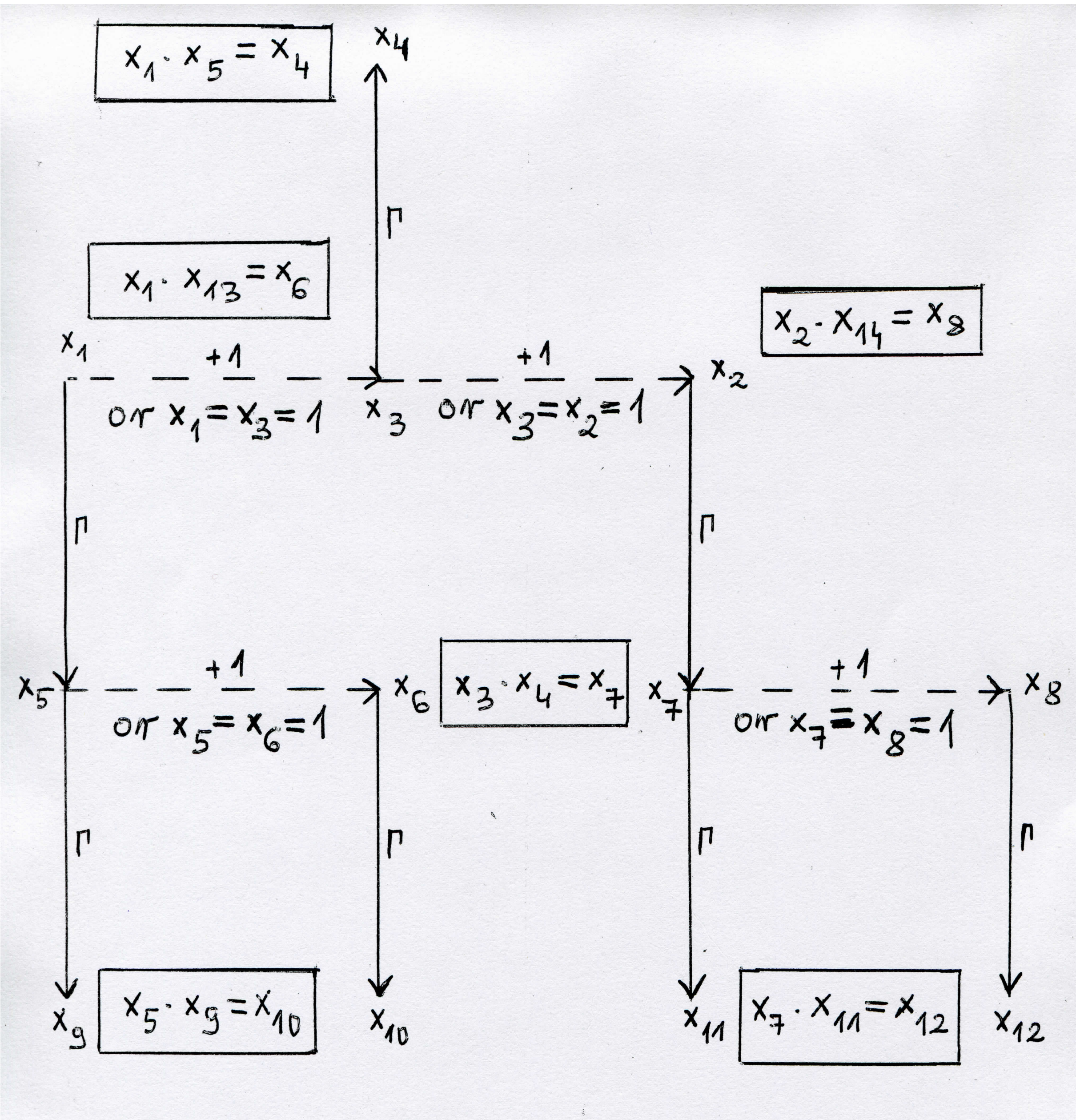

Fig. 4 Construction of the system C

Lemma 7. *The system C has only finitely many solutions $(x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ with $x_1 \in \{1, 2, 3, 4\}$. For every integer $x_1 \geqslant 5$, the system C is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if x_1 and $x_1 + 2$ are prime. In this case, the numbers $x_2, \ldots, x_{14}$ are uniquely determined by x_1, and $x_1 = \min(x_1, \ldots, x_{14})$.*

Proof. By Lemma 1, for every integer $x_1 \geqslant 5$, the system C is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if x_1 divides $\Gamma(x_1) + 1$ and $x_1 + 2$ divides $\Gamma(x_1 + 2) + 1$. By Lemma 2, the last is true if and only if x_1 and $x_1 + 2$ are prime. The inequality $x_1 \geqslant 5$ and Lemma 4 imply that $x_1 = \min(x_1, \ldots, x_{14})$. □

Theorem 4. *For every integer $b \geqslant 4$, the statement $\Psi(14, b)$ proves the implication: if there exists a twin prime greater than $b + 2$, then there are infinitely many twin primes.*

Proof. Let us assume that there exists a prime number x_1 such that $x_1 + 2$ is prime and $x_1 + 2 > b + 2$. Since $b \geqslant 4$, we obtain that $x_1 \geqslant 5$. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{13}$ such that the tuple $(x_1, \ldots, x_{14})$ solves the system C. Lemma 7 guarantees that $x_1 = \min(x_1, \ldots, x_{14})$. Since $C \subseteq G_{14}$, we conclude that the statement $\Psi(14, b)$ and the inequality $b < x_1 = \min(x_1, \ldots, x_{14})$ imply that the system C has infinitely many solutions in positive integers $x_1, \ldots, x_{14}$. According to Lemma 7, there are infinitely many twin primes. $\square$

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