A conjecture which implies that any twin prime greater than (((((((24!)!)!)!)!)!)!)!)!)! proves that the set of twin primes is infinite

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Abstract

For a positive integer, let $\Gamma(n) \equiv (n-1)!$. Let $f(5) = 24!$, and let $f(n + 1) = \Gamma(f(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let $T(n)$ denote the statement: if a system of equations $S \subseteq \{ \Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leq f(n)$. We conjecture that the statements $T(5), \ldots, T(14)$ are true. The statement $T(6)$ implies that if $x! + 1$ is a square for at most finitely many non-negative integers $x$ then each such $x$ satisfies $x \leq f(6)$. The statement $T(9)$ proves the implication: if there exists an integer $x > f(9)$ such that $x^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$. The statement $T(14)$ proves the implication: if there exists a twin prime greater than $f(14) + 2$, then there are infinitely many twin primes.

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1. Introduction and basic lemmas

In this article, we study a conjecture which applies to Brocard’s problem, the problem of the infinitude of primes of the form $n^2 + 1$, and the twin prime problem. The conjecture allows us to compute an integer $b_6$ such that if $x! + 1$ is a square for at most finitely many non-negative integers $x$ then each such $x$ satisfies $x \leq b_6$. The conjecture allows us to compute an integer $b_9$ such that any prime number of the form $n^2 + 1$ which is greater than $b_9$ proves that the set of prime numbers of the form $n^2 + 1$ is infinite. The conjecture allows us to compute an integer $b_{14}$ such that any twin prime greater than $b_{14} + 2$ proves that the set of twin primes is infinite.

For a positive integer, let $\Gamma(n) \equiv (n-1)!$.

Lemma 1. For every positive integers $x$ and $y$, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if

$$(x + 1 = y) \lor (x = y = 1)$$

Lemma 2. (Wilson’s theorem, [1] p. 89). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $\Gamma(x) + 1$.

Lemma 3. For every integer $x \geq 5$, we have $x \leq \sqrt{\Gamma(x) + 1}$.

Lemma 4. For every integer $x \geq 5$, we have $x \leq \frac{\Gamma(x) + 1}{x}$. 
2. A conjecture on the statements $Ψ(n, b)$

For a positive integer $n$, let $G_n$ denote the following system of equations:

$$\{ \Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \}$$

For positive integers $n$ and $b$, let $Ψ(n, b)$ denote the statement: if a system $S \subseteq G_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$ then each such solution $(x_1, \ldots, x_n)$ satisfies $\min(x_1, \ldots, x_n) \leq b$.

**Theorem 1.** For every positive integer $n$, there exists an integer $b \geq 4$ such that the statement $Ψ(n, b)$ is true.

**Proof.** It follows from the fact that the system $G_n$ has a finite number of subsystems. $\square$

Let $f(5) = 24!$, and let $f(n + 1) = \Gamma(f(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let $U_n \subseteq G_n$ be the system of equations illustrated in Figure 1. Lemma[1] explains the construction of the system $U_n$.

![Fig. 1](image-url)  
Fig. 1 Construction of the system $U_n$

For every integer $n \geq 5$, the system $U_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, f(5), \ldots, f(n))$.

**Conjecture.** For every integer $n \in \{5, \ldots, 14\}$, the statement $Ψ(n, f(n))$ is true.

We present a heuristic reasoning that leads to the Conjecture. Let $n \in \{5, \ldots, 14\}$. We consider subsystems of the system $G_n$ which have only finitely many solutions in positive integers $x_1, \ldots, x_n$. We conjecture that the largest number in the largest known solution majorizes $\min(x_1, \ldots, x_n)$ for every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves a subsystem of $G_n$. 

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3. Brocard’s problem

A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $\Gamma(x) + 1 = y^2$, see [3]. It is conjectured that $\Gamma(x) + 1$ is a square only for $x \in \{5, 6, 8\}$, see [4, p. 297].

Let $\mathcal{A} \subseteq G_6$ be the system of equations illustrated in Figure 2. Lemma 1 explains the construction of the system $\mathcal{A}$.

![Fig. 2 Construction of the system $\mathcal{A}$](image)

**Lemma 5.** The system $\mathcal{A}$ has only finitely many solutions $(x_1, \ldots, x_6) \in (\mathbb{N} \setminus \{0\})^6$ with $x_1 \in \{1, 2\}$. For every integer $x_1 \geq 3$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_6$ if and only if $\Gamma(x_1) + 1$ is a square. In this case, $x_1 \geq 5$, the numbers $x_2, \ldots, x_6$ are uniquely determined by $x_1$, and $x_1 = \min(x_1, \ldots, x_6)$.

**Proof.** All the statements in this Lemma, except the equality $x_1 = \min(x_1, \ldots, x_6)$, follow from Lemma 1, Lemma 3, and the inequality $x_1 \geq 5$ imply that $x_1 = \min(x_1, \ldots, x_6)$. \[ \square \]

**Theorem 2.** For every positive integer $b$, if $\Gamma(x_1) + 1$ is a square for at most finitely many positive integers $x_1$, then the statement $\Psi(6, b)$ implies that each such $x_1$ satisfies $x_1 \leq b$.

**Proof.** Let us assume that for a positive integer $x_1$ there exists a positive integer $x_2$ such that $\Gamma(x_1) + 1 = x_2^2$. Then, $x_1 \geq 5$. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_6) \in (\mathbb{N} \setminus \{0\})^5$ such that the tuple $(x_1, \ldots, x_6)$ solves the system $\mathcal{A}$. Lemma 5 guarantees that $x_1 = \min(x_1, \ldots, x_6)$. By the antecedent and Lemma 5, the system $\mathcal{A}$ has only finitely many solutions in positive integers $x_1, \ldots, x_6$. Therefore, the statement $\Psi(6, b)$ implies that $x_1 = \min(x_1, \ldots, x_6) \leq b$. \[ \square \]

4. Are there infinitely many prime numbers of the form $n^2 + 1$?

Landau’s conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [2, pp. 37–38].

Let $\mathcal{B} \subseteq G_9$ be the system of equations illustrated in Figure 3. Lemma 1 explains the construction of the system $\mathcal{B}$.
Lemma 6. The system $\mathcal{B}$ has only finitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ with $x_1 = 1$. For every integer $x_1 \geq 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the numbers $x_2, \ldots, x_9$ are uniquely determined by $x_1$, and $x_1 = \min(x_1, \ldots, x_9)$.

Proof. By Lemma 1 for every integer $x_1 \geq 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $\Gamma(x_1^2 + 1) + 1$. By Lemma 2 the last is true if and only if $x_1^2 + 1$ is prime. The inequality $x_1 \geq 2$ and Lemma 4 imply that $x_1 = \min(x_1, \ldots, x_9)$.  

Theorem 3. For every positive integer $b$, the statement $\Psi(9, b)$ proves the implication: if there exists an integer $x_1 > b$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.

Proof. Let us assume that a positive integer $x_1$ is greater than $b$ and $x_1^2 + 1$ is prime. Since $b \geq 1$, we obtain that $x_1 \geq 2$. By Lemma 6 there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, \ldots, x_9)$ solves the system $\mathcal{B}$. Lemma 4 guarantees that $x_1 = \min(x_1, \ldots, x_9)$. Since $\mathcal{B} \subseteq G_9$, we obtain that the statement $\Psi(9, b)$ and the inequality $b < x_1 = \min(x_1, \ldots, x_9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemma 4 there are infinitely many primes of the form $n^2 + 1$.

5. The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [2, p. 39].

Let $\mathcal{C} \subseteq G_{14}$ be the system of equations illustrated in Figure 4. Lemma 1 explains the construction of the system $\mathcal{C}$. 

**Fig. 3** Construction of the system $\mathcal{B}$
Lemma 7. The system $C$ has only finitely many solutions $(x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ with $x_1 \in \{1, 2, 3, 4\}$. For every integer $x_1 \geq 5$, the system $C$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if $x_1$ and $x_1 + 2$ are prime. In this case, the numbers $x_2, \ldots, x_{14}$ are uniquely determined by $x_1$, and $x_1 = \min(x_1, \ldots, x_{14})$.

Proof. By Lemma [1] for every integer $x_1 \geq 5$, the system $C$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if $x_1$ divides $\Gamma(x_1) + 1$ and $x_1 + 2$ divides $\Gamma(x_1 + 2) + 1$. By Lemma [2] the last is true if and only if $x_1$ and $x_1 + 2$ are prime. The inequality $x_1 \geq 5$ and Lemma [3] imply that $x_1 = \min(x_1, \ldots, x_{14})$. $\square$
Theorem 4. For every integer $b \geq 4$, the statement $\Psi(14,b)$ proves the implication: if there exists a twin prime greater than $b + 2$, then there are infinitely many twin primes.

Proof. Let us assume that there exists a prime number $x_1$ such that $x_1 + 2$ is prime and $x_1 + 2 > b + 2$. Since $b \geq 4$, we obtain that $x_1 \geq 5$. By Lemma 7 there exists a unique tuple $(x_2, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{13}$ such that the tuple $(x_1, \ldots, x_{14})$ solves the system $C$. Lemma 7 guarantees that $x_1 = \min(x_1, \ldots, x_{14})$. Since $C \subseteq G_{14}$, we conclude that the statement $\Psi(14,b)$ and the inequality $b < x_1 = \min(x_1, \ldots, x_{14})$ imply that the system $C$ has infinitely many solutions in positive integers $x_1, \ldots, x_{14}$. According to Lemma 7 there are infinitely many twin primes. \qed

The inequality $f(14) + 2 < ((((((((24!)!)!)!)!)!)!)!)!$ together with the Conjecture and Theorem 4 justifies the title of the article.

References