# On $ZFC$-formulae $\varphi(x)$ for which we know a non-negative integer $n$ such that $\max(\{x \in \mathbb{N}\colon \varphi(x)\}) \leqslant n$ if the set $\{x \in \mathbb{N}\colon \varphi(x)\}$ is finite

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**Abstract**

Let $\Gamma(k)$ denote $(k-1)!$, and let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3,\ldots,16\}$ and $k \in \{2\} \cup [2^{2^{n-3}}+1,\infty) \cap \mathbb{N}$. For an integer $n \in \{3,\ldots,16\}$, let $\Sigma_n$ denote the following statement: if a system of equations $\mathcal{S} \subseteq \{\Gamma_n(x_i) = x_k:\ i,k \in \{1,\ldots,n\}\} \cup \{x_i \cdot x_j = x_k:\ i,j,k \in \{1,\ldots,n\}\}$ with $\Gamma$ instead of $\Gamma_n$ has only finitely many solutions in positive integers $x_1,\ldots,x_n$, then every tuple $(x_1,\ldots,x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $\mathcal{S}$ satisfies $x_1,\ldots,x_n \leqslant 2^{2^{n-2}}$. Our hypothesis claims that the statements $\Sigma_3,\ldots,\Sigma_{16}$ are true. The statement $\Sigma_6$ proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x,y)$ belongs to the set $\{(1,2),(2,3)\}$. The statement $\Sigma_6$ proves the following implication: if the equation $x!+1 = y^2$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x,y)$ belongs to the set $\{(4,5),(5,11),(7,71)\}$. The statement $\Sigma_9$ implies the infinitude of primes of the form $n^2+1$. The statement $\Sigma_9$ implies that any prime of the form $n!+1$ with $n \geqslant 2^{2^{9-3}}$ proves the infinitude of primes of the form $n!+1$. The statement $\Sigma_{14}$ implies the infinitude of twin primes. The statement $\Sigma_{16}$ implies the infinitude of Sophie Germain primes.

**Key words and phrases:** Brocard's problem, Brocard-Ramanujan equation $x!+1 = y^2$, composite Fermat numbers, Erdös' equation $x(x+1) = y!$, prime numbers of the form $n^2+1$, prime numbers of the form $n!+1$, Sophie Germain primes, twin primes.

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## 1 Introduction and basic lemmas

The phrase "*we know a non-negative integer n*" in the title means that we know an algorithm which returns $n$. The title of the article cannot be formalized in $ZFC$ because the phrase "*we know a non-negative integer n*" refers to currently known non-negative integers $n$ with some property. A formally stated title may look like this: *On ZFC-formulae $\varphi(x)$ for which there exists a non-negative integer n such that ZFC proves that*

$$\operatorname{card}(\{x \in \mathbb{N}\colon \varphi(x)\}) < \infty \Longrightarrow \max(\{x \in \mathbb{N}\colon \varphi(x)\}) \leqslant n$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer $n$ such that $ZFC$ proves the above implication.

**Lemma 1.** *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1 = y) \vee (x = y = 1)$$

Let $\Gamma(k)$ denote $(k-1)!$.

**Lemma 2.** *For every positive integers x and y, $x \cdot \Gamma(x) = \Gamma(y)$ if and only if*

$$(x+1 = y) \vee (x = y = 1)$$

**Lemma 3.** *For every non-negative integers b and c, $b+1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.*

**Lemma 4.** *(Wilson's theorem, [4, p. 89]). For every positive integer x, x divides $(x-1)!+1$ if and only if $x = 1$ or x is prime.*

## 2 Threshold numbers of a set $\mathcal{X} \subseteq \mathbb{N}$

We say that a non-negative integer $m$ is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than $m$, cf. [21] and [22]. If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer $m$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $\{\max(\mathcal{X}), \max(\mathcal{X})+1, \max(\mathcal{X})+2, \ldots\}$.

Let $\rho : \mathbb{N} \to \mathbb{Q}^2$ be a computable bijection.

**Lemma 5.** *The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [8, p. 212]. The known rational solutions are $(x, y) = (-1, 0)$, $(-1, 1)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, -5)$, $(2, 6)$, $(3, -15)$, $(3, 16)$, $(30, -4929)$, $(30, 4930)$, $\left(\frac{1}{4}, \frac{15}{32}\right)$, $\left(\frac{1}{4}, \frac{17}{32}\right)$, $\left(-\frac{15}{16}, -\frac{185}{1024}\right)$, $\left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [14, pp. 223–224].*

**Corollary 1.** *The set $\mathcal{T} = \{n \in \mathbb{N} : \rho(n)$ solves the equation $x^5 - x = y^2 - y\}$ is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of $\mathcal{T}$.*

Let $\tau : \mathbb{N} \to (\mathbb{N} \setminus \{0\})^7$ be a computable bijection. Let $\mathcal{D}$ denote the following system of equations:

$$\left\{\begin{array}{rcl} x^2 + y^2 &=& s^2 \\ x^2 + z^2 &=& t^2 \\ y^2 + z^2 &=& u^2 \\ x^2 + y^2 + z^2 &=& v^2 \end{array}\right.$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

**Lemma 6.** *([18]). No perfect cuboids are known.*

**Corollary 2.** *The set $\mathcal{F} = \{n \in \mathbb{N} : \tau(n)$ solves the system $\mathcal{D}\}$ is empty or infinite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{F}$. Every non-negative integer $n$ is a threshold number of $\mathcal{F}$.*

It is conjectured that the set of prime numbers of the form $n^2 + 1$ is infinite, see [9, pp. 37–38]. It is conjectured that the set of prime numbers of the form $n! + 1$ is infinite, see [2, p. 443] and [15]. It is conjectured that the set of twin primes is infinite, see [9, p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^n} + 1$ is infinite, see [5, p. 23] and [6, pp. 158–159]. For each of these sets, we do not know any threshold number.

## 3 A variant of Chess and the set of all positive integers $n$ such that an appropriate strategy of `Black` guarantees that `White` cannot enforce a win in less than $n$ moves

Let us assume that there are no draws, castlings, and en passant captures. Let us assume that a player with no moves loses. As such, the game may continue forever. Let $\mathcal{H}$ denote the set of all positive integers $n$ such that an appropriate strategy of `Black` guarantees that `White` cannot enforce a win in less than $n$ moves.

**Lemma 7.** *([13, p. 128]). A player who is in a winning position is always able to enforce a win in a number of moves that is less than the number of positions in the game.*

**Lemma 8.** *The number of positions does not exceed* $13^{64}$.

*Proof.* With castlings or en passant captures, a legality of a move depends not only on the positions of the pieces on the board. Without castlings and en passant captures, we observe that 13 corresponds to 12 distinct pieces and the empty square. 64 is the number of squares on the chessboard. □

Lemmas 7 and 8 imply the following corollary.

**Corollary 3.** *If* `White` *have a winning strategy, then* $\mathcal{H} \subseteq [1, 13^{64} - 1]$. *Otherwise,* $\mathcal{H} = \mathbb{N} \setminus \{0\}$. *The number* $13^{64} - 1$ *is a threshold number of* $\mathcal{H}$*, and we can decide the equality* $\mathcal{H} = \mathbb{N} \setminus \{0\}$. *If* $\mathcal{H} \neq \mathbb{N} \setminus \{0\}$*, then we can compute* $\mathcal{H}$ *and* $\max(\mathcal{H})$.

## 4 A Diophantine equation whose non-solvability expresses the consistency of $ZFC$

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

**Theorem 1.** *([3, p. 35]). There exists a polynomial* $D(x_1, \ldots, x_m)$ *with integer coefficients such that if ZFC is arithmetically consistent, then the sentences* "The equation $D(x_1, \ldots, x_m) = 0$ is solvable in non-negative integers" *and* "The equation $D(x_1, \ldots, x_m) = 0$ is not solvable in non-negative integers" *are not provable in ZFC.*

Let $\mathcal{Y}$ denote the set of all non-negative integers $k$ such that the equation $D(x_1, \ldots, x_m) = 0$ has no solutions in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Let $\mathcal{E}$ denote the set of all non-negative integers $k$ such that the equation $D(x_1, \ldots, x_m) = 0$ has a solution in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{E}$. Theorem 1 implies Theorems 2 and 3.

**Theorem 2.** *For every* $n \in \mathbb{N}$*, ZFC proves that* $n \in \mathcal{Y}$. *If ZFC is arithmetically consistent, then the sentences* "$\mathcal{Y}$ is finite" *and* "$\mathcal{Y}$ is infinite" *are not provable in ZFC. If ZFC is arithmetically consistent, then for every* $n \in \mathbb{N}$ *the sentences* "$n$ is a threshold number of $\mathcal{Y}$" *and* "$n$ is not a threshold number of $\mathcal{Y}$" *are not provable in ZFC.*

**Theorem 3.** *The set* $\mathcal{E}$ *is empty or infinite. In both cases, every non-negative integer* $n$ *is a threshold number of* $\mathcal{E}$. *If ZFC is arithmetically consistent, then the sentences* "$\mathcal{E}$ is empty", "$\mathcal{E}$ is not empty", "$\mathcal{E}$ is finite", *and* "$\mathcal{E}$ is infinite" *are not provable in ZFC.*

## 5 Hypothetical statements $\Psi_3, \ldots, \Psi_{16}$

For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{2\} \; x_i! &= x_{i+1} \\ x_1 \cdot x_2 &= x_3 \\ x_2 \cdot x_2 &= x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

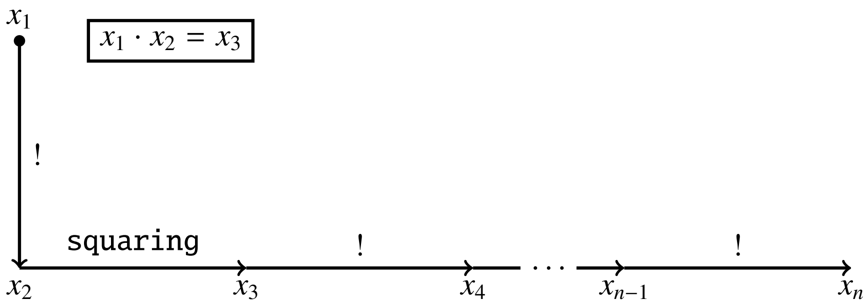


**Fig. 1** Construction of the system $\mathcal{U}_n$

Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geqslant 3$.

**Lemma 9.** *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $\big(2, 2, g(3), \dots, g(n)\big)$.*

Let

$$B_n = \Big\{x_i! = x_k :\ (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \dots, n\}\Big\}$$

For an integer $n \geqslant 3$, let $\Psi_n$ denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant g(n)$.* The statement $\Psi_n$ says that for subsystems of $B_n$ the largest known solution is indeed the largest possible.

**Hypothesis 1.** *The statements $\Psi_3, \dots, \Psi_{16}$ are true.*

**Theorem 4.** *Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.*

*Proof.* For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

**Theorem 5.** *For every statement $\Psi_n$, the bound $g(n)$ cannot be decreased.*

*Proof.* It follows from Lemma 9 because $\mathcal{U}_n \subseteq B_n$. □

## 6 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 1 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

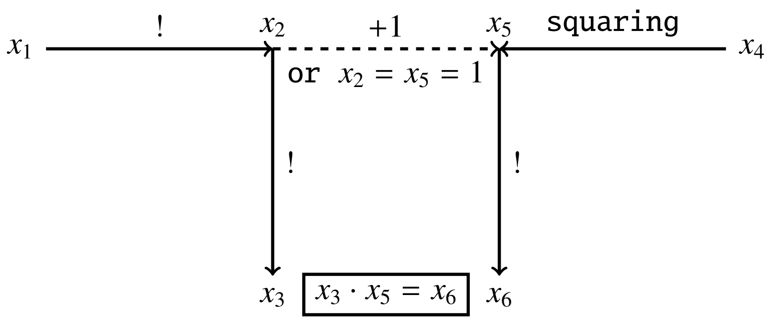


**Fig. 2** Construction of the system $\mathcal{A}$

**Lemma 10.** *For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system $\mathcal{A}$ is solvable in positive integers $x_2$, $x_3$, $x_5$, $x_6$ if and only if $x_1! + 1 = x_4^2$. In this case, the integers $x_2, x_3, x_5, x_6$ are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_2 &= x_1! \\
x_3 &= (x_1!)! \\
x_5 &= x_1! + 1 \\
x_6 &= (x_1! + 1)!
\end{aligned}
$$

*Proof.* It follows from Lemma 1. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [17, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [10].

**Theorem 6.** *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement $\Psi_6$ guarantees that each such solution $(x_1, x_4)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

*Proof.* Suppose that the antecedent holds. Let positive integers $x_1$ and $x_4$ satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 10, the system $\mathcal{A}$ is solvable in positive integers $x_2, x_3, x_5, x_6$. Since $\mathcal{A} \subseteq B_6$, the statement $\Psi_6$ implies that $x_6 = (x_1! + 1)! \leqslant g(6) = g(5)!$. Hence, $x_1! + 1 \leqslant g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \dots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

## 7 Are there infinitely many prime numbers of the form $n^2 + 1$?

Let $\mathcal{B}$ denote the following system of equations:

$$
\begin{cases}
x_2! = x_3 \\
x_3! = x_4 \\
x_5! = x_6 \\
x_8! = x_9 \\
x_1 \cdot x_1 = x_2 \\
x_3 \cdot x_5 = x_6 \\
x_4 \cdot x_8 = x_9 \\
x_5 \cdot x_7 = x_8
\end{cases}
$$

Lemma 1 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.

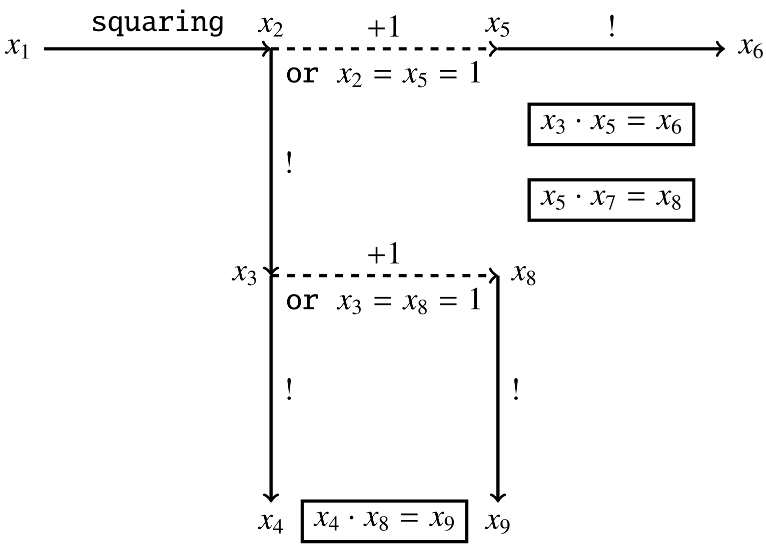


**Fig. 3** Construction of the system $\mathcal{B}$

**Lemma 11.** *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \dots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

*Proof.* By Lemma 1, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 11 follows from Lemma 4. □

**Lemma 12.** *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.*

*Proof.* If a tuple $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \dots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [9, pp. 37–38].

**Theorem 7.** *The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

*Proof.* Suppose that the antecedent holds. By Lemma 11, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geqslant g(7)$. Hence, $(x_1^2)! \geqslant g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement $\Psi_9$ and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 11 and 12, there are infinitely many primes of the form $n^2 + 1$. □

**Corollary 4.** *Let $\mathcal{X}_9$ denote the set of primes of the form $n^2 + 1$. The statement $\Psi_9$ implies that we know an algorithm such that it returns a threshold number of $\mathcal{X}_9$, and this number equals* $\max(\mathcal{X}_9)$*, if $\mathcal{X}_9$ is finite.*

*Proof.* We consider an algorithm which computes $\max(\mathcal{X}_9 \cap [1, g(7)])$. □

# 8 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [2, p. 443] and [15].

**Theorem 8.** *(cf. Theorem 12). The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geqslant g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

*Proof.* We leave the analogous proof to the reader. □

# 9 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [9, p. 39]. Let $C$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_4! &= x_5 \\ x_6! &= x_7 \\ x_7! &= x_8 \\ x_9! &= x_{10} \\ x_{12}! &= x_{13} \\ x_{15}! &= x_{16} \\ x_2 \cdot x_4 &= x_5 \\ x_5 \cdot x_6 &= x_7 \\ x_7 \cdot x_9 &= x_{10} \\ x_4 \cdot x_{11} &= x_{12} \\ x_3 \cdot x_{12} &= x_{13} \\ x_9 \cdot x_{14} &= x_{15} \\ x_8 \cdot x_{15} &= x_{16} \end{cases}$$

Lemma 1 and the diagram in Figure 4 explain the construction of the system $C$.

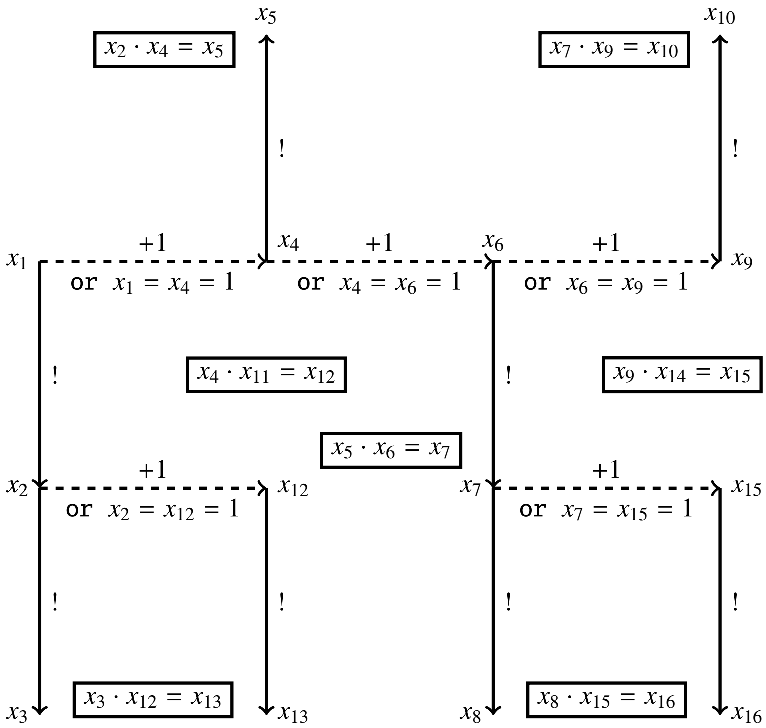


**Fig. 4** Construction of the system $C$

**Lemma 13.** *For every* $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$*, the system* $C$ *is solvable in positive integers* $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ *if and only if* $x_4$ *and* $x_9$ *are prime and* $x_4 + 2 = x_9$*. In this case, the integers* $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ *are uniquely determined by the following equalities:*

$$\begin{array}{rcl}
x_1 & = & x_4 - 1 \\
x_2 & = & (x_4 - 1)! \\
x_3 & = & ((x_4 - 1)!)! \\
x_5 & = & x_4! \\
x_6 & = & x_9 - 1 \\
x_7 & = & (x_9 - 1)! \\
x_8 & = & ((x_9 - 1)!)! \\
x_{10} & = & x_9! \\
x_{11} & = & \dfrac{(x_4 - 1)! + 1}{x_4} \\
x_{12} & = & (x_4 - 1)! + 1 \\
x_{13} & = & ((x_4 - 1)! + 1)! \\
x_{14} & = & \dfrac{(x_9 - 1)! + 1}{x_9} \\
x_{15} & = & (x_9 - 1)! + 1 \\
x_{16} & = & ((x_9 - 1)! + 1)!
\end{array}$$

*Proof.* By Lemma 1, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system $C$ is solvable in positive integers $x_1$, $x_2$, $x_3$, $x_5$, $x_6$, $x_7$, $x_8$, $x_{10}$, $x_{11}$, $x_{12}$, $x_{13}$, $x_{14}$, $x_{15}$, $x_{16}$ if and only if

$$\Big(x_4 + 2 = x_9\Big) \wedge \Big(x_4 | (x_4 - 1)! + 1\Big) \wedge \Big(x_9 | (x_9 - 1)! + 1\Big)$$

Hence, the claim of Lemma 13 follows from Lemma 4. $\square$

**Lemma 14.** *There are only finitely many tuples* $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ *which solve the system* $C$ *and satisfy* $(x_4 \in \{1,2\}) \vee (x_9 \in \{1,2\})$.

*Proof.* If a tuple $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system $C$ and $(x_4 \in \{1,2\}) \vee (x_9 \in \{1,2\})$, then $x_1, \dots, x_{16} \leqslant 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. □

**Theorem 9.** *The statement* $\Psi_{16}$ *proves the following implication: if there exists a twin prime greater than* $g(14)$*, then there are infinitely many twin primes.*

*Proof.* Suppose that the antecedent holds. Then, there exist prime numbers $x_4$ and $x_9$ such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0,1,2\}$. By Lemma 13, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$ such that the tuple $(x_1, \dots, x_{16})$ solves the system $C$. Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geqslant g(14)$. Therefore, $(x_9 - 1)! \geqslant g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement $\Psi_{16}$ and the inequality $x_{16} > g(16)$ imply that the system $C$ has infinitely many solutions in positive integers $x_1, \dots, x_{16}$. According to Lemmas 13 and 14, there are infinitely many twin primes. □

**Corollary 5.** *Let* $\mathcal{X}_{16}$ *denote the set of twin primes. The statement* $\Psi_{16}$ *implies that we know an algorithm such that it returns a threshold number of* $\mathcal{X}_{16}$*, and this number equals* $\max(\mathcal{X}_{16})$*, if* $\mathcal{X}_{16}$ *is finite.*

*Proof.* We consider an algorithm which computes $\max(\mathcal{X}_{16} \cap [1, g(14)])$. □

## 10 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n+1) = \Gamma(\lambda(n))$ for every integer $n \geqslant 5$. For an integer $n \geqslant 5$, let $\mathcal{J}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{3\} \; \Gamma(x_i) & = & x_{i+1} \\ x_1 \cdot x_1 & = & x_4 \\ x_2 \cdot x_3 & = & x_5 \end{cases}$$

Lemma 2 and the diagram in Figure 5 explain the construction of the system $\mathcal{J}_n$.

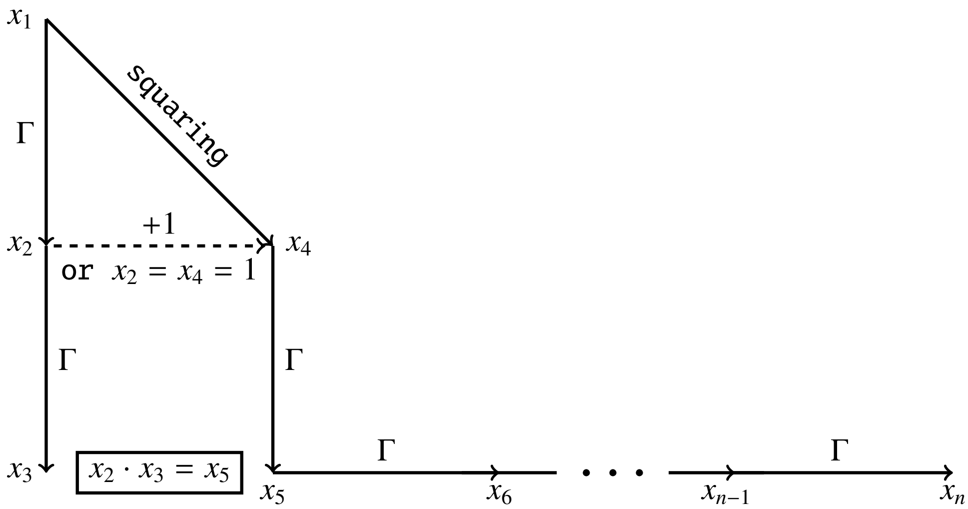


**Fig. 5** Construction of the system $\mathcal{J}_n$

For every integer $n \geqslant 5$, the system $\mathcal{J}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$. For an integer $n \geqslant 5$, let $\Delta_n$ denote the following statement: *if a system of equations* $\mathcal{S} \subseteq \Big\{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\Big\}$ *has only finitely many solutions in positive integers* $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant \lambda(n)$.

**Hypothesis 2.** *The statements* $\Delta_5, \ldots, \Delta_{14}$ *are true.*

Lemmas 2 and 4 imply that the statements $\Delta_n$ have similar consequences as the statements $\Psi_n$.

**Theorem 10.** *The statement* $\Delta_6$ *implies that any prime number* $p \geqslant 25$ *proves the infinitude of primes.*

*Proof.* It follows from Lemmas 2 and 4. We leave the details to the reader. □

## 11 Hypothetical statements $\Sigma_3, \ldots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \ldots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : \ i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\}\}$$

For an integer $n \in \{3, \ldots, 16\}$, let $P_n$ denote the following system of equations:

$$\begin{cases} \qquad\qquad x_1 \cdot x_1 &= x_1 \\ \qquad\qquad \Gamma_n(x_2) &= x_1 \\ \forall i \in \{2, \ldots, n-1\} \; x_i \cdot x_i &= x_{i+1} \end{cases}$$

**Lemma 15.** *For every integer* $n \in \{3, \ldots, 16\}$*,* $P_n \subseteq Q_n$ *and the system* $P_n$ *with* $\Gamma$ *instead of* $\Gamma_n$ *has exactly one solution in positive integers* $x_1, \ldots, x_n$*, namely* $\left(1, 2^{2^0},\ 2^{2^1}, 2^{2^2}, \ldots, 2^{2^{n-2}}\right)$.

For an integer $n \in \{3, \ldots, 16\}$, let $\Sigma_n$ denote the following statement: if a system of equations $\mathcal{S} \subseteq Q_n$ with $\Gamma$ instead of $\Gamma_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $\mathcal{S}$ satisfies $x_1, \ldots, x_n \leqslant 2^{2^{n-2}}$.

**Hypothesis 3.** *The statements* $\Sigma_3, \ldots, \Sigma_{16}$ *are true.*

**Lemma 16.** *(cf. Lemma 2). For every integer* $n \in \{4, \ldots, 16\}$ *and for every positive integers* $x$ *and* $y$*,* $x \cdot \Gamma_n(x) = \Gamma_n(y)$ *if and only if* $(x + 1 = y) \wedge \left(x \geqslant 2^{2^{n-3}} + 1\right)$.

Let $\mathcal{Z}_9 \subseteq Q_9$ be the system of equations in Figure 6.

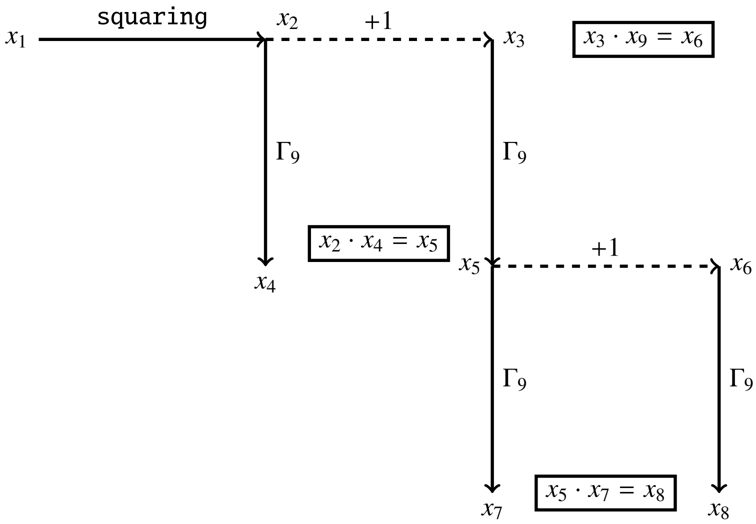


**Fig. 6** Construction of the system $\mathcal{Z}_9$

**Lemma 17.** *For every positive integer $x_1$, the system $\mathcal{Z}_9$ is solvable in positive integers $x_2, \dots, x_9$ if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers $x_2, \dots, x_9$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with $n$ and solve the system $\mathcal{Z}_9$ with $\Gamma$ instead of $\Gamma_9$.*

*Proof.* It follows from Lemmas 2, 4, and 16. □

**Lemma 18.** *([16]). The number $(13!)^2 + 1 = 38775788043632640001$ is prime.*

**Lemma 19.** $\Big((13!)^2 \geqslant 2^{2^{9-3}} + 1 = 18446744073709551617\Big) \wedge \Big(\Gamma_9((13!)^2) > 2^{2^{9-2}}\Big)$.

**Theorem 11.** *The statement $\Sigma_9$ implies the infinitude of primes of the form $n^2 + 1$.*

*Proof.* It follows from Lemmas 17–19. □

**Theorem 12.** *(cf. Theorem 8). The statement $\Sigma_9$ implies that any prime of the form $n! + 1$ with $n \geqslant 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$.*

*Proof.* We leave the proof to the reader. □

**Corollary 6.** *Let $\mathcal{Y}_9$ denote the set of primes of the form $n! + 1$. The statement $\Sigma_9$ implies that we know an algorithm such that it returns a threshold number of $\mathcal{Y}_9$, and this number equals $\max(\mathcal{Y}_9)$, if $\mathcal{Y}_9$ is finite.*

*Proof.* We consider an algorithm which computes $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$. □

Let $\mathcal{Z}_{14} \subseteq Q_{14}$ be the system of equations in Figure 7.

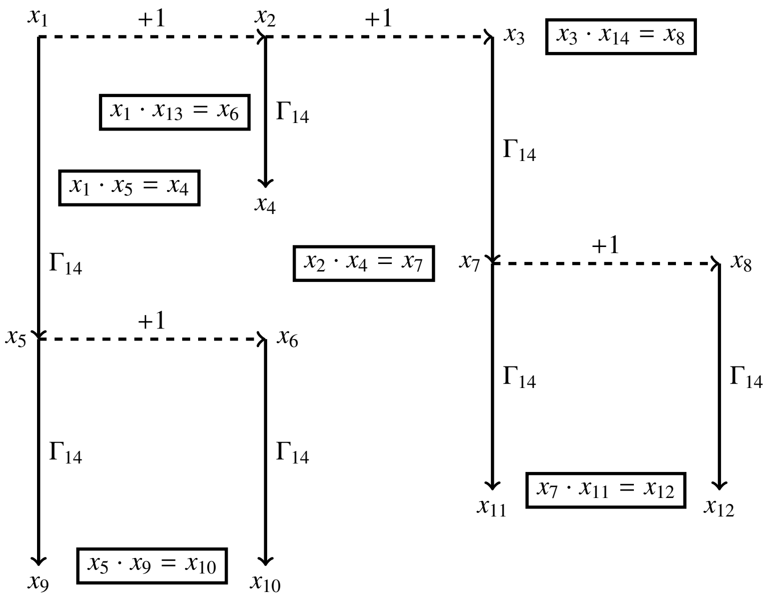


**Fig. 7** Construction of the system $\mathcal{Z}_{14}$

**Lemma 20.** *For every positive integer $x_1$, the system $\mathcal{Z}_{14}$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if $x_1$ and $x_1 + 2$ are prime and $x_1 \geqslant 2^{2^{14-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{14}$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with $n$ and solve the system $\mathcal{Z}_{14}$ with $\Gamma$ instead of $\Gamma_{14}$.*

*Proof.* It follows from Lemmas 2, 4, and 16. □

**Lemma 21.** *([20, p. 87]). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).*

**Lemma 22.** $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

**Theorem 13.** *The statement $\Sigma_{14}$ implies the infinitude of twin primes.*

*Proof.* It follows from Lemmas 20–22. □

A prime $p$ is said to be a Sophie Germain prime if both $p$ and $2p+1$ are prime, see [19]. Let $\mathcal{Z}_{16} \subseteq Q_{16}$ be the system of equations in Figure 8.

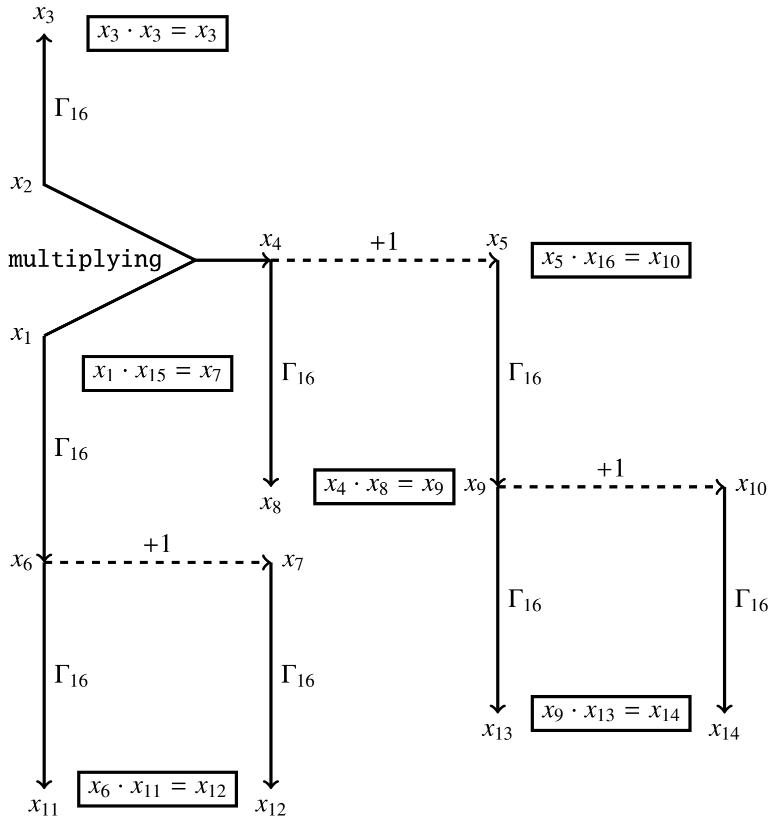


**Fig. 8** Construction of the system $\mathcal{Z}_{16}$

**Lemma 23.** *For every positive integer $x_1$, the system $\mathcal{Z}_{16}$ is solvable in positive integers $x_2, \ldots, x_{16}$ if and only if $x_1$ is a Sophie Germain prime and $x_1 \geqslant 2^{2^{16-3}}+1$. In this case, positive integers $x_2, \ldots, x_{16}$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with $n$ and solve the system $\mathcal{Z}_{16}$ with $\Gamma$ instead of $\Gamma_{16}$.*

*Proof.* It follows from Lemmas 2, 4, and 16. □

**Lemma 24.** *([12, p. 330]).* $8069496435 \cdot 10^{5072} - 1$ *is a Sophie Germain prime (Harvey Dubner).*

**Lemma 25.** $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$.

**Theorem 14.** *The statement $\Sigma_{16}$ implies the infinitude of Sophie Germain primes.*

*Proof.* It follows from Lemmas 23–25. □

**Theorem 15.** *The statement $\Sigma_6$ proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(1,2),(2,3)\}$.*

*Proof.* We leave the proof to the reader. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdös, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [7].

**Theorem 16.** *The statement $\Sigma_6$ proves the following implication: if the equation $x!+1 = y^2$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(4,5),(5,11),(7,71)\}$.*

*Proof.* We leave the proof to the reader. □

## 12 Hypothetical statements $\Omega_3, \ldots, \Omega_{16}$ and their consequences

For an integer $n \in \{3, \ldots, 16\}$, let $\Omega_n$ denote the following statement: *if a system of equations* $\mathcal{S} \subseteq \Big\{\Gamma(x_i) = x_k : i,k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k : i,j,k \in \{1, \ldots, n\}\Big\}$ *has a solution in integers* $x_1, \ldots, x_n$ *greater than* $2^{2^{n-2}}$, *then* $\mathcal{S}$ *has infinitely many solutions in positive integers* $x_1, \ldots, x_n$. For every $n \in \{3, \ldots, 16\}$, the statement $\Sigma_n$ implies the statement $\Omega_n$.

**Lemma 26.** *The number* $(65!)^2 + 1$ *is prime and* $65! > 2^{2^{9-2}}$.

*Proof.* The following PARI/GP ([11]) command

```
(21:46) gp > isprime((65!)^2+1,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([20, p. 226]). It rigorously shows that the number $(65!)^2 + 1$ is prime. □

**Lemma 27.** *If positive integers* $x_1, \ldots, x_9$ *solve the system* $\mathcal{Z}_9$ *and* $x_1 > 2^{2^{9-2}}$, *then* $x_1 = \min(x_1, \ldots, x_9)$.

**Theorem 17.** *The statement* $\Omega_9$ *implies the infinitude of primes of the form* $n^2 + 1$.

*Proof.* It follows from Lemmas 17 and 26–27. □

**Lemma 28.** *If positive integers* $x_1, \ldots, x_{14}$ *solve the system* $\mathcal{Z}_{14}$ *and* $x_1 > 2^{2^{14-2}}$, *then* $x_1 = \min(x_1, \ldots, x_{14})$.

**Theorem 18.** *The statement* $\Omega_{14}$ *implies the infinitude of twin primes.*

*Proof.* It follows from Lemmas 20–22 and 28. □

## 13 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [6, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [6, p. 1].

**Open Problem.** ([6, p. 159]). *Are there infinitely many composite numbers of the form* $2^{2^n} + 1$*?*
Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [5, p. 23]. Let

$$H_n = \Big\{x_i \cdot x_j = x_k : \ i,j,k \in \{1, \ldots, n\}\Big\} \cup \Big\{2^{2^{x_i}} = x_k : \ i,k \in \{1, \ldots, n\}\Big\}$$

Let $h(1) = 1$, and let $h(n+1) = 2^{2^{h(n)}}$ for every positive integer $n$.

**Lemma 29.** *The following subsystem of* $H_n$

$$\begin{cases} \hfill x_1 \cdot x_1 &= x_1 \\ \forall i \in \{1, \ldots, n-1\} \; 2^{2^{x_i}} &= x_{i+1} \end{cases}$$

*has exactly one solution* $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, *namely* $(h(1), \ldots, h(n))$.

For a positive integer $n$, let $\xi_n$ denote the following statement: *if a system of equations* $S \subseteq H_n$ *has only finitely many solutions in positive integers* $x_1, \ldots, x_n$, *then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant h(n)$. The statement $\xi_n$ says that for subsystems of $H_n$ the largest known solution is indeed the largest possible.

**Hypothesis 4.** *The statements* $\xi_1, \ldots, \xi_{13}$ *are true.*

**Theorem 19.** *Every statement* $\xi_n$ *is true with an unknown integer bound that depends on* $n$.

*Proof.* For every positive integer $n$, the system $H_n$ has a finite number of subsystems. □

**Theorem 20.** *The statement* $\xi_{13}$ *proves the following implication: if* $z \in \mathbb{N} \setminus \{0\}$ *and* $2^{2^z} + 1$ *is composite and greater than* $h(12)$*, then* $2^{2^z} + 1$ *is composite for infinitely many positive integers* $z$.

*Proof.* Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{1}$$

in positive integers. By Lemma 3, we can transform equation (1) into an equivalent system of equations $\mathcal{G}$ which has 13 variables ($x$, $y$, $z$, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 9.

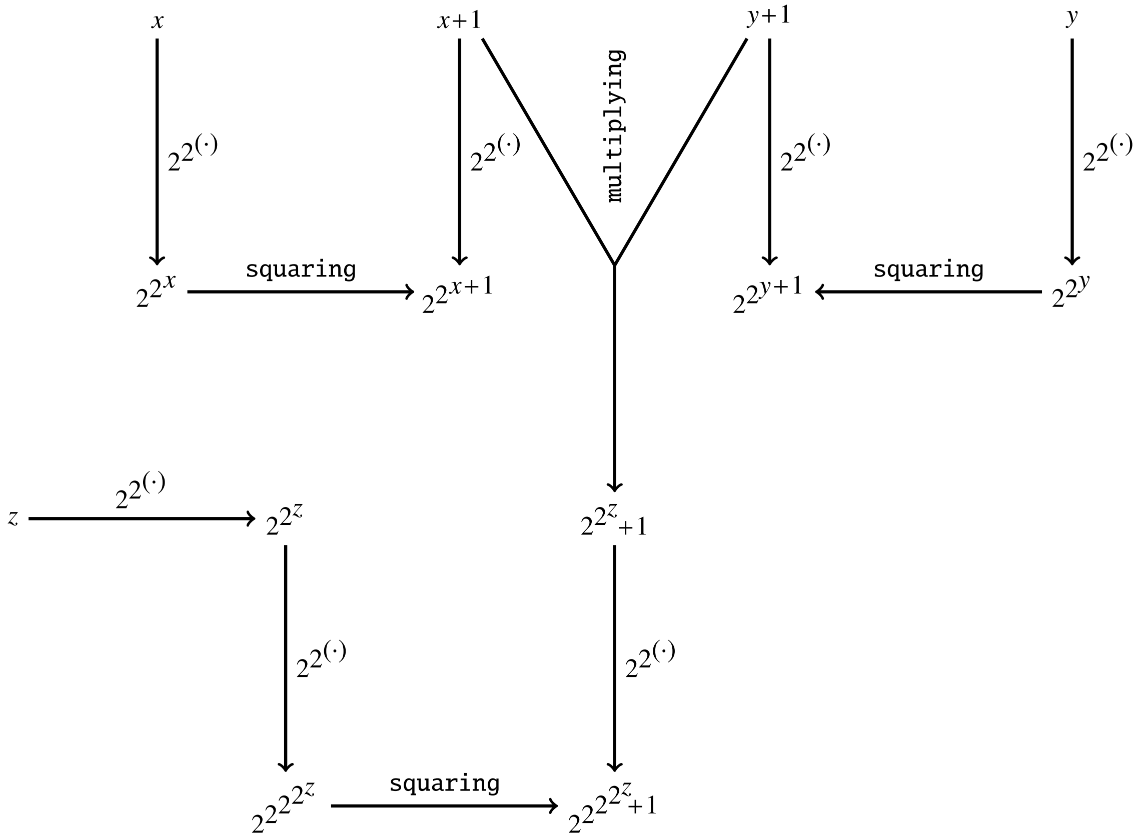


**Fig. 9** Construction of the system $\mathcal{G}$

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z}+1}} > h(13)$. By this, the statement $\xi_{13}$ implies that the system $\mathcal{G}$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. □

**Corollary 7.** *Let* $\mathcal{W}_{13}$ *denote the set of composite Fermat numbers. The statement* $\xi_{13}$ *implies that we know an algorithm such that it returns a threshold number of* $\mathcal{W}_{13}$*, and this number equals* $\max(\mathcal{W}_{13})$*, if* $\mathcal{W}_{13}$ *is finite.*

*Proof.* We consider an algorithm which computes $\max(\mathcal{W}_{13} \cap [1, h(12)])$. □

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