A common approach to solving the equation $x(x + 1) = y!$ and proving the infinitude of Wilson primes

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Abstract

For a positive integer $x$, let $\Gamma(x)$ denote $(x - 1)!$. Let $\Gamma^{-1}: \{1, 2, 6, 24, \ldots \} \to \mathbb{N} \setminus \{0\}$ denote the inverse function that satisfies $\Gamma^{-1}(1) = 2$. For a positive integer $n$, by a $\Gamma$-computation of length $n$ we understand any sequence of terms $x_1, \ldots, x_n$ such that $x_1$ is identical to the variable $x$ and for every integer $i \in \{2, \ldots, n\}$ there exist integers $j, k \in \{1, \ldots, i - 1\}$ such that $x_i$ is identical to $x_j \cdot x_k$, or $\Gamma(x_j)$, or $\Gamma^{-1}(x_j)$. For a positive integer $n$, by a $Q$-computation of length $n$ we understand any sequence of terms $x_1, \ldots, x_n$ such that $x_1$ is identical to the variable $x$ and for every integer $i \in \{2, \ldots, n\}$ there exist integers $j, k \in \{1, \ldots, i - 1\}$ such that $x_i$ is identical to $x_j \cdot x_k$, or $\frac{x_j}{x_k}$, or $\Gamma(x_j)$, or $\Gamma^{-1}(x_j)$. Let $f(6) = 15!$, and let $f(n + 1) = \Gamma(f(n))$ for every integer $n \geq 6$. Let $g(6) = 24!$, and let $g(n + 1) = \Gamma(g(n))$ for every integer $n \geq 6$. For an integer $n \geq 6$, let $\Psi_n$ denote the following statement: if a $Q$-computation of length $n$ produces positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers $x$, then $\max(x_1, \ldots, x_n) \leq f(n)$ for every such $x$. For an integer $n \geq 6$, let $\Phi_n$ denote the following statement: if a $Q$-computation of length $n$ produces positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers $x$, then $\max(x_1, \ldots, x_n) \leq g(n)$ for every such $x$. We prove: (1) the statement $\Psi_6$ implies that if the equation $x(x + 1) = y!$ has at most finitely many solutions in positive integers, then each such solution $(x, y)$ belongs to the set $\{(1, 2), (2, 3)\}$; (2) if $y! + 1$ is a square for at most finitely many positive integers $y$, then the statement $\Psi_8$ implies that every such $y$ is smaller than $f(7)$; (3) the statement $\Phi_7$ implies that the set of Wilson primes is infinite.

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For a positive integer $x$, let $\Gamma(x)$ denote $(x - 1)!$. Let $\Gamma^{-1}: \{1, 2, 6, 24, \ldots \} \to \mathbb{N} \setminus \{0\}$ denote the inverse function that satisfies $\Gamma^{-1}(1) = 2$. For positive integers $x$ and $y$, let $\text{rest}(x, y)$ denote the rest from dividing $x$ by $y$.

Definition 1. For a positive integer $n$, by a $\Gamma$-computation of length $n$ we understand any sequence of terms $x_1, \ldots, x_n$ such that $x_1$ is identical to the variable $x$ and for every integer $i \in \{2, \ldots, n\}$ there exist integers $j, k \in \{1, \ldots, i - 1\}$ such that $x_i$ is identical to $x_j \cdot x_k$, or $\Gamma(x_j)$, or $\Gamma^{-1}(x_j)$.
Definition 2. For a positive integer \( n \), by a Q-computation of length \( n \) we understand any sequence of terms \( x_1, \ldots, x_n \) such that \( x_1 \) is identical to the variable \( x \) and for every integer \( i \in \{2, \ldots, n\} \) there exist integers \( j, k \in \{1, \ldots, i-1\} \) such that \( x_i \) is identical to \( x_j \cdot x_k \), or \( \frac{x_j}{x_k} \), or \( \Gamma(x_j) \), or \( \Gamma^{-1}(x_j) \).

Definition 3. For a positive integer \( n \), by a R-computation of length \( n \) we understand any sequence of terms \( x_1, \ldots, x_n \) such that \( x_1 \) is identical to the variable \( x \) and for every integer \( i \in \{2, \ldots, n\} \) there exist integers \( j, k \in \{1, \ldots, i-1\} \) such that \( x_i \) is identical to \( x_j \cdot x_k \), or \( \text{rest}(x_j, x_k) \), or \( \Gamma(x_j) \), or \( \Gamma^{-1}(x_j) \).

Let \( f(6) = 15! \), and let \( f(n + 1) = \Gamma(f(n)) \) for every integer \( n \geq 6 \). For an integer \( n \geq 6 \), let \( \Psi_n \) denote the following statement: if a \( \Gamma \)-computation of length \( n \) produces positive integers \( x_1, \ldots, x_n \) for at most finitely many positive integers \( x \), then \( \max(x_1, \ldots, x_n) \leq f(n) \) for every such \( x \).

Theorem 1. For every integer \( n \geq 6 \) and for every positive integer \( x \), the following \( \Gamma \)-computation

\[
\begin{align*}
x_1 &:= x \\
x_2 &:= x_1 \cdot x_1 \\
x_3 &:= \Gamma^{-1}(x_2) \\
x_4 &:= x_3 \cdot x_3 \\
x_5 &:= x_4 \cdot x_4 \\
\forall i \in \{6, \ldots, n\} & x_i := \Gamma(x_{i-1})
\end{align*}
\]

produces positive integers \( x_1, \ldots, x_n \) if and only if \( x = 1 \). If \( x = 1 \), then \( \max(x_1, \ldots, x_n) = f(n) \).

Proof. If \( x = 1 \), then \( x_1 = x_2 = 1, x_3 = 2, x_4 = 4, x_5 = 16, \) and \( x_i = f(i) \) for every integer \( i \in \{6, \ldots, n\} \). Hence, \( \max(x_1, \ldots, x_n) = f(n) \). If an integer \( x \) is greater than 1, then the term \( x_3 \) (that is identical to \( \Gamma^{-1}(x^2) \)) is not a positive integer, see [3] for a more general result. \( \square \)

Theorem 2. For every integer \( n \geq 6 \), the bound \( f(n) \) in the statement \( \Psi_n \) cannot be decreased.

Proof. It follows from Theorem 1. \( \square \)

Let \( g(6) = 24! \), and let \( g(n + 1) = \Gamma(g(n)) \) for every integer \( n \geq 6 \). For an integer \( n \geq 6 \), let \( \Phi_n \) denote the following statement: if a Q-computation of length \( n \) produces positive integers \( x_1, \ldots, x_n \) for at most finitely many positive integers \( x \), then \( \max(x_1, \ldots, x_n) \leq g(n) \) for every such \( x \).

Theorem 3. For every integer \( n \geq 6 \) and for every positive integer \( x \), the following Q-computation

\[
\begin{align*}
x_1 &:= x \\
x_2 &:= x_1 \cdot x_1 \\
x_3 &:= \Gamma(x_2) \\
x_4 &:= \Gamma(x_1) \\
x_5 &:= \Gamma(x_4) \\
x_6 &:= \frac{x_3}{x_5} \\
x_7 &:= \Gamma(x_3) \quad \text{(if } n \geq 7) \\
\forall i \in \{8, \ldots, n\} & x_i := \Gamma(x_{i-1}) \quad \text{(if } n \geq 8) 
\end{align*}
\]

produces positive integers \( x_1, \ldots, x_n \) if and only if \( x \in \{1, 2, 3, 4, 5\} \). If \( x \in \{1, 2, 3, 4\} \), then \( \max(x_1, \ldots, x_n) < g(n) \). If \( x = 5 \), then \( \max(x_1, \ldots, x_n) = g(n) \).
Proof. If \( x = 1 \), then \( x_1 = \ldots = x_6 = 1 \). Since \( x_3 \) is a positive integer, we obtain that \( x_7, \ldots, x_n \) are positive integers, if \( n \geq 7 \). Since \( \max(x_1, \ldots, x_6) < 24! \), we obtain that \( \max(x_1, \ldots, x_n) < g(n) \).

If \( x = 2 \), then \( x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 1, x_5 = 1, x_6 = 6 \). Since \( x_3 \) is a positive integer, we obtain that \( x_7, \ldots, x_n \) are positive integers, if \( n \geq 7 \). Since \( \max(x_1, \ldots, x_6) < 24! \), we obtain that \( \max(x_1, \ldots, x_n) < g(n) \).

If \( x = 3 \), then \( x_1 = 3, x_2 = 9, x_3 = 3! \), \( x_4 = 2, x_5 = 1, x_6 = 8! \). Since \( x_3 \) is a positive integer, we obtain that \( x_7, \ldots, x_n \) are positive integers, if \( n \geq 7 \). Since \( \max(x_1, \ldots, x_6) < 24! \), we obtain that \( \max(x_1, \ldots, x_n) < g(n) \).

If \( x = 4 \), then \( x_1 = 4, x_2 = 16, x_3 = 15!, x_4 = 6, x_5 = 120, x_6 = \frac{15!}{120} = 10897286400 \). Since \( x_3 \) is a positive integer, we obtain that \( x_7, \ldots, x_n \) are positive integers, if \( n \geq 7 \). Since \( \max(x_1, \ldots, x_6) < 24! \), we obtain that \( \max(x_1, \ldots, x_n) < g(n) \).

If \( x = 5 \), then
\[
\begin{align*}
x_1 &= 5 \\
x_2 &= x_1 \cdot x_1 = 25 \\
x_3 &= \Gamma(x_2) = 24! \\
x_4 &= \Gamma(x_1) = 24 \\
x_5 &= \Gamma(x_4) = 23! \\
x_6 &= \frac{x_3}{x_5} = \frac{24!}{23!} = 24
\end{align*}
\]

Since \( x_3 \) is a positive integer, we obtain that \( x_7, \ldots, x_n \) are positive integers, if \( n \geq 7 \). Since \( x_3 = \max(x_1, \ldots, x_6) = 24! \), we obtain that \( \max(x_1, \ldots, x_n) = g(n) \).

If an integer \( x \) is greater than 5, then
\[
x_6 = \frac{x_3}{x_5} = \frac{\Gamma(x^2)}{\Gamma(\Gamma(x))} < 1
\]
\(\square\)

**Theorem 4.** For every integer \( n \geq 6 \), the bound \( g(n) \) in the statement \( \Phi_n \) cannot be decreased.

**Proof.** It follows from Theorem 3. \(\square\)

Let \( h(6) = 119! \), and let \( h(n + 1) = \Gamma(h(n)) \) for every integer \( n \geq 6 \). For an integer \( n \geq 6 \), let \( \Theta_n \) denote the following statement: if a R-computation of length \( n \) produces positive integers \( x_1, \ldots, x_n \) for at most finitely many positive integers \( x \), then \( \max(x_1, \ldots, x_n) \leq h(n) \) for every such \( x \).

**Lemma 1.** ([4] pp. 214–215). For every positive integer \( x \), \( x \) does not divide \( \Gamma(x) \) if and only if \( x = 4 \) or \( x \) is prime.

**Theorem 5.** For every integer \( n \geq 6 \) and for every positive integer \( x \), the following R-computation
\[
\begin{align*}
x_1 &:= x \\
x_2 &:= x_1 \cdot x_1 \\
x_3 &:= \Gamma(x_2) \\
x_4 &:= \text{rest}(x_3, x_2) \\
x_5 &:= \Gamma(x_3) \\
\forall i \in \{6, \ldots, n\} \ i &:= \Gamma(x_{i-1})
\end{align*}
\]
produces positive integers \( x_1, \ldots, x_n \) if and only if \( x = 2 \). If \( x = 2 \), then \( \max(x_1, \ldots, x_n) = h(n) \).
Proof. If \( x = 1 \), then \( x_1 = x_2 = x_3 = 1 \) and \( x_4 = 0 \). If \( x = 2 \), then \( x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 2, x_5 = 120, \) and \( x_i = h(i) \) for every integer \( i \in \{6, \ldots, n\} \). Therefore, \( \max(x_1, \ldots, x_n) = h(n) \). If an integer \( x \) is greater than 2, then \( x^2 \) is composite and greater than 4. By Lemma 1

\[
x_4 = \text{rest}(x_3, x_2) = \text{rest}(\Gamma(x_2), x_2) = \text{rest}\left(\Gamma(x_2^2), x_2^2\right) = 0
\]

\[\square\]

**Theorem 6.** For every integer \( n \geq 6 \), the bound \( h(n) \) in the statement \( \Theta_n \) cannot be decreased.

**Proof.** It follows from Theorem [5] \[\square\]

**Lemma 2.** For every positive integer \( n \), there are only finitely many \( \Gamma \)-computations of length \( n \). For every positive integer \( n \), there are only finitely many \( Q \)-computations of length \( n \). For every positive integer \( n \), there are only finitely many \( R \)-computations of length \( n \).

**Theorem 7.** For every integer \( n \geq 6 \), the statement \( \Psi_n \) is true with an unknown integer bound that depends on \( n \). For every integer \( n \geq 6 \), the statement \( \Phi_n \) is true with an unknown integer bound that depends on \( n \). For every integer \( n \geq 6 \), the statement \( \Theta_n \) is true with an unknown integer bound that depends on \( n \).

**Proof.** It follows from Lemma [2] \[\square\]

**Theorem 8.** For every integer \( n \geq 6 \), the statement \( \Psi_{n+1} \) implies the statement \( \Psi_n \). For every integer \( n \geq 6 \), the statement \( \Phi_{n+1} \) implies the statement \( \Phi_n \). For every integer \( n \geq 6 \), the statement \( \Theta_{n+1} \) implies the statement \( \Theta_n \).

**Proof.** We present only the proof for the statement \( \Psi_{n+1} \) as the proofs for the statements \( \Phi_{n+1} \) and \( \Theta_{n+1} \) are essentially the same. Let \( n \in \{6, 7, 8, \ldots\} \). Let us assume that a \( \Gamma \)-computation \( \mathcal{W} \) of length \( n \) produces positive integers \( x_1, \ldots, x_n \) for at most finitely many positive integers \( x \). This implies that for every integer \( i \in \{1, \ldots, n\} \) the \( \Gamma \)-computation \( \mathcal{W} \) with added instruction \( x_{n+1} := \Gamma(x_i) \) produces positive integers \( x_1, \ldots, x_{n+1} \) for at most finitely many positive integers \( x \). The statement \( \Psi_{n+1} \) implies that

\[\forall i \in \{1, \ldots, n\} \quad \Gamma(x_i) = x_{n+1} \leq f(n + 1) = \Gamma(f(n))\]

Since \( f(n) > 1 \), we obtain that \( x_i \leq f(n) \) for every integer \( i \in \{1, \ldots, n\} \). \[\square\]

**Lemma 3.** For every positive integer \( x \), the term \( \Gamma^{-1}(x \cdot \Gamma(x)) \) represents \( x + 1 \).

**Lemma 4.** For every positive integer \( x \), \( x(x+1) \) is a factorial of a positive integer if and only if the following \( \Gamma \)-computation \( \mathcal{A} \)

\[
\begin{align*}
    x_1 &:= x \\
    x_2 &:= \Gamma(x_1) \\
    x_3 &:= x_1 \cdot x_2 \\
    x_4 &:= \Gamma^{-1}(x_3) \\
    x_5 &:= x_1 \cdot x_4 \\
    x_6 &:= \Gamma^{-1}(x_5)
\end{align*}
\]

produces positive integers \( x_1, \ldots, x_6 \).
Lemma 6. For every positive integer $x$ the terms $x_1, \ldots, x_5$ represent positive integers and $x_5 = x(x + 1)$. Hence, $x_6$ that is identical to $\Gamma^{-1}(x_5)$ represents a positive integer if and only if $\Gamma^{-1}(x(x + 1))$ represents a positive integer. The last means that $x(x + 1)$ equals $y!$ for some positive integer $y$. □

Theorem 9. The statement $\Psi_6$ implies that if the equation $x(x + 1) = y!$ has at most finitely many solutions in positive integers, then each such solution $(x, y)$ belongs to the set $\{(1, 2), (2, 3)\}$.

Proof. Let us assume that the equation $x(x + 1) = y!$ has at most finitely many solutions in positive integers. By Lemma 4 the $\Gamma$-computation $A$ produces positive integers $x_1, \ldots, x_6$, for at most finitely many positive integers $x$. We take positive integers $n$ and $m$ that satisfy $n(n + 1) = m!$. By Lemma 4 the $\Gamma$-computation $A$ for $x = n$ produces positive integers $x_1, \ldots, x_6$. The statement $\Psi_6$ implies that $x_2 = \Gamma(n) \leq f(6) = \Gamma(16)$. Since $16 > 1$, we obtain that $n \leq 16$. For every integer $n \in \{1, \ldots, 16\}$, $n(n + 1)$ is a factorial of a positive integer if and only if $n \in \{1, 2\}$. □

The question of solving the equation $x(x + 1) = y!$ was posed by P. Erdős, see [1]. F. Luca proved that the $abc$ conjecture implies that the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers, see [5].

A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the equation $y! + 1 = x^2$, see [6]. Let

$$F_1 = \left\{ y \in \mathbb{N} \setminus \{0\} : \exists x \in \mathbb{N} \setminus \{0\} \ y! + 1 = x^2 \right\}$$

It is conjectured that $F_1 = \{4, 5, 7\}$, see [9, p. 297].

Lemma 5. The set $F_1$ is finite if and only if the set

$$F_2 = \{ x \in \mathbb{N} \setminus \{0\} : \exists y \in \mathbb{N} \setminus \{0\} \ x(x + 2) = y! \}$$

is finite.

Proof. If $y! + 1 = x^2$, then $x \geq 5$ and $(x - 1)((x - 1) + 2) = y!$. If $x(x + 2) = y!$, then $y! + 1 = (x + 1)^2$. □

Lemma 6. For every positive integer $x$, the following $\Gamma$-computation $B$

$$
\begin{align*}
  x_1 &:= x \\
  x_2 &:= \Gamma(x_1) \\
  x_3 &:= x_1 \cdot x_2 \\
  x_4 &:= \Gamma^{-1}(x_3) \\
  x_5 &:= x_4 \cdot x_3 \\
  x_6 &:= \Gamma^{-1}(x_5) \\
  x_7 &:= x_1 \cdot x_6 \\
  x_8 &:= \Gamma^{-1}(x_7) 
\end{align*}
$$

produces positive integers $x_1, \ldots, x_8$ if and only if $x(x + 2)$ is a factorial of a positive integer.

Proof. By Lemma 5, for every positive integer $x$, the terms $x_1, \ldots, x_7$ represent positive integers and $x_5 = x \cdot (x + 2)$. The term $x_8$ (that is identical to $\Gamma^{-1}(x(x + 2))$) represents a positive integer if and only if $x(x + 2)$ is a factorial of a positive integer. □
Theorem 10. If $y! + 1$ is a square for at most finitely many positive integers $y$, then the statement $\Psi_8$ implies that every such $y$ is smaller than $f(7)$.

Proof. If positive integers $n$ and $m$ satisfy $n! + 1 = m^2$, then $m \geq 5$ and

$$(m - 1) \cdot ((m - 1) + 2) = \Gamma(n + 1)$$

By this and Lemma 6, the $\Gamma$-computation $B$ produces for $x = m - 1$ positive integers $x_1, \ldots, x_8$. The antecedent and Lemma 5 imply that the set $F_2$ is finite. Therefore, the statement $\Psi_8$ guarantees that $\Gamma(n + 1) = x_7 \leq f(8) = \Gamma(f(7))$. Since $f(7) > 1$, we obtain that $n + 1 \leq f(7)$. Thus, $n < f(7)$. □

Lemma 7. (Wilson’s theorem, [4, p. 89]). For every positive integer $x$, $x$ divides $\Gamma(x) + 1$ if and only if $x = 1$ or $x$ is prime.

A Wilson prime is a prime number $p$ such that $p^2$ divides $(p - 1)! + 1$. It is conjectured that the set of Wilson primes is infinite, see [2].

Lemma 8. For every positive integer $x$, the following Q-computation $C$

$$
\begin{align*}
  x_1 &:= x \\
  x_2 &:= \Gamma(x_1) \\
  x_3 &:= \Gamma(x_2) \\
  x_4 &:= x_2 \cdot x_3 \\
  x_5 &:= \Gamma^{-1}(x_4) \\
  x_6 &:= \frac{x_5}{x_1} \\
  x_7 &:= \frac{x_6}{x_1}
\end{align*}
$$

produces positive integers $x_1, \ldots, x_7$ if and only if $x = 1$ or $x$ is a Wilson prime.

Proof. By Lemma 3, for every positive integer $x$, the terms $x_1, \ldots, x_5$ represent positive integers and $x_5 = \Gamma(x) + 1$. By Lemma 7, the term $x_6$ (that is identical to $\frac{\Gamma(x) + 1}{x}$) and the term $x_7$ (that is identical to $\frac{\Gamma(x) + 1}{x^2}$) represent positive integers if and only if $x = 1$ or $x$ is a Wilson prime. □

Theorem 11. The statement $\Phi_7$ implies that the set of Wilson primes is infinite.

Proof. The number 563 is a Wilson prime, see [2] and [8]. By Lemma 8 for $x = 563$ the Q-computation $C$ produces positive integers $x_1, \ldots, x_7$. We have:

$$
\begin{align*}
  x_1 &= 563 \\
  x_2 &= \Gamma(563) \\
  x_3 &= \Gamma(563) \\
  x_4 &= \Gamma(563) \cdot \Gamma(563) = \Gamma(563)^2 \\
  x_5 &= \Gamma(563)^2 + 1 \\
  x_6 &= \frac{563}{563 + 1} \\
  x_7 &= \frac{563 + 1}{563^2}
\end{align*}
$$

Since $\max(x_1, \ldots, x_7) = x_4 = \Gamma(563)^2 + 1 > \Gamma(24!) = \Gamma(g(6)) = g(7)$, the statement $\Phi_7$ implies that the Q-computation $C$ produces positive integers $x_1, \ldots, x_7$ for infinitely many positive integers $x$. By Lemma 8 we obtain that the set of Wilson primes is infinite. □
Let

\[ T_1 = \{ n \in \mathbb{N} \setminus \{0\} : n! + 1 \text{ is prime} \} \]
\[ T_2 = \{ n \in \mathbb{N} \setminus \{0\} : (n \text{ is prime}) \land (n! + 1 \text{ is prime}) \} \]
\[ T_3 = \{ n \in \mathbb{N} \setminus \{0\} : (n \text{ is prime}) \land (n + 2 \text{ is prime}) \} \]
\[ T_4 = \{ n \in \mathbb{N} \setminus \{0\} : n^2 + 1 \text{ is prime} \} \]
\[ T_5 = \{ n \in \mathbb{N} \setminus \{0\} : (n^2 + 1 \text{ is prime}) \land (n^2 + 3 \text{ is prime}) \} \]
\[ T_6 = \{ n \in \mathbb{N} \setminus \{0\} : (n \text{ is prime}) \land (2n + 1 \text{ is prime}) \} \]

It is commonly conjectured that the sets \( T_1, \ldots, T_6 \) are infinite.

**Theorem 12.** For every integer \( i \in \{1, \ldots, 6\} \) we can compute positive integers \( j \) and \( k \) such that the statement \( \Phi_j \) implies that any element of \( T_i \) that is greater than \( k \) proves that the set \( T_i \) is infinite.

**Proof.** The proof is left to the reader, because for every integer \( i \in \{1, \ldots, 6\} \) the proof essentially goes as in the proof of Theorem 11. \( \square \)

**Lemma 9.** For every positive integer \( x \), the following R-computation \( D \)

\[
\begin{align*}
x_1 &:= x \\
x_2 &:= x_1 \cdot x_1 \\
x_3 &:= \Gamma(x_2) \\
x_4 &:= x_2 \cdot x_3 \\
x_5 &:= \Gamma^{-1}(x_4) \\
x_6 &:= \text{rest}(x_4, x_5)
\end{align*}
\]

produces positive integers \( x_1, \ldots, x_6 \) if and only if \( x^2 + 1 \) is prime.

**Proof.** It follows from Lemma 1 because \( x^2 + 1 \neq 4 \). \( \square \)

**Theorem 13.** The statement \( \Theta_6 \) implies that there are infinitely many primes of the form \( n^2 + 1 \).

**Proof.** The number \( 14^2 + 1 \) is prime. By Lemma 9, for \( x = 14 \) the R-computation \( D \) produces positive integers \( x_1, \ldots, x_6 \). Since \( x_4 = \Gamma(14^2 + 1) > \Gamma(120) = h(6) \), the statement \( \Theta_6 \) guarantees that the R-computation \( D \) produces positive integers \( x_1, \ldots, x_6 \) for infinitely many positive integers \( x \). By Lemma 9, we obtain that there are infinitely many primes of the form \( n^2 + 1 \). \( \square \)

**Lemma 10.** For every positive integer \( x \), the following R-computation \( E \)

\[
\begin{align*}
x_1 &:= x \\
x_2 &:= \Gamma(x_1) \\
x_3 &:= \Gamma(x_2) \\
x_4 &:= x_2 \cdot x_3 \\
x_5 &:= \Gamma^{-1}(x_4) \\
x_6 &:= \text{rest}(x_4, x_5)
\end{align*}
\]

produces positive integers \( x_1, \ldots, x_6 \) if and only if \( \Gamma(x) + 1 \) is prime.

**Proof.** It follows from Lemma 1 because \( \Gamma(x) + 1 \neq 4 \). \( \square \)
Theorem 14. The statement $\Theta_6$ implies that there are infinitely many primes of the form $n! + 1$.

Proof. The number $\Gamma(12) + 1$ is prime. By Lemma [10] for $x = 12$ the $R$-computation $E$ produces positive integers $x_1, \ldots, x_6$. Since $x_4 = \Gamma(12) + 1 > \Gamma(120) = h(6)$, the statement $\Theta_6$ guarantees that the $R$-computation $E$ produces positive integers $x_1, \ldots, x_6$ for infinitely many positive integers $x$. By Lemma [10] we obtain that there are infinitely many primes of the form $\Gamma(x) + 1$.

Lemma 11. For every positive integer $x$, the following $R$-computation

\[
\begin{align*}
x_1 &:= x \\
x_2 &:= \Gamma(x_1) \\
x_3 &:= x_1 \cdot x_2 \\
x_4 &:= \Gamma^{-1}(x_3) \\
x_5 &:= x_4 \cdot x_3 \\
x_6 &:= \Gamma^{-1}(x_5) \\
x_7 &:= \text{rest}(x_2, x_1) \\
x_8 &:= \text{rest}(x_5, x_6)
\end{align*}
\]

produces positive integers $x_1, \ldots, x_8$ if and only if $x = 2$ or both $x$ and $x + 2$ are prime.


Theorem 15. The statement $\Theta_8$ implies that any twin prime greater than $h(7)-2$ proves that the set of twin primes is infinite.

Proof. The proof is based on Lemma [11]. We omit this proof because is similar to the proof of Theorem [11].

References


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