On ZFC-formulae \( \varphi(x) \) for which we know a non-negative integer \( n \) such that
\[
\{ x \in \mathbb{N} : \varphi(x) \} \subseteq \{ x \in \mathbb{N} : x \leq n - 1 \}
\] if the set \( \{ x \in \mathbb{N} : \varphi(x) \} \) is finite

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Abstract

We say that a non-negative integer \( m \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if \( X \) is infinite if and only if \( X \) contains an element greater than \( m \). We formulate hypothetical statements \( \Psi_3, \ldots, \Psi_{16} \). The statement \( \Psi_9 \) implies that a simple formula computes a gigantic threshold number of the set of primes of the form \( n^2 + 1 \). The statement \( \Psi_9 \) implies that a simple formula computes a gigantic threshold number of the set of primes of the form \( n! + 1 \). The statement \( \Psi_{16} \) implies that a simple formula computes a gigantic threshold number of the set of twin primes. The following problem is open: define a set \( X \subseteq \mathbb{N} \) that satisfies the following conditions:

1. a known algorithm for every \( n \in \mathbb{N} \) decides whether or not \( n \in X \),
2. a known algorithm returns a threshold number of \( X \),
3. new elements of \( X \) are still discovered,
4. we do not know any algorithm deciding the inequality \( \text{card}(X) < \infty \).

Key words and phrases: arithmetical consistency of ZFC, Brocard-Ramanujan equation \( x! + 1 = y^2 \), composite Fermat numbers, finiteness/infiniteness of a set, incompleteness of ZFC, oracle for the halting problem, prime numbers of the form \( n^2 + 1 \), prime numbers of the form \( n! + 1 \), twin primes, Sophie Germain primes.

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1 Introduction

The phrase “we know a non-negative integer \( n \)” in the title means that we know an algorithm which returns \( n \). The title cannot be formalised in ZFC because the phrase “we know a non-negative integer \( n \)” refers to currently known non-negative integers \( n \) with some property. A formally stated title may look like this: On ZFC-formulae \( \varphi(x) \) for which there exists a non-negative integer \( n \) such that ZFC proves that
\[
\text{card}(\{ x \in \mathbb{N} : \varphi(x) \}) < \infty \implies \{ x \in \mathbb{N} : \varphi(x) \} \subseteq \{ x \in \mathbb{N} : x \leq n - 1 \}
\]
Unfortunately, this formulation admits formulae \( \varphi(x) \) without any known non-negative integer \( n \) such that ZFC proves the above implication.

Lemma 1. For every non-negative integer \( n \), \( \text{card}(\{ x \in \mathbb{N} : x \leq n - 1 \}) = n \).

Corollary 1. The title altered to “ZFC-formulae \( \varphi(x) \) for which we know a non-negative integer \( n \) such that \( \text{card}(\{ x \in \mathbb{N} : \varphi(x) \}) \leq n \) if the set \( \{ x \in \mathbb{N} : \varphi(x) \} \) is finite” involves a weaker assumption on \( \varphi(x) \).
2 Subsets of \( \mathbb{N} \) and their threshold numbers

We say that a non-negative integer \( m \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if \( X \) is infinite if and only if \( X \) contains an element greater than \( m \), cf. \([25]\) and \([26]\). If a set \( X \subseteq \mathbb{N} \) is empty or finite, then any non-negative integer \( m \) is a threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is non-empty and finite, then the all threshold numbers of \( X \) form the set \( \{ \max(X), \max(X)+1, \max(X)+2, \ldots \} \).

It is conjectured that the set of prime numbers of the form \( n^2 + 1 \) is infinite, see \([15\) pp. 37–38]. It is conjectured that the set of prime numbers of the form \( n!+1 \) is infinite, see \([3\) p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see \([13\) p. 39]. It is conjectured that the set of composite numbers of the form \( 2^{2^n} + 1 \) is infinite, see \([11\) p. 23] and \([12\) pp. 158–159]. A prime \( p \) is said to be a Sophie Germain prime if both \( p \) and \( 2p+1 \) are prime, see \([23\]. It is conjectured that the set of Sophie Germain primes is infinite, see \([18\) p. 330]. For each of these sets, we do not know any threshold number.

**Open Problem 1.** Define a set \( X \subseteq \mathbb{N} \) that satisfies the following conditions: (1) a known algorithm for every \( n \in \mathbb{N} \) decides whether or not \( n \in X \), (2) a known algorithm returns a threshold number of \( X \), (3) new elements of \( X \) are still discovered, (4) we do not know any algorithm deciding the inequality \( \text{card}(X) < \infty \).

The following statement:

\[
\text{for every non-negative integer } n \text{ there exist prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in \left[ 10^n, 10^n+1 \right] \tag{T}
\]

is a \( \Pi_1 \) statement which strengthens the twin prime conjecture, see \([4\) p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger \( \Pi_1 \) statements, see \([1\). The statement \( (T) \) is equivalent to the non-halting of a Turing machine. If a set \( X \subseteq \mathbb{N} \) is computable and we know a threshold number of \( X \), then the infinity of \( X \) is equivalent to the halting of a Turing machine.

The height of a rational number \( \frac{p}{q} \) is denoted by \( H\left(\frac{p}{q}\right) \) and equals \( \max(|p|,|q|) \) provided \( \frac{p}{q} \) is written in lowest terms. The height of a rational tuple \( (x_1, \ldots, x_n) \) is denoted by \( H(x_1, \ldots, x_n) \) and equals \( \max(H(x_1), \ldots, H(x_n)) \).

**Observation 1.** The equation \( x^5 - x = y^2 - y \) has only finitely many rational solutions, see \([14\) p. 212]. The known rational solutions are \( (x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left( \frac{15}{16}, \frac{1209}{1024} \right), \left( \frac{15}{16}, \frac{185}{1024} \right) \), and the existence of other solutions is an open question, see \([79\] pp. 223–224].

**Corollary 2.** The set

\[ \mathcal{T} = \{ n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n \} \]

is finite. We know an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in \mathcal{T} \). We do not know any algorithm which returns a threshold number of \( \mathcal{T} \).
Let $L$ denote the following system of equations:

$$
\begin{align*}
    x^2 + y^2 &= s^2 \\
    x^2 + z^2 &= t^2 \\
    y^2 + z^2 &= u^2 \\
    x^2 + y^2 + z^2 &= v^2
\end{align*}
$$

Let

$$
\mathcal{F} = \{ n \in \mathbb{N} \setminus \{0\} : \text{the system } L \text{ has no solutions in } \{1, \ldots, n\}^3 \} \land \\
\text{the system } L \text{ has a solution in } \{1, \ldots, n + 1\}^3 \}
$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

**Observation 2.** ([22]) No perfect cuboids are known.

**Corollary 3.** We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{F}$. ZFC proves that $\text{card}(\mathcal{F}) \in \{0, 1\}$. We do not know any algorithm which returns $\text{card}(\mathcal{F})$. We do not know any algorithm which returns a threshold number of $\mathcal{F}$.

Let

$$
\mathcal{H} = \begin{cases} 
\mathbb{N}, & \text{if } \sin \left( \frac{99999}{99999} \right) < 0 \\
\mathbb{N} \cap \left[ 0, \sin \left( \frac{99999}{99999} \right) \right], & \text{otherwise}
\end{cases}
$$

We do not know whether or not the set $\mathcal{H}$ is finite.

**Observation 3.** The number 99999 is a threshold number of $\mathcal{H}$. We know an algorithm which decides the equality $\mathcal{H} = \mathbb{N}$. If $\mathcal{H} \neq \mathbb{N}$, then the set $\mathcal{H}$ consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{H}$.

Let

$$
\mathcal{K} = \begin{cases} 
\{ n \}, & \text{if } (n \in \mathbb{N}) \land \left( 2^n = n+1 \right) \\
\{ 0 \}, & \text{if } 2^n \geq n+1
\end{cases}
$$

**Theorem 1.** ZFC proves that $\text{card}(\mathcal{K}) = 1$. If ZFC is consistent, then for every $n \in \mathbb{N}$ the sentences ”$n$ is a threshold number of $\mathcal{K}$” and ”$n$ is not a threshold number of $\mathcal{K}$” are not provable in ZFC.

**Proof.** It suffices to observe that $2^n$ can attain every value from the set $\{N_1, N_2, N_3, \ldots \}$, see [7] and [10, p. 232].
3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel’s second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

Theorem 2. ([5] p. 35) There exists a polynomial \( D(x_1, \ldots, x_m) \) with integer coefficients such that if ZFC is arithmetically consistent, then the sentences “The equation \( D(x_1, \ldots, x_m) = 0 \) is solvable in non-negative integers” and “The equation \( D(x_1, \ldots, x_m) = 0 \) is not solvable in non-negative integers” are not provable in ZFC.

Observation 4. ([9] p. 53) The polynomial \( D(x_1, \ldots, x_m) \) is not effectively known.

Let \( Y \) denote the set of all non-negative integers \( k \) such that the equation \( D(x_1, \ldots, x_m) = 0 \) has no solutions in \( \{0, \ldots, k\}^m \). Since the set \( \{0, \ldots, k\}^m \) is finite, there exists an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in Y \). Theorem 2 implies the next theorem.

Theorem 3. For every \( n \in \mathbb{N} \), ZFC proves that \( n \in Y \). If ZFC is arithmetically consistent, then the sentences “\( Y \) is finite” and “\( Y \) is infinite” are not provable in ZFC. If ZFC is arithmetically consistent, then for every \( n \in \mathbb{N} \) the sentences “\( n \) is a threshold number of \( Y \)” and “\( n \) is not a threshold number of \( Y \)” are not provable in ZFC.

Let \( E \) denote the set of all non-negative integers \( k \) such that the equation \( D(x_1, \ldots, x_m) = 0 \) has a solution in \( \{0, \ldots, k\}^m \). Since the set \( \{0, \ldots, k\}^m \) is finite, there exists an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in E \). Theorem 2 implies the next theorem.

Theorem 4. The set \( E \) is empty or infinite. In both cases, every non-negative integer \( n \) is a threshold number of \( E \). If ZFC is arithmetically consistent, then the sentences “\( E \) is empty”, “\( E \) is not empty”, “\( E \) is finite”, and “\( E \) is infinite” are not provable in ZFC.

Let

\[
\mathcal{V} = \left\{ n \in \mathbb{N} : \left( \text{the polynomial } D(x_1, \ldots, x_m) \text{ has no solutions in } \{0, \ldots, n\}^m \right) \land \left( \text{the polynomial } D(x_1, \ldots, x_m) \text{ has a solution in } \{0, \ldots, n+1\}^m \right) \right\}
\]

Since the sets \( \{0, \ldots, n\}^m \) and \( \{0, \ldots, n+1\}^m \) are finite, there exists an algorithm which for every \( n \in \mathbb{N} \) decides whether or not \( n \in \mathcal{V} \). According to Observation 4, at present we are not able to write a computer program that realizes such an algorithm. Theorem 2 implies the next theorem.

Theorem 5. (5) ZFC proves that \( \text{card}(\mathcal{V}) \in \{0, 1\} \). (6) For every \( n \in \mathbb{N} \), ZFC proves that \( n \notin \mathcal{V} \). (7) ZFC does not prove the emptiness of \( \mathcal{V} \) if ZFC is arithmetically consistent. (8) For every \( n \in \mathbb{N} \), the sentence “\( n \) is a threshold number of \( \mathcal{V} \)” is not provable in ZFC, if ZFC is arithmetically consistent. (9) For every \( n \in \mathbb{N} \), the sentence “\( n \) is not a threshold number of \( \mathcal{V} \)” is not provable in ZFC, if ZFC is arithmetically consistent.

Open Problem 2. Define a simple algorithm \( A \) such that \( A \) returns 0 or 1 on every input \( k \in \mathbb{N} \) and the set

\[
\mathcal{V} = \{ k \in \mathbb{N} : \text{ the program } A \text{ returns 1 on input } k \}
\]

satisfies conditions (5)–(9).
4 Number-theoretic lemmas

Lemma 2. For every positive integers \(x\) and \(y\), \(x! \cdot y = y!\) if and only if
\[(x + 1 = y) \lor (x = y = 1)\]

Let \(\Gamma(k)\) denote \((k - 1)!\).

Lemma 3. For every positive integers \(x\) and \(y\), \(x \cdot \Gamma(x) = \Gamma(y)\) if and only if
\[(x + 1 = y) \lor (x = y = 1)\]

Lemma 4. For every non-negative integers \(b\) and \(c\), \(b + 1 = c\) if and only if
\[2^{2^b} \cdot 2^{2^b} = 2^{2^c}\]

Lemma 5. (Wilson’s theorem, [8, p. 89]). For every positive integer \(x\), \(x\) divides \((x - 1)! + 1\) if and only if \(x = 1\) or \(x\) is prime.

5 Hypothetical statements \(\Psi_3, \ldots, \Psi_{16}\)

For an integer \(n \geq 3\), let \(\mathcal{U}_n\) denote the following system of equations:

\[
\begin{aligned}
\forall i \in \{1, \ldots, n - 1\} \setminus \{2\} & \quad x_i! = x_{i+1} \\
 x_1 \cdot x_2 & = x_3 \\
x_2 \cdot x_2 & = x_3
\end{aligned}
\]

The diagram in Figure 1 illustrates the construction of the system \(\mathcal{U}_n\).

![Fig. 1 Construction of the system \(\mathcal{U}_n\)](image)

Let \(g(3) = 4\), and let \(g(n + 1) = g(n)!\) for every integer \(n \geq 3\).

Lemma 6. For every integer \(n \geq 3\), the system \(\mathcal{U}_n\) has exactly two solutions in positive integers, namely \((1, \ldots, 1)\) and \((2, 2, g(3), \ldots, g(n))\).

Let
\[
B_n = \left\{ x_i! = x_k : (i, k \in \{1, \ldots, n\}) \land (i \neq k) \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\}
\]

For an integer \(n \geq 3\), let \(\Psi_n\) denote the following statement: if a system of equations \(S \subseteq B_n\) has only finitely many solutions in positive integers \(x_1, \ldots, x_n\), then each such solution \((x_1, \ldots, x_n)\) satisfies \(x_1, \ldots, x_n \leq g(n)\). The statement \(\Psi_n\) says that for subsystems of \(B_n\) the largest known solution is indeed the largest possible.
Hypothesis 1. The statements $\Psi_3, \ldots, \Psi_{16}$ are true.

Lemma 7. Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

Proof. For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

Lemma 8. For every statement $\Psi_n$, the bound $g(n)$ cannot be decreased.

Proof. It follows from Lemma 6 because $U_n \subseteq B_n$. □

6 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let $\mathcal{A}$ denote the following system of equations:

$$
\begin{align*}
  x_1! &= x_2 \\
  x_2! &= x_3 \\
  x_5! &= x_6 \\
  x_4 \cdot x_4 &= x_5 \\
  x_3 \cdot x_5 &= x_6
\end{align*}
$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 9. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system $\mathcal{A}$ is solvable in positive integers $x_2, x_3, x_5, x_6$ if and only if $x_1! + 1 = x_4^2$. In this case, the integers $x_2, x_3, x_5, x_6$ are uniquely determined by the following equalities:

$$
\begin{align*}
  x_2 &= x_1! \\
  x_3 &= (x_1!)! \\
  x_5 &= x_1! + 1 \\
  x_6 &= (x_1! + 1)!
\end{align*}
$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [21, p. 297]. A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [16].

Theorem 6. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement $\Psi_6$ guarantees that each such solution $(x_1, x_4)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$. 


Proof. Suppose that the antecedent holds. Let positive integers $x_1$ and $x_4$ satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 9, the system $A$ is solvable in positive integers $x_2, x_3, x_5, x_6$. Since $A \subseteq B_6$, the statement $\Psi_6$ implies that $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$. Hence, $x_1! + 1 \leq g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$.

\[\square\]

7 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau’s conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [15, pp. 37–38]. Let $B$ denote the following system of equations:

\[
\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_3 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}
\]

Lemma 2 and the diagram in Figure 3 explain the construction of the system $B$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{construction_of_system_B.png}
\caption{Construction of the system $B$}
\end{figure}

**Lemma 10.** For every integer $x_1 \geq 2$, the system $B$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the
following equalities:

\[
\begin{align*}
x_2 &= x_1^2 + 1 \\
x_3 &= (x_1^2)! \\
x_4 &= (((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= (((x_1^2)! + 1)! \\
\end{align*}
\]

**Proof.** By Lemma 2, for every integer \( n \geq 2 \), the system \( B \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) divides \((x_1^2)! + 1\). Hence, the claim of Lemma 10 follows from Lemma 5. \( \square \)

**Lemma 11.** There are only finitely many tuples \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\) which solve the system \( B \) and satisfy \( x_1 = 1 \).

**Proof.** If a tuple \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\) solves the system \( B \) and \( x_1 = 1 \), then \( x_1, \ldots, x_9 \leq 2 \). Indeed, \( x_1 = 1 \) implies that \( x_2 = x_1^2 = 1 \). Hence, for example, \( x_3 = x_2! = 1 \). Therefore, \( x_8 = x_3 + 1 = 2 \) or \( x_8 = 1 \). Consequently, \( x_9 = x_8! \leq 2 \). \( \square \)

**Theorem 7.** The statement \( \Psi_0 \) proves the following implication: if there exists an integer \( x_1 \geq 2 \) such that \( x_1^2 + 1 \) is prime and greater than \( g(7) \), then there are infinitely many primes of the form \( n^2 + 1 \).

**Proof.** Suppose that the antecedent holds. By Lemma 10 there exists a unique tuple \((x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\) such that the tuple \((x_1, x_2, \ldots, x_9)\) solves the system \( B \). Since \( x_1^2 + 1 > g(7) \), we obtain that \( x_1^2 \geq g(7) \). Hence, \((x_1^2)! \geq g(7)! = g(8) \). Consequently, \( x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9) \).

Since \( B \subseteq B_0 \), the statement \( \Psi_0 \) and the inequality \( x_9 > g(9) \) imply that the system \( B \) has infinitely many solutions \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\). According to Lemmas 10 and 11 there are infinitely many primes of the form \( n^2 + 1 \). \( \square \)

**Corollary 4.** Let \( X_0 \) denote the set of primes of the form \( n^2 + 1 \). The statement \( \Psi_0 \) implies that we know an algorithm such that it returns a threshold number of \( X_0 \), and this number equals \( \text{max}(X_0) \), if \( X_0 \) is finite. Assuming the statement \( \Psi_0 \), a single query to an oracle for the halting problem decides the infinity of \( X_0 \). Assuming the statement \( \Psi_0 \), the infinity of \( X_0 \) is decidable in the limit.

**Proof.** We consider an algorithm which computes \( \text{max}(X_0 \cap [1, g(7)]) \). \( \square \)

8 Are there infinitely many prime numbers of the form \( n! + 1 \)?

It is conjectured that there are infinitely many primes of the form \( n! + 1 \), see [3, p. 443].

**Theorem 8.** (cf. Theorem 72). The statement \( \Psi_9 \) proves the following implication: if there exists an integer \( x_1 \geq g(6) \) such that \( x_1! + 1 \) is prime, then there are infinitely many primes of the form \( n! + 1 \).

**Proof.** We leave the analogous proof to the reader. \( \square \)
9 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [15, p. 39]. Let $C$ denote the following system of equations:

\[
\begin{align*}
x_1 & = x_2 \\
x_2 & = x_3 \\
x_4 & = x_5 \\
x_6 & = x_7 \\
x_7 & = x_8 \\
x_9 & = x_{10} \\
x_{12} & = x_{13} \\
x_{14} & = x_{15} \\
x_2 \cdot x_4 & = x_5 \\
x_5 \cdot x_6 & = x_7 \\
x_7 \cdot x_9 & = x_{10} \\
x_4 \cdot x_{11} & = x_{12} \\
x_3 \cdot x_{12} & = x_{13} \\
x_9 \cdot x_{14} & = x_{15} \\
x_8 \cdot x_{15} & = x_{16}
\end{align*}
\]

Lemma 2 and the diagram in Figure 4 explain the construction of the system $C$.

![Diagram](image-url)

Fig. 4 Construction of the system $C$
Lemma 12. For every \( x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\} \), the system \( C \) is solvable in positive integers \( x_4, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \) if and only if \( x_4 \) and \( x_9 \) are prime and \( x_4 + 2 = x_9 \). In this case, the integers \( x_4, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \) are uniquely determined by the following equalities:

\[
\begin{align*}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{align*}
\]

Proof. By Lemma 2, for every \( x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\} \), the system \( C \) is solvable in positive integers \( x_4, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \) if and only if

\[
(x_4 + 2 = x_9) \land (x_4((x_4 - 1)! + 1) \land (x_9((x_9 - 1)! + 1))
\]

Hence, the claim of Lemma 12 follows from Lemma 5.

Lemma 13. There are only finitely many tuples \( (x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16} \) which solve the system \( C \) and satisfy \( x_4 \in \{1, 2\} \) or \( x_9 \in \{1, 2\} \).

Proof. If a tuple \( (x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16} \) solves the system \( C \) and \( (x_4, x_9) \in \{1, 2\} \), then \( x_1, \ldots, x_{16} \leq 7! \). Indeed, for example, if \( x_4 = 2 \) then \( x_6 = x_4 + 1 = 3 \). Hence, \( x_7 = x_6! = 6 \). Therefore, \( x_15 = x_7 + 1 = 7 \). Consequently, \( x_{16} = x_15! = 7! \).

Theorem 9. The statement \( \Psi_{16} \) proves the following implication: if there exists a twin prime greater than \( g(14) \), then there are infinitely many twin primes.

Proof. Suppose that the antecedent holds. Then, there exist prime numbers \( x_4 \) and \( x_9 \) such that \( x_9 = x_4 + 2 > g(14) \). Hence, \( x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\} \). By Lemma 12 there exists a unique tuple \( (x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16} \) such that the tuple \( (x_1, \ldots, x_{16}) \) solves the system \( C \). Since \( x_9 > g(14) \), we obtain that \( x_9 - 1 \geq g(14) \). Therefore, \( (x_9 - 1)! \geq g(14)! = g(15) \). Hence, \( (x_9 - 1)! + 1 > g(15) \). Consequently,

\[
x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)
\]

Since \( C \subseteq B_{16} \), the statement \( \Psi_{16} \) and the inequality \( x_{16} > g(16) \) imply that the system \( C \) has infinitely many solutions in positive integers \( x_1, \ldots, x_{16} \). According to Lemmas 12 and 13 there are infinitely many twin primes.

Corollary 5. (cf. [9]). Let \( X_{16} \) denote the set of twin primes. The statement \( \Psi_{16} \) implies that we know an algorithm such that it returns a threshold number of \( X_{16} \), and this number equals \( \max(X_{16}) \), if \( X_{16} \) is finite. Assuming the statement \( \Psi_{16} \), a single query to an oracle for the halting problem decides the infinity of \( X_{16} \). Assuming the statement \( \Psi_{16} \), the infinity of \( X_{16} \) is decidable in the limit.

Proof. We consider an algorithm which computes \( \max(X_{16} \cap [1, g(14)]) \).
10 Hypothetical statements $\Delta_5, \ldots, \Delta_{14}$ and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n + 1) = \Gamma(\lambda(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let $\mathcal{J}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{3\} \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{cases}$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system $\mathcal{J}_n$.

![Diagram](image)

**Fig. 5** Construction of the system $\mathcal{J}_n$

For every integer $n \geq 5$, the system $\mathcal{J}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$. For an integer $n \geq 5$, let $\Delta_n$ denote the following statement: if a system of equations $S \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq \lambda(n)$.

**Hypothesis 2.** The statements $\Delta_5, \ldots, \Delta_{14}$ are true.

Lemmas 3 and 5 imply that the statements $\Delta_n$ have similar consequences as the statements $\Psi_n$.

**Theorem 10.** The statement $\Delta_6$ implies that any prime number $p \geq 25$ proves the infinitude of primes.

**Proof.** It follows from Lemmas 3 and 5. We leave the details to the reader. \(\square\)
11 Hypothetical statements $\Sigma_3, \ldots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup [2^{2n-3} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \ldots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

For an integer $n \in \{3, \ldots, 16\}$, let $P_n$ denote the following system of equations:

$$\left\{ \begin{array}{l}
x_1 \cdot x_1 = x_1 \\
\Gamma_n(x_2) = x_1 \\
\forall i \in \{2, \ldots, n-1\} \ x_i \cdot x_i = x_{i+1}
\end{array} \right.$$

**Lemma 14.** For every integer $n \in \{3, \ldots, 16\}$, $P_n \subseteq Q_n$ and the system $P_n$ with $\Gamma$ instead of $\Gamma_n$ has exactly one solution in positive integers $x_1, \ldots, x_n$, namely $(1, 2^{20}, 2^{21}, 2^{22}, \ldots, 2^{2n-2})$.

For an integer $n \in \{3, \ldots, 16\}$, let $\Sigma_n$ denote the following statement: if a system of equations $S \subseteq Q_n$ with $\Gamma$ instead of $\Gamma_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then every tuple $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system $S$ satisfies $x_1, \ldots, x_n \leq 2^{2n-2}$.

**Hypothesis 3.** The statements $\Sigma_3, \ldots, \Sigma_{16}$ are true.

**Lemma 15.** (cf. Lemma 3). For every integer $n \in \{4, \ldots, 16\}$ and for every positive integers $x$ and $y$, $x \cdot \Gamma_n(x) = \Gamma_n(y)$ if and only if $(x + 1 = y) \land \left(x \geq 2^{2n-3} + 1\right)$.

Let $Z_9 \subseteq Q_9$ be the system of equations in Figure 6.

![Fig. 6 Construction of the system $Z_9$](image)

**Lemma 16.** For every positive integer $x_1$, the system $Z_9$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1 > 2^{2^9-4}$ and $x_1^2 + 1$ is prime. In this case, positive integers $x_2, \ldots, x_9$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with $n$ and solve the system $Z_9$ with $\Gamma$ instead of $\Gamma_9$. 
Proof. It follows from Lemmas \([3, 5, 15]\). □

**Lemma 17.** \((\ref{lem:prime})\). The number \((13!)^2 + 1 = 38775788043632640001\) is prime.

**Lemma 18.** \((13!)^2 \geq 2^{9^3} + 1 = 18446744073709551617 \land (\Gamma_9((13!)^2) > 2^{9^3-2}).\)

**Theorem 11.** The statement \(\Sigma_9\) implies the infinitude of primes of the form \(n^2 + 1\).

**Proof.** It follows from Lemmas \([16, 18]\). □

**Theorem 12.** \((\text{cf. Theorem 8})\). The statement \(\Sigma_9\) implies that any prime of the form \(n! + 1\) with \(n \geq 2^{9^3-3}\) proves the infinitude of primes of the form \(n! + 1\).

**Proof.** We leave the proof to the reader. □

**Corollary 6.** Let \(\mathcal{Y}_9\) denote the set of primes of the form \(n! + 1\). The statement \(\Sigma_9\) implies that we know an algorithm such that it returns a threshold number of \(\mathcal{Y}_9\), and this number equals max(\(\mathcal{Y}_9\)), if \(\mathcal{Y}_9\) is finite. Assuming the statement \(\Sigma_9\), a single query to an oracle for the halting problem decides the infinity of \(\mathcal{Y}_9\). Assuming the statement \(\Sigma_9\), the infinity of \(\mathcal{Y}_9\) is decidable in the limit.

**Proof.** We consider an algorithm which computes max(\(\mathcal{Y}_9 \cap [1, (2^{9^3-3} - 1)! + 1]\)). □

Let \(Z_{14} \subseteq Q_{14}\) be the system of equations in Figure 7.

![Fig. 7 Construction of the system \(Z_{14}\)](image)

**Lemma 19.** For every positive integer \(x_1\), the system \(Z_{14}\) is solvable in positive integers \(x_2, \ldots, x_{14}\) if and only if \(x_1\) and \(x_1 + 2\) are prime and \(x_1 \geq 2^{14^3-3} + 1\). In this case, positive integers \(x_2, \ldots, x_{14}\) are uniquely determined by \(x_1\). For every positive integer \(n\), at most finitely many tuples \((x_1, \ldots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}\) begin with \(n\) and solve the system \(Z_{14}\) with \(\Gamma\) instead of \(\Gamma_{14}\).
Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 20. ([24 p. 87]). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).

Lemma 21. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 13. The statement $\Sigma_{14}$ implies the infinitude of twin primes.

Proof. It follows from Lemmas 19–21. □

A prime $p$ is said to be a Sophie Germain prime if both $p$ and $2p + 1$ are prime, see [23]. It is conjectured that there are infinitely many Sophie Germain primes, see [18 p. 330]. Let $\mathcal{Z}_{16} \subseteq Q_{16}$ be the system of equations in Figure 8.

![Fig. 8 Construction of the system $\mathcal{Z}_{16}$](image)

Lemma 22. For every positive integer $x_1$, the system $\mathcal{Z}_{16}$ is solvable in positive integers $x_2, \ldots, x_{16}$ if and only if $x_1$ is a Sophie Germain prime and $x_1 \geq 2^{2^{16-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{16}$ are uniquely determined by $x_1$. For every positive integer $n$, at most finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with $n$ and solve the system $\mathcal{Z}_{16}$ with $\Gamma$ instead of $\Gamma_{16}$.
Proof. It follows from Lemmas 3, 5, and 15.

**Lemma 23.** ([18, p. 330]). \[8069496435 \cdot 10^{5072} - 1\] is a Sophie Germain prime (Harvey Dubner).

**Lemma 24.** \[8069496435 \cdot 10^{5072} - 1 > 2^{16^2} - 2\.

**Theorem 14.** The statement \(\Sigma_{16}\) implies the infinitude of Sophie Germain primes.

**Proof.** It follows from Lemmas 22–24.

**Theorem 15.** The statement \(\Sigma_6\) proves the following implication: if the equation \(x(x + 1) = y!\) has only finitely many solutions in positive integers \(x\) and \(y\), then each such solution \((x, y)\) belongs to the set \(((1, 2), (2, 3))\).

**Proof.** We leave the proof to the reader.

The question of solving the equation \(x(x + 1) = y!\) was posed by P. Erdős, see [2]. F. Luca proved that the abc conjecture implies that the equation \(x(x + 1) = y!\) has only finitely many solutions in positive integers, see [13].

**Theorem 16.** The statement \(\Sigma_6\) proves the following implication: if the equation \(x! + 1 = y^2\) has only finitely many solutions in positive integers \(x\) and \(y\), then each such solution \((x, y)\) belongs to the set \(((4, 5), (5, 11), (7, 71))\).

**Proof.** We leave the proof to the reader.

12 Hypothetical statements \(\Omega_3, \ldots, \Omega_{16}\) and their consequences

For an integer \(n \in \{3, \ldots, 16\}\), let \(\Omega_n\) denote the following statement: if a system of equations \(S \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}\) has a solution in integers \(x_1, \ldots, x_n\) greater than \(2^{2^{n^2}}\), then \(S\) has infinitely many solutions in positive integers \(x_1, \ldots, x_n\).

For every \(n \in \{3, \ldots, 16\}\), the statement \(\Sigma_n\) implies the statement \(\Omega_n\).

**Lemma 25.** The number \((65!)^2 + 1\) is prime and \(65! > 2^{2^9^2} - 2\).

**Proof.** The following PARI/GP ([17]) command

\[
(04:04) \text{gp} > \text{isprime}((65!)^2+1,\{\text{flag}=2\})
\]

\%1 = 1

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([24, p. 226]). It rigorously shows that the number \((65!)^2 + 1\) is prime.

**Lemma 26.** If positive integers \(x_1, \ldots, x_9\) solve the system \(\mathcal{Z}_9\) and \(x_1 > 2^{2^9^2} - 2\), then \(x_1 = \min(x_1, \ldots, x_9)\).

**Theorem 17.** The statement \(\Omega_9\) implies the infinitude of primes of the form \(n^2 + 1\).

Lemma 27. If positive integers \( x_1, \ldots, x_{14} \) solve the system \( Z_{14} \) and \( x_1 > 2^{14-2} \), then \( x_1 = \min(x_1, \ldots, x_{14}) \).

Theorem 18. The statement \( \Omega_{14} \) implies the infinitude of twin primes.

Proof. It follows from Lemmas 19, 21, and 27. □

13 Are there infinitely many composite Fermat numbers?

Integers of the form \( 2^{2^n} + 1 \) are called Fermat numbers. Primes of the form \( 2^{2^n} + 1 \) are called Fermat primes, as Fermat conjectured that every integer of the form \( 2^{2^n} + 1 \) is prime, see [12, p. 1]. Fermat correctly remarked that \( 2^{2^0} + 1 = 3 \), \( 2^{2^1} + 1 = 5 \), \( 2^{2^2} + 1 = 17 \), \( 2^{2^3} + 1 = 257 \), and \( 2^{2^4} + 1 = 65537 \) are all prime, see [12, p. 1].

Open Problem 3. ([12, p. 159]) Are there infinitely many composite numbers of the form \( 2^{2^n} + 1 \)?

Most mathematicians believe that \( 2^{2^n} + 1 \) is composite for every integer \( n \geq 5 \), see [11, p. 23]. Let

\[
H_n = \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\} \cup \left\{ 2^{2^x_i} = x_k : i, k \in \{1, \ldots, n\} \right\}
\]

Let \( h(1) = 1 \), and let \( h(n + 1) = 2^{2^{h(n)}} \) for every positive integer \( n \).

Lemma 28. The following subsystem of \( H_n \)

\[
\begin{align*}
    x_1 \cdot x_1 &= x_1 \\
    \forall i \in \{1, \ldots, n-1\} \quad 2^{2^{x_i}} &= x_{i+1}
\end{align*}
\]

has exactly one solution \( (x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n \), namely \( (h(1), \ldots, h(n)) \).

For a positive integer \( n \), let \( \xi_n \) denote the following statement: if a system of equations \( S \subseteq H_n \) has only finitely many solutions in positive integers \( x_1, \ldots, x_n \), then each such solution \( (x_1, \ldots, x_n) \) satisfies \( x_1, \ldots, x_n \leq h(n) \). The statement \( \xi_n \) says that for subsystems of \( H_n \) the largest known solution is indeed the largest possible.

Hypothesis 4. The statements \( \xi_1, \ldots, \xi_{13} \) are true.

Lemma 29. Every statement \( \xi_n \) is true with an unknown integer bound that depends on \( n \).

Proof. For every positive integer \( n \), the system \( H_n \) has a finite number of subsystems. □

Theorem 19. The statement \( \xi_{13} \) proves the following implication: if \( z \in \mathbb{N} \setminus \{0\} \) and \( 2^{2^z} + 1 \) is composite and greater than \( h(12) \), then \( 2^{2^z} + 1 \) is composite for infinitely many positive integers \( z \).
Proof. Let us consider the equation

\[(x + 1)(y + 1) = 2^{2^z} + 1\]  

(E)

in positive integers. By Lemma 4, we can transform the equation (E) into an equivalent system of equations \(G\) which has 13 variables \((x, y, z,\) and 10 other variables\) and which consists of equations of the forms \(\alpha \cdot \beta = \gamma\) and \(2^{2^\alpha} = \gamma\), see the diagram in Figure 9.

Since \(2^{2^z} + 1 > h(12)\), we obtain that \(2^{2^{2^{2^z}}} + 1 > h(13)\). By this, the statement \(\xi_{13}\) implies that the system \(G\) has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers.

\[\square\]

Corollary 7. Let \(W_{13}\) denote the set of composite Fermat numbers. The statement \(\xi_{13}\) implies that we know an algorithm such that it returns a threshold number of \(W_{13}\), and this number equals \(\max(W_{13})\), if \(W_{13}\) is finite. Assuming the statement \(\xi_{13}\), a single query to an oracle for the halting problem decides the infinity of \(W_{13}\). Assuming the statement \(\xi_{13}\), the infinity of \(W_{13}\) is decidable in the limit.

Proof. We consider an algorithm which computes \(\max(W_{13} \cap [1, h(12)])\). \[\square\]

References


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