

On *ZFC*-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite

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Abstract

We say that a non-negative integer m is a threshold number of a set $X \subseteq \mathbb{N}$, if X is infinite if and only if X contains an element greater than m . We formulate hypothetical statements Ψ_3, \dots, Ψ_{16} . The statement Ψ_9 implies that a simple formula computes a gigantic threshold number of the set of primes of the form $n^2 + 1$ ($n! + 1$). The statement Ψ_{16} implies that a simple formula computes a gigantic threshold number of the set of twin primes. The following problem is open: define a set $X \subseteq \mathbb{N}$ that satisfies the following conditions: (1) a known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$, (2) a known algorithm returns a threshold number of X , (3) new elements of X are still discovered, (4) we do not know any algorithm deciding the inequality $\text{card}(X) < \infty$.

Key words and phrases: arithmetical consistency of *ZFC*, composite Fermat numbers, finiteness of a set, incompleteness of *ZFC*, infiniteness of a set, oracle for the halting problem, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, twin primes, Sophie Germain primes.

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1 Introduction

The phrase “we know a non-negative integer n ” in the title means that we know an algorithm which returns n . The title cannot be formalised in *ZFC* because the phrase “we know a non-negative integer n ” refers to currently known non-negative integers n with some property. A formally stated title may look like this: *On ZFC-formulae $\varphi(x)$ for which there exists a non-negative integer n such that ZFC proves that*

$$\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) < \infty \implies \{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer n such that *ZFC* proves the above implication.

Lemma 1. *For every non-negative integer n , $\text{card}(\{x \in \mathbb{N} : x \leq n - 1\}) = n$.*

Corollary 1. *The title altered to “ZFC-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) \leq n$ if the set $\{x \in \mathbb{N} : \varphi(x)\}$ is finite” involves a weaker assumption on $\varphi(x)$.*

2 Subsets of \mathbb{N} and their threshold numbers

We say that a non-negative integer m is a threshold number of a set $X \subseteq \mathbb{N}$, if X is infinite if and only if X contains an element greater than m , cf. [25] and [26]. If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer m is a threshold number of X . If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $\{\max(X), \max(X) + 1, \max(X) + 2, \dots\}$.

It is conjectured that the set of prime numbers of the form $n^2 + 1$ is infinite, see [15, pp. 37–38]. It is conjectured that the set of prime numbers of the form $n! + 1$ is infinite, see [3, p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [15, p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^n} + 1$ is infinite, see [11, p. 23] and [12, pp. 158–159]. A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [23]. It is conjectured that the set of Sophie Germain primes is infinite, see [18, p. 330]. For each of these sets, we do not know any threshold number.

Open Problem 1. *Define a set $X \subseteq \mathbb{N}$ that satisfies the following conditions: (1) a known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$, (2) a known algorithm returns a threshold number of X , (3) new elements of X are still discovered, (4) we do not know any algorithm deciding the inequality $\text{card}(X) < \infty$.*

The following statement:

for every non-negative integer n there exist

$$\text{prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in \left[10^n, 10^n + 1\right] \quad (\text{T})$$

is a Π_1 statement which strengthens the twin prime conjecture, see [4, p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger Π_1 statements, see [1]. The statement (T) is equivalent to the non-halting of a Turing machine. If a set $X \subseteq \mathbb{N}$ is computable and we know a threshold number of X , then the infinity of X is equivalent to the halting of a Turing machine.

The height of a rational number $\frac{p}{q}$ is denoted by $H\left(\frac{p}{q}\right)$ and equals $\max(|p|, |q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple (x_1, \dots, x_n) is denoted by $H(x_1, \dots, x_n)$ and equals $\max(H(x_1), \dots, H(x_n))$.

Observation 1. *The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [14, p. 212]. The known rational solutions are $(x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left(\frac{1}{4}, \frac{15}{32}\right), \left(\frac{1}{4}, \frac{17}{32}\right), \left(-\frac{15}{16}, -\frac{185}{1024}\right), \left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [19, pp. 223–224].*

Corollary 2. *The set*

$$\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$$

is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of \mathcal{T} .

Let \mathcal{L} denote the following system of equations:

$$\begin{cases} x^2 + y^2 = s^2 \\ x^2 + z^2 = t^2 \\ y^2 + z^2 = u^2 \\ x^2 + y^2 + z^2 = v^2 \end{cases}$$

Let

$$\mathcal{F} = \left\{ n \in \mathbb{N} \setminus \{0\} : \left(\text{the system } \mathcal{L} \text{ has no solutions in } \{1, \dots, n\}^7 \right) \wedge \right. \\ \left. \left(\text{the system } \mathcal{L} \text{ has a solution in } \{1, \dots, n+1\}^7 \right) \right\}$$

A perfect cuboid is a cuboid having integer side lengths, integer face diagonals, and an integer space diagonal.

Observation 2. ([22]) *No perfect cuboids are known.*

Corollary 3. *We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{F}$. ZFC proves that $\text{card}(\mathcal{F}) \in \{0, 1\}$. We do not know any algorithm which returns $\text{card}(\mathcal{F})$. We do not know any algorithm which returns a threshold number of \mathcal{F} .*

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, & \text{if } \sin\left(9^{99999}\right) < 0 \\ \mathbb{N} \cap \left[0, \sin\left(9^{99999}\right) \cdot 9^{99999} \right) & \text{otherwise} \end{cases}$$

We do not know whether or not the set \mathcal{H} is finite.

Observation 3. *The number 9^{99999} is a threshold number of \mathcal{H} . We know an algorithm which decides the equality $\mathcal{H} = \mathbb{N}$. If $\mathcal{H} \neq \mathbb{N}$, then the set \mathcal{H} consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{H}$.*

Let

$$\mathcal{K} = \begin{cases} \{n\}, & \text{if } (n \in \mathbb{N}) \wedge (2^{\aleph_0} = \aleph_{n+1}) \\ \{0\}, & \text{if } 2^{\aleph_0} \geq \aleph_\omega \end{cases}$$

Theorem 1. *ZFC proves that $\text{card}(\mathcal{K}) = 1$. If ZFC is consistent, then for every $n \in \mathbb{N}$ the sentences "n is a threshold number of \mathcal{K} " and "n is not a threshold number of \mathcal{K} " are not provable in ZFC.*

Proof. It suffices to observe that 2^{\aleph_0} can attain every value from the set $\{\aleph_1, \aleph_2, \aleph_3, \dots\}$, see [7] and [10, p. 232]. \square

3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

Theorem 2. ([5, p. 35]) *There exists a polynomial $D(x_1, \dots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation $D(x_1, \dots, x_m) = 0$ is solvable in non-negative integers" and "The equation $D(x_1, \dots, x_m) = 0$ is not solvable in non-negative integers" are not provable in ZFC.*

Observation 4. ([9, p. 53]) *The polynomial $D(x_1, \dots, x_m)$ is not effectively known.*

Let \mathcal{Y} denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has no solutions in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, there exists an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Theorem 2 implies the next theorem.

Theorem 3. *For every $n \in \mathbb{N}$, ZFC proves that $n \in \mathcal{Y}$. If ZFC is arithmetically consistent, then the sentences " \mathcal{Y} is finite" and " \mathcal{Y} is infinite" are not provable in ZFC. If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences " n is a threshold number of \mathcal{Y} " and " n is not a threshold number of \mathcal{Y} " are not provable in ZFC.*

Let \mathcal{E} denote the set of all non-negative integers k such that the equation $D(x_1, \dots, x_m) = 0$ has a solution in $\{0, \dots, k\}^m$. Since the set $\{0, \dots, k\}^m$ is finite, there exists an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{E}$. Theorem 2 implies the next theorem.

Theorem 4. *The set \mathcal{E} is empty or infinite. In both cases, every non-negative integer n is a threshold number of \mathcal{E} . If ZFC is arithmetically consistent, then the sentences " \mathcal{E} is empty", " \mathcal{E} is not empty", " \mathcal{E} is finite", and " \mathcal{E} is infinite" are not provable in ZFC.*

Let

$$\mathcal{V} = \left\{ n \in \mathbb{N} : \left(\text{the polynomial } D(x_1, \dots, x_m) \text{ has no solutions in } \{0, \dots, n\}^m \right) \wedge \right. \\ \left. \left(\text{the polynomial } D(x_1, \dots, x_m) \text{ has a solution in } \{0, \dots, n+1\}^m \right) \right\}$$

Since the sets $\{0, \dots, n\}^m$ and $\{0, \dots, n+1\}^m$ are finite, there exists an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{V}$. According to Observation 4, at present we are not able to write a computer program that realizes such an algorithm. Theorem 2 implies the next theorem.

Theorem 5. (5) *ZFC proves that $\text{card}(\mathcal{V}) \in \{0, 1\}$.* (6) *For every $n \in \mathbb{N}$, ZFC proves that $n \notin \mathcal{V}$.* (7) *ZFC does not prove the emptiness of \mathcal{V} , if ZFC is arithmetically consistent.* (8) *For every $n \in \mathbb{N}$, the sentence " n is a threshold number of \mathcal{V} " is not provable in ZFC, if ZFC is arithmetically consistent.* (9) *For every $n \in \mathbb{N}$, the sentence " n is not a threshold number of \mathcal{V} " is not provable in ZFC, if ZFC is arithmetically consistent.*

Open Problem 2. *Define a simple algorithm A such that A returns 0 or 1 on every input $k \in \mathbb{N}$ and the set*

$$\mathcal{V} = \{k \in \mathbb{N} : \text{the program A returns 1 on input } k\}$$

satisfies conditions (5)–(9).

4 Number-theoretic lemmas

Lemma 2. For every positive integers x and y , $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Let $\Gamma(k)$ denote $(k - 1)!$.

Lemma 3. For every positive integers x and y , $x \cdot \Gamma(x) = \Gamma(y)$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. For every non-negative integers b and c , $b + 1 = c$ if and only if $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.

Lemma 5. (Wilson's theorem, [8, p. 89]). For every positive integer x , x divides $(x - 1)! + 1$ if and only if $x = 1$ or x is prime.

5 Hypothetical statements Ψ_3, \dots, Ψ_{16}

For an integer $n \geq 3$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{2\} & x_i! = x_{i+1} \\ & x_1 \cdot x_2 = x_3 \\ & x_2 \cdot x_2 = x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{U}_n .

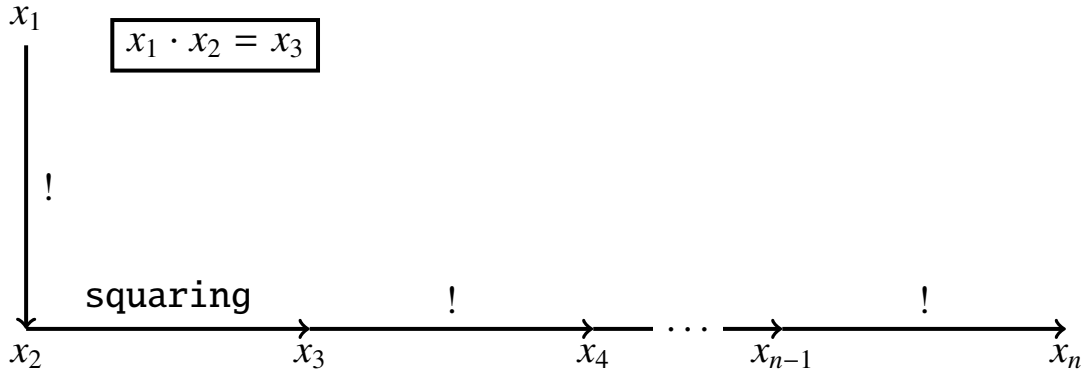


Fig. 1 Construction of the system \mathcal{U}_n

Let $g(3) = 4$, and let $g(n + 1) = g(n)!$ for every integer $n \geq 3$.

Lemma 6. For every integer $n \geq 3$, the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(2, 2, g(3), \dots, g(n))$.

Let

$$B_n = \{x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer $n \geq 3$, let Ψ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq g(n)$. The statement Ψ_n says that for subsystems of B_n the largest known solution is indeed the largest possible.

Hypothesis 1. The statements Ψ_3, \dots, Ψ_{16} are true.

Lemma 7. Every statement Ψ_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

Lemma 8. For every statement Ψ_n , the bound $g(n)$ cannot be decreased.

Proof. It follows from Lemma 6 because $\mathcal{U}_n \subseteq B_n$. □

6 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let \mathcal{A} denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

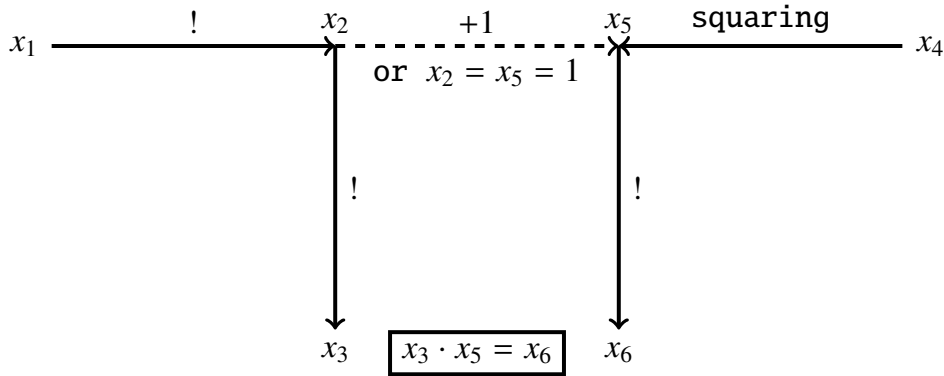


Fig. 2 Construction of the system \mathcal{A}

Lemma 9. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system \mathcal{A} is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [21, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [16].

Theorem 6. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 9, the system \mathcal{A} is solvable in positive integers x_2, x_3, x_5, x_6 . Since $\mathcal{A} \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$. Hence, $x_1! + 1 \leq g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \dots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. \square

7 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [15, pp. 37–38]. Let \mathcal{B} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system \mathcal{B} .

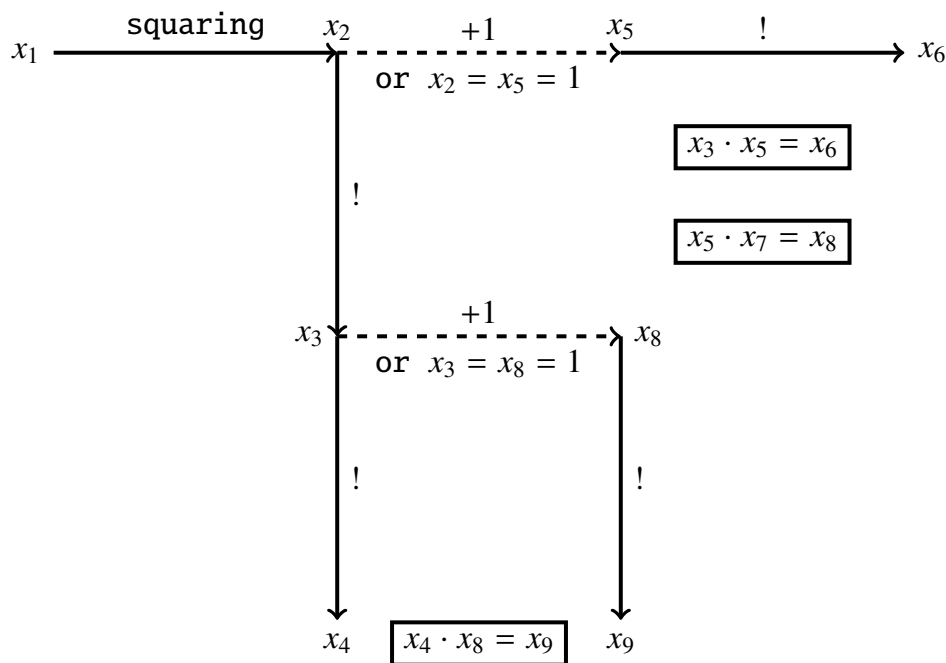


Fig. 3 Construction of the system \mathcal{B}

Lemma 10. For every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the

following equalities:

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

Proof. By Lemma 2, for every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 10 follows from Lemma 5. \square

Lemma 11. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system \mathcal{B} and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system \mathcal{B} and $x_1 = 1$, then $x_1, \dots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. \square

Theorem 7. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Suppose that the antecedent holds. By Lemma 10, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{B} . Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geq g(7)$. Hence, $(x_1^2)! \geq g(7)! = g(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)$$

Since $\mathcal{B} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > g(9)$ imply that the system \mathcal{B} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 10 and 11, there are infinitely many primes of the form $n^2 + 1$. \square

Corollary 4. *Let \mathcal{X}_9 denote the set of primes of the form $n^2 + 1$. The statement Ψ_9 implies that we know an algorithm such that it returns a threshold number of \mathcal{X}_9 , and this number equals $\max(\mathcal{X}_9)$, if \mathcal{X}_9 is finite. Assuming the statement Ψ_9 , a single query to an oracle for the halting problem decides the infinity of \mathcal{X}_9 . Assuming the statement Ψ_9 , the infinity of \mathcal{X}_9 is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_9 \cap [1, g(7)])$. \square

8 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [3, p. 443].

Theorem 8. (cf. Theorem 12). *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. We leave the analogous proof to the reader. \square

9 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [15, p. 39]. Let C denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_4! = x_5 \\ x_6! = x_7 \\ x_7! = x_8 \\ x_9! = x_{10} \\ x_{12}! = x_{13} \\ x_{15}! = x_{16} \\ x_2 \cdot x_4 = x_5 \\ x_5 \cdot x_6 = x_7 \\ x_7 \cdot x_9 = x_{10} \\ x_4 \cdot x_{11} = x_{12} \\ x_3 \cdot x_{12} = x_{13} \\ x_9 \cdot x_{14} = x_{15} \\ x_8 \cdot x_{15} = x_{16} \end{array} \right.$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system C .

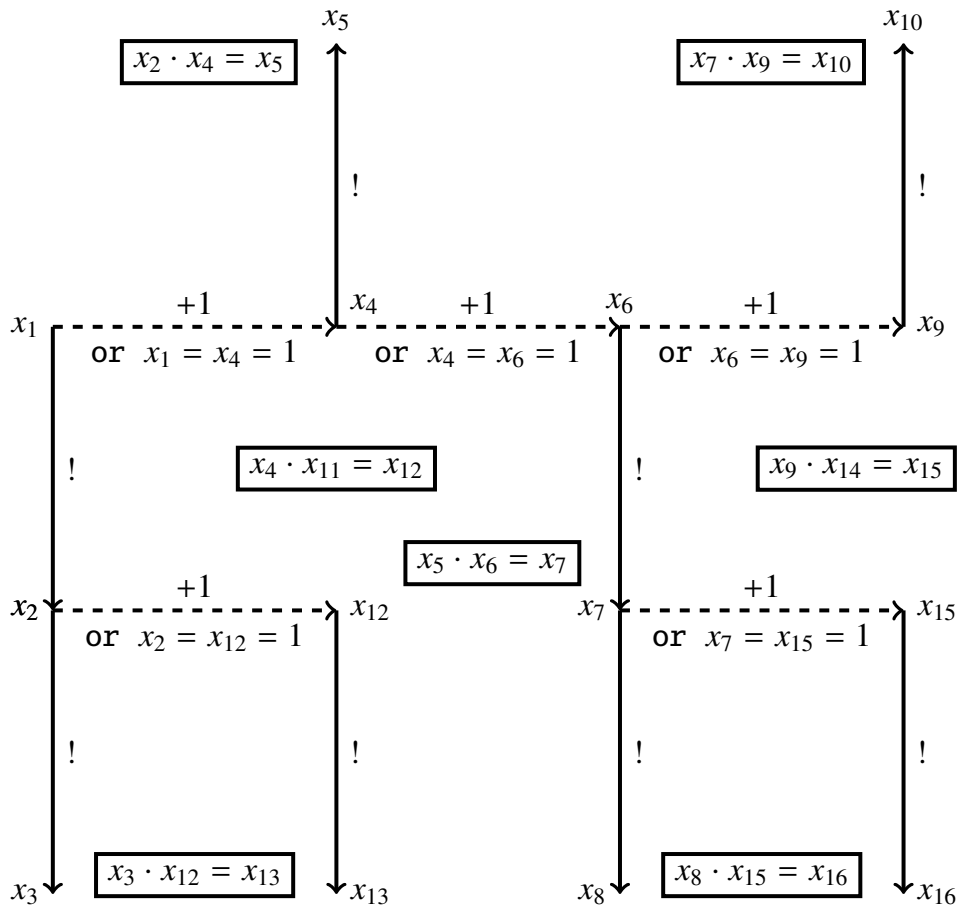


Fig. 4 Construction of the system C

Lemma 12. For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:

$$\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9! \\
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}$$

Proof. By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | (x_4 - 1)! + 1) \wedge (x_9 | (x_9 - 1)! + 1)$$

Hence, the claim of Lemma 12 follows from Lemma 5. \square

Lemma 13. There are only finitely many tuples $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system C and satisfy $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$.

Proof. If a tuple $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system C and $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$, then $x_1, \dots, x_{16} \leq 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. \square

Theorem 9. The statement Ψ_{16} proves the following implication: if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.

Proof. Suppose that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma 12, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ such that the tuple (x_1, \dots, x_{16}) solves the system C . Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geq g(14)$. Therefore, $(x_9 - 1)! \geq g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since $C \subseteq B_{16}$, the statement Ψ_{16} and the inequality $x_{16} > g(16)$ imply that the system C has infinitely many solutions in positive integers x_1, \dots, x_{16} . According to Lemmas 12 and 13, there are infinitely many twin primes. \square

Corollary 5. (cf. [6]). Let \mathcal{X}_{16} denote the set of twin primes. The statement Ψ_{16} implies that we know an algorithm such that it returns a threshold number of \mathcal{X}_{16} , and this number equals $\max(\mathcal{X}_{16})$, if \mathcal{X}_{16} is finite. Assuming the statement Ψ_{16} , a single query to an oracle for the halting problem decides the infinity of \mathcal{X}_{16} . Assuming the statement Ψ_{16} , the infinity of \mathcal{X}_{16} is decidable in the limit.

Proof. We consider an algorithm which computes $\max(\mathcal{X}_{16} \cap [1, g(14)])$. \square

10 Hypothetical statements $\Delta_5, \dots, \Delta_{14}$ and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n+1) = \Gamma(\lambda(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let \mathcal{J}_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{3\} \Gamma(x_i) = x_{i+1} \\ x_1 \cdot x_1 = x_4 \\ x_2 \cdot x_3 = x_5 \end{cases}$$

Lemma 3 and the diagram in Figure 5 explain the construction of the system \mathcal{J}_n .

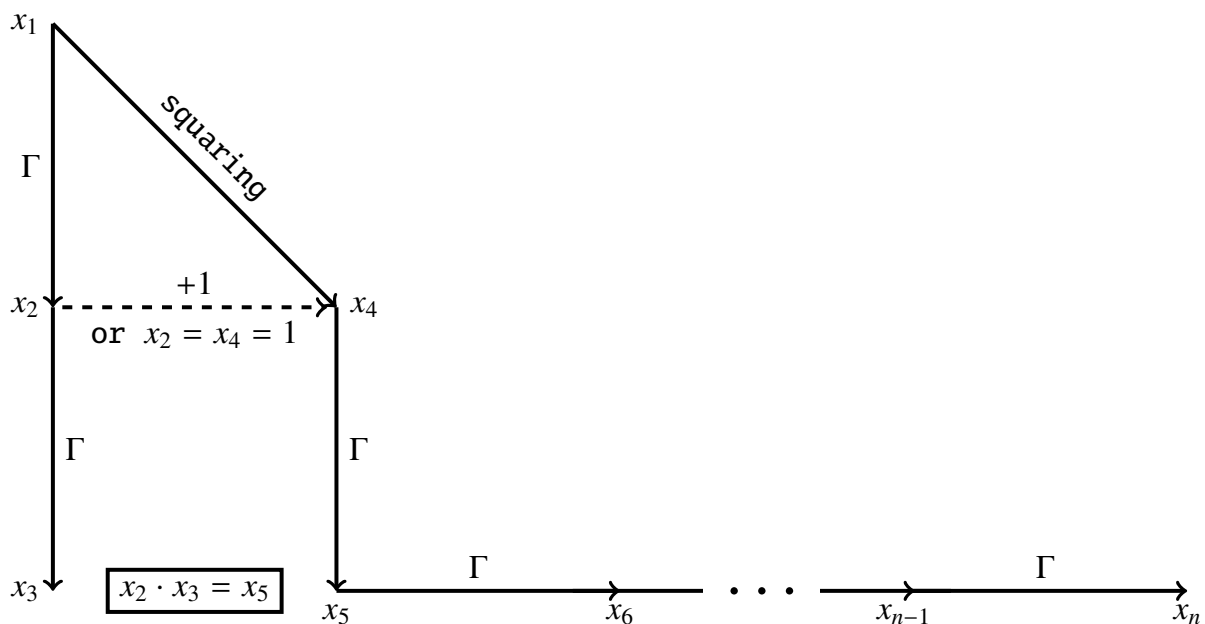


Fig. 5 Construction of the system \mathcal{J}_n

For every integer $n \geq 5$, the system \mathcal{J}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \dots, \lambda(n))$. For an integer $n \geq 5$, let Δ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq \lambda(n)$.*

Hypothesis 2. *The statements $\Delta_5, \dots, \Delta_{14}$ are true.*

Lemmas 3 and 5 imply that the statements Δ_n have similar consequences as the statements Ψ_n .

Theorem 10. *The statement Δ_6 implies that any prime number $p \geq 25$ proves the infinitude of primes.*

Proof. It follows from Lemmas 3 and 5. We leave the details to the reader. \square

11 Hypothetical statements $\Sigma_3, \dots, \Sigma_{16}$ and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \dots, 16\}$ and $k \in \{2\} \cup [2^{2^{n-3}} + 1, \infty) \cap \mathbb{N}$. For an integer $n \in \{3, \dots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer $n \in \{3, \dots, 16\}$, let P_n denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \Gamma_n(x_2) = x_1 \\ \forall i \in \{2, \dots, n-1\} x_i \cdot x_i = x_{i+1} \end{cases}$$

Lemma 14. For every integer $n \in \{3, \dots, 16\}$, $P_n \subseteq Q_n$ and the system P_n with Γ instead of Γ_n has exactly one solution in positive integers x_1, \dots, x_n , namely $(1, 2^{2^0}, 2^{2^1}, 2^{2^2}, \dots, 2^{2^{n-2}})$.

For an integer $n \in \{3, \dots, 16\}$, let Σ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq Q_n$ with Γ instead of Γ_n has only finitely many solutions in positive integers x_1, \dots, x_n , then every tuple $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ that solves the original system \mathcal{S} satisfies $x_1, \dots, x_n \leq 2^{2^{n-2}}$.

Hypothesis 3. The statements $\Sigma_3, \dots, \Sigma_{16}$ are true.

Lemma 15. (cf. Lemma 3). For every integer $n \in \{4, \dots, 16\}$ and for every positive integers x and y , $x \cdot \Gamma_n(x) = \Gamma_n(y)$ if and only if $(x+1 = y) \wedge (x \geq 2^{2^{n-3}} + 1)$.

Let $\mathcal{Z}_9 \subseteq Q_9$ be the system of equations in Figure 6.

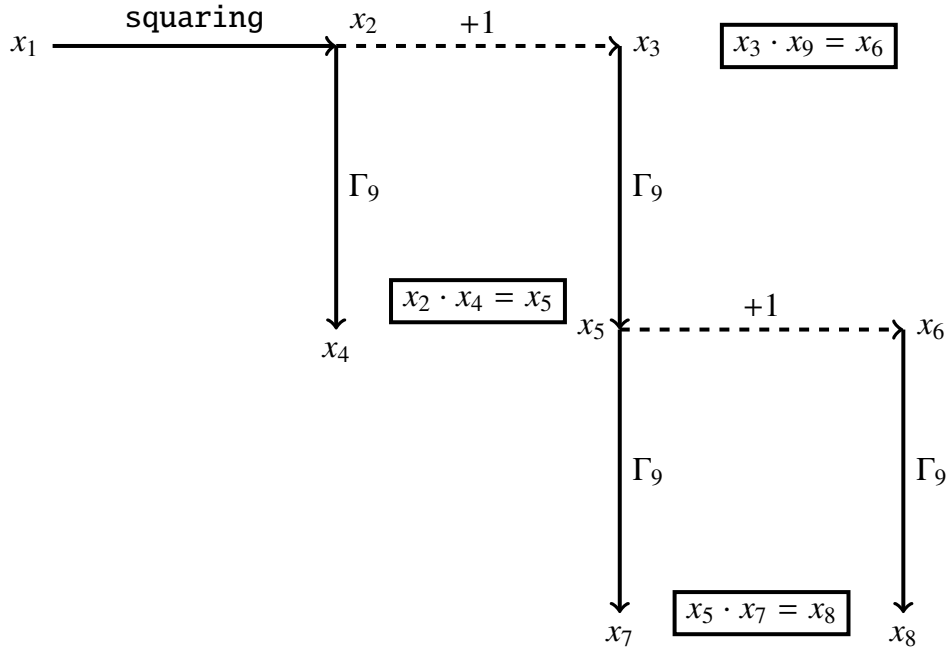


Fig. 6 Construction of the system \mathcal{Z}_9

Lemma 16. For every positive integer x_1 , the system \mathcal{Z}_9 is solvable in positive integers x_2, \dots, x_9 if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers x_2, \dots, x_9 are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ begin with n and solve the system \mathcal{Z}_9 with Γ instead of Γ_9 .

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 17. ([20]). *The number $(13!)^2 + 1 = 38775788043632640001$ is prime.*

Lemma 18. $\left((13!)^2 \geq 2^{2^{9-3}} + 1 = 18446744073709551617\right) \wedge \left(\Gamma_9((13!)^2) > 2^{2^{9-2}}\right)$.

Theorem 11. *The statement Σ_9 implies the infinitude of primes of the form $n^2 + 1$.*

Proof. It follows from Lemmas 16–18. □

Theorem 12. (cf. Theorem 8). *The statement Σ_9 implies that any prime of the form $n! + 1$ with $n \geq 2^{2^{9-3}}$ proves the infinitude of primes of the form $n! + 1$.*

Proof. We leave the proof to the reader. □

Corollary 6. *Let \mathcal{Y}_9 denote the set of primes of the form $n! + 1$. The statement Σ_9 implies that we know an algorithm such that it returns a threshold number of \mathcal{Y}_9 , and this number equals $\max(\mathcal{Y}_9)$, if \mathcal{Y}_9 is finite. Assuming the statement Σ_9 , a single query to an oracle for the halting problem decides the infinity of \mathcal{Y}_9 . Assuming the statement Σ_9 , the infinity of \mathcal{Y}_9 is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{Y}_9 \cap [1, (2^{2^{9-3}} - 1)! + 1])$. □

Let $\mathcal{Z}_{14} \subseteq \mathcal{Q}_{14}$ be the system of equations in Figure 7.

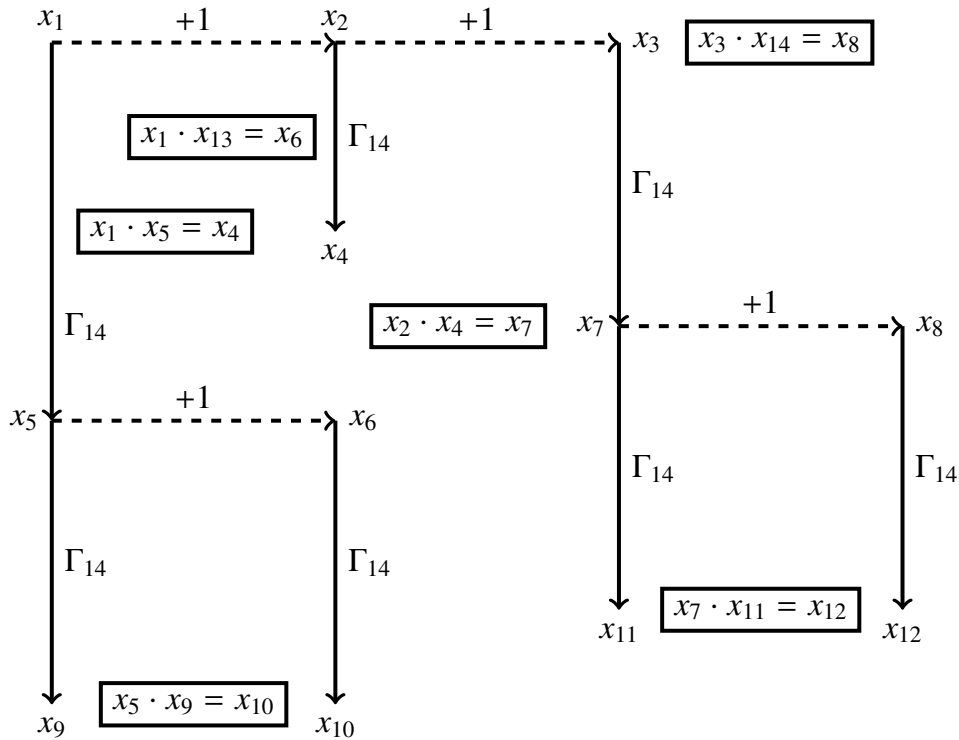


Fig. 7 Construction of the system \mathcal{Z}_{14}

Lemma 19. *For every positive integer x_1 , the system \mathcal{Z}_{14} is solvable in positive integers x_2, \dots, x_{14} if and only if x_1 and $x_1 + 2$ are prime and $x_1 \geq 2^{2^{14-3}} + 1$. In this case, positive integers x_2, \dots, x_{14} are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_{14}) \in (\mathbb{N} \setminus \{0\})^{14}$ begin with n and solve the system \mathcal{Z}_{14} with Γ instead of Γ_{14} .*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 20. ([24, p. 87]). *The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).*

Lemma 21. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 13. *The statement Σ_{14} implies the infinitude of twin primes.*

Proof. It follows from Lemmas 19–21. □

A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [23]. It is conjectured that there are infinitely many Sophie Germain primes, see [18, p. 330]. Let $\mathcal{Z}_{16} \subseteq \mathcal{Q}_{16}$ be the system of equations in Figure 8.

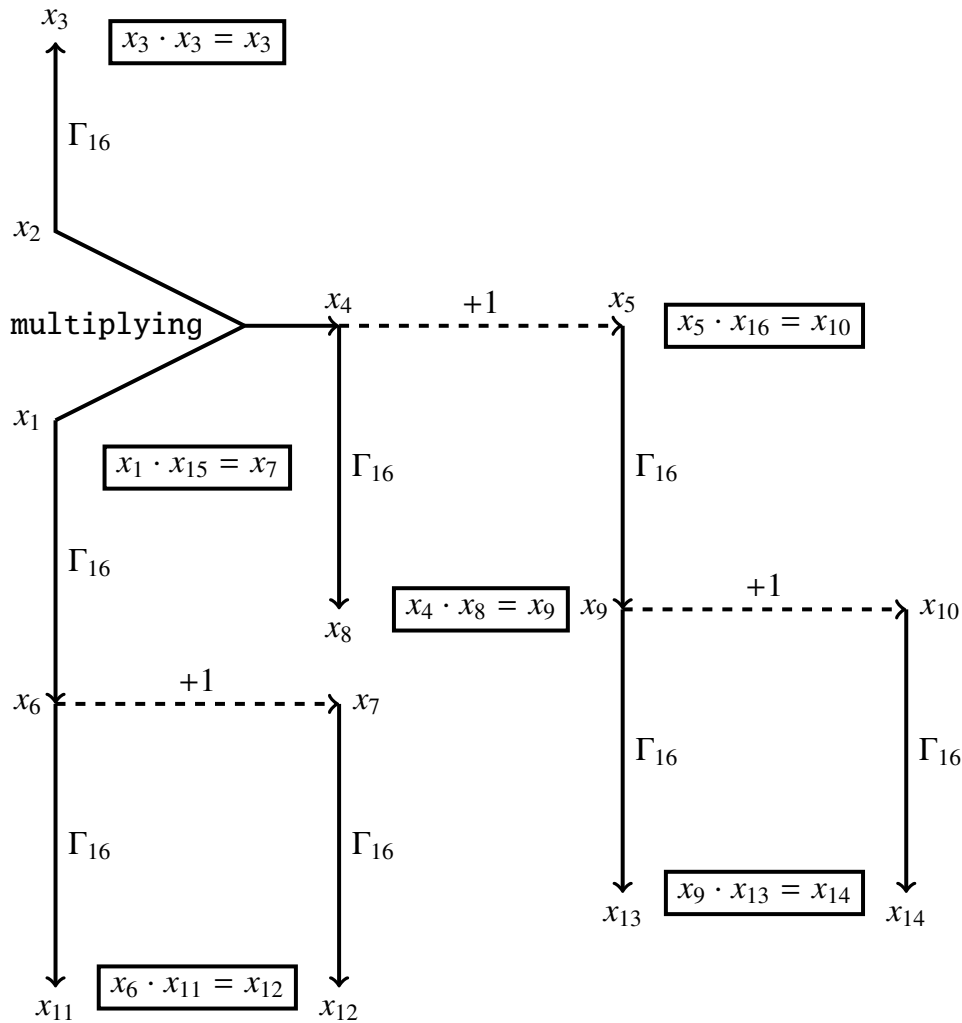


Fig. 8 Construction of the system \mathcal{Z}_{16}

Lemma 22. *For every positive integer x_1 , the system \mathcal{Z}_{16} is solvable in positive integers x_2, \dots, x_{16} if and only if x_1 is a Sophie Germain prime and $x_1 \geq 2^{2^{16-3}} + 1$. In this case, positive integers x_2, \dots, x_{16} are uniquely determined by x_1 . For every positive integer n , at most finitely many tuples $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ begin with n and solve the system \mathcal{Z}_{16} with Γ instead of Γ_{16} .*

Proof. It follows from Lemmas 3, 5, and 15. □

Lemma 23. ([18, p. 330]). $8069496435 \cdot 10^{5072} - 1$ is a Sophie Germain prime (Harvey Dubner).

Lemma 24. $8069496435 \cdot 10^{5072} - 1 > 2^{2^{16-2}}$.

Theorem 14. The statement Σ_{16} implies the infinitude of Sophie Germain primes.

Proof. It follows from Lemmas 22–24. □

Theorem 15. The statement Σ_6 proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(1, 2), (2, 3)\}$.

Proof. We leave the proof to the reader. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdős, see [2]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [13].

Theorem 16. The statement Σ_6 proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers x and y , then each such solution (x, y) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. We leave the proof to the reader. □

12 Hypothetical statements $\Omega_3, \dots, \Omega_{16}$ and their consequences

For an integer $n \in \{3, \dots, 16\}$, let Ω_n denote the following statement: if a system of equations $S \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has a solution in integers x_1, \dots, x_n greater than $2^{2^{n-2}}$, then S has infinitely many solutions in positive integers x_1, \dots, x_n . For every $n \in \{3, \dots, 16\}$, the statement Σ_n implies the statement Ω_n .

Lemma 25. The number $(65!)^2 + 1$ is prime and $65! > 2^{2^{9-2}}$.

Proof. The following PARI/GP ([17]) command

```
(04:04) gp > isprime((65!)^2+1,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([24, p. 226]). It rigorously shows that the number $(65!)^2 + 1$ is prime. □

Lemma 26. If positive integers x_1, \dots, x_9 solve the system \mathcal{Z}_9 and $x_1 > 2^{2^{9-2}}$, then $x_1 = \min(x_1, \dots, x_9)$.

Theorem 17. The statement Ω_9 implies the infinitude of primes of the form $n^2 + 1$.

Proof. It follows from Lemmas 16 and 25–26. □

Lemma 27. *If positive integers x_1, \dots, x_{14} solve the system \mathcal{Z}_{14} and $x_1 > 2^{2^{14-2}}$, then $x_1 = \min(x_1, \dots, x_{14})$.*

Theorem 18. *The statement Ω_{14} implies the infinitude of twin primes.*

Proof. It follows from Lemmas 19–21 and 27. □

13 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [12, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [12, p. 1].

Open Problem 3. ([12, p. 159]) *Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [11, p. 23]. Let

$$H_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Let $h(1) = 1$, and let $h(n+1) = 2^{2^{h(n)}}$ for every positive integer n .

Lemma 28. *The following subsystem of H_n*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \dots, h(n))$.

For a positive integer n , let ξ_n denote the following statement: *if a system of equations $S \subseteq H_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq h(n)$. The statement ξ_n says that for subsystems of H_n the largest known solution is indeed the largest possible.*

Hypothesis 4. *The statements ξ_1, \dots, ξ_{13} are true.*

Lemma 29. *Every statement ξ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system H_n has a finite number of subsystems. □

Theorem 19. *The statement ξ_{13} proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers z .*

Proof. Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \quad (\text{E})$$

in positive integers. By Lemma 4, we can transform the equation (E) into an equivalent system of equations \mathcal{G} which has 13 variables (x, y, z , and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 9.

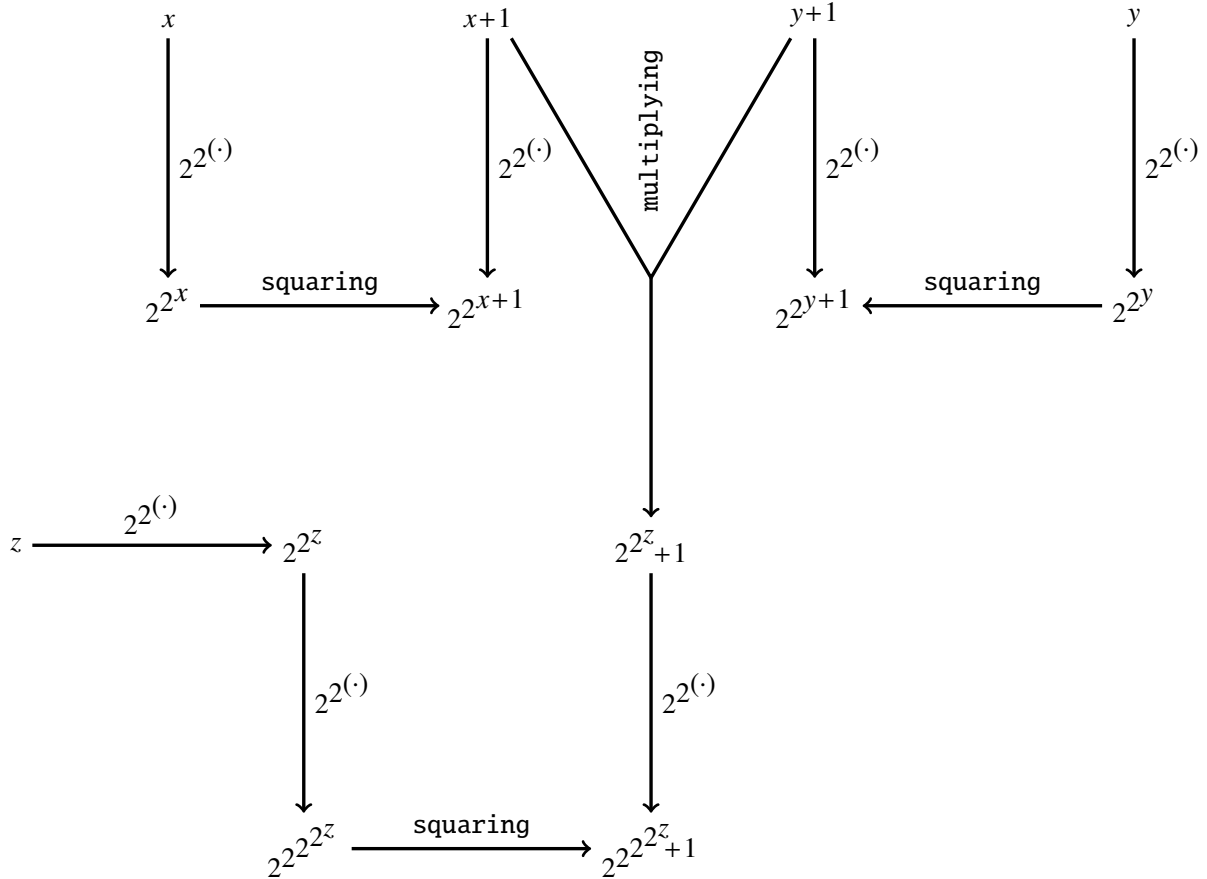


Fig. 9 Construction of the system \mathcal{G}

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z}+1}} > h(13)$. By this, the statement ξ_{13} implies that the system \mathcal{G} has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. \square

Corollary 7. Let \mathcal{W}_{13} denote the set of composite Fermat numbers. The statement ξ_{13} implies that we know an algorithm such that it returns a threshold number of \mathcal{W}_{13} , and this number equals $\max(\mathcal{W}_{13})$, if \mathcal{W}_{13} is finite. Assuming the statement ξ_{13} , a single query to an oracle for the halting problem decides the infinity of \mathcal{W}_{13} . Assuming the statement ξ_{13} , the infinity of \mathcal{W}_{13} is decidable in the limit.

Proof. We consider an algorithm which computes $\max(\mathcal{W}_{13} \cap [1, h(12)])$. \square

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